Unit 8 - Week 7: Properties of measures of Information-2 and Information Theoretic lower bounds

Course outline

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- Week 3: Randomness and entropy
- Week 4: Information and statistical inference-1
- Week 5: Information and statistical inference-2
- Week 6: Properties of measures of Information-1
- Week 7: Properties of measures of Information-2 and Information Theoretic lower bounds
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  - Proof of Fano's inequality
  - Variational formulae
  - Capacity as information radius
  - Proof of Pinsker's inequality
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- Week 8: Information theoretic lower bounds and Data Compression-1
- Week 9: Data Compression-1 (Unit 9) and Data Compression-2 (Unit 10)
- Week 10: Data Compression-2 (Unit 10) and Data Compression-3 (Unit 11)
- Week 11: Channel coding and capacity (Unit 12) and Shannon's channel coding theorem proof (Unit 13)
- Week 12: Shannon's channel coding theorem proof (Unit 13) and Gaussian channels (Unit 14)
- VIDEO DOWNLOAD
- Live Session

Weekly Assignment 7

The due date for submitting this assignment has passed.
As per our records you have not submitted this assignment.

Due on 2020-11-04, 23:59 IST.

Unless stated otherwise, assume that probability distributions are defined over a finite alphabet, and that 'log' is to the base 2.

1) Let X, Y be random variables with joint distribution P_{XY}, where X is distributed uniformly over $\{1, \ldots, M\}$. Let $g : \mathcal{Y} \to \mathcal{X}$ be a deterministic function and let $p_e = P_{XY}(X \neq g(Y))$. Which of the following are true? *1 point*
($\mathrm{Ber}(p)$ denotes the Bernoulli distribution with parameter p.)

- $I(X \wedge Y) \geq D(\mathrm{Ber}(1-p_e)||\mathrm{Ber}(\frac{1}{M}))$
- $I(X \wedge Y) \leq D(\mathrm{Ber}(p_e)||\mathrm{Ber}(1-\frac{1}{M}))$
- If $1-p_e > \frac{1}{M}$, then X and Y are not independent.
- If X and Y are not independent, then $1-p_e > \frac{1}{M}$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$I(X \wedge Y) \geq D(\mathrm{Ber}(1-p_e)||\mathrm{Ber}(\frac{1}{M}))$
If $1-p_e > \frac{1}{M}$, then X and Y are not independent.

2) Let $\mathcal{X} = \{1, 2, 3\}$. Let $W : \mathcal{X} \to \mathcal{X}$ be a channel defined as follows: $W_1 = (0, 1, 0), W_2 = (0, 0, 1), W_3 = (1, 0, 0)$. Let P and Q be two distributions on $\mathcal{X}$, with $\mathrm{supp}(P) \subset \mathrm{supp}(Q)$. Which of the following are true? *1 point*
($W \circ P$ denotes the output distribution when the input distribution is P and the channel is W.)

- $D(W \circ P||W \circ Q) \leq D(P||Q)$
- $D(W \circ P||W \circ Q) = D(P||Q)$
- $D(W \circ P||W \circ Q) = 0$
- $D(W \circ P||W \circ Q)$ might be ∞

No, the answer is incorrect.
Score: 0
Accepted Answers:
$D(W \circ P||W \circ Q) \leq D(P||Q)$
$D(W \circ P||W \circ Q) = D(P||Q)$

3) Let X and Y be random variables taking values in alphabet $\mathcal{X}$. Let P_{XY} be their joint distribution (P_{XY} is a distribution on $\mathcal{X} \times \mathcal{X}$). Which of the following are true? *1 point*
($h(p) = -p \log p - (1-p) \log(1-p)$.)

- $H(X|Y) \leq P_{XY}(X \neq Y) \log|\mathcal{X}| + h(P_{XY}(X \neq Y))$
- $H(Y|X) \leq P_{XY}(X \neq Y) \log|\mathcal{X}| + h(P_{XY}(X \neq Y))$
- $I(X \wedge Y) \geq H(X) - P_{XY}(X \neq Y) \log|\mathcal{X}| - h(P_{XY}(X \neq Y))$
- $I(X \wedge Y) \geq H(Y) - P_{XY}(X \neq Y) \log|\mathcal{X}| - h(P_{XY}(X \neq Y))$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$H(X|Y) \leq P_{XY}(X \neq Y) \log|\mathcal{X}| + h(P_{XY}(X \neq Y))$
$H(Y|X) \leq P_{XY}(X \neq Y) \log|\mathcal{X}| + h(P_{XY}(X \neq Y))$
$I(X \wedge Y) \geq H(X) - P_{XY}(X \neq Y) \log|\mathcal{X}| - h(P_{XY}(X \neq Y))$
$I(X \wedge Y) \geq H(Y) - P_{XY}(X \neq Y) \log|\mathcal{X}| - h(P_{XY}(X \neq Y))$

4) Let X be a random variable taking values in a finite alphabet $\mathcal{X} \subset \mathbb{R}$. Let P, Q be distributions on $\mathcal{X}$, such that $D(P||Q) \leq 2$. It is given that $\mathbb{E}_Q[X^2] \leq 4$. What can we infer about $\mathbb{E}_P[\log|X|]$? *1 point*

- $\mathbb{E}_P[\log|X|] \leq 4$
- $\mathbb{E}_P[\log|X|] \leq 2$
- $\mathbb{E}_P[\log|X|] \geq 6$
- $\mathbb{E}_P[\log|X|] \geq 3$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\mathbb{E}_P[\log|X|] \leq 4$
$\mathbb{E}_P[\log|X|] \leq 2$

5) Let $\mathcal{X} = \{1, 2, 3\}$. Let $W : \mathcal{X} \to \mathcal{Y}$ be a channel, and let Q be a distribution on $\mathcal{Y}$. Define $\alpha = \max_P \mathbb{E}_P[D(W_X||Q)]$, where $\max$ is over all probability distributions on $\mathcal{X}$. Let $\mathcal{D}$ be a collection of probability distributions on $\mathcal{X}$ defined as $\mathcal{D} = \{(1,0,0), (0,1,0), (0,0,1)\}$. Which of the following are true? *1 point*

- $\alpha < \max_{x \in \mathcal{X}} D(W_x||Q)$
- $\alpha = \max_{x \in \mathcal{X}} D(W_x||Q)$
- $\alpha \geq \max_{P \in \mathcal{D}} \mathbb{E}_P[D(W_X||Q)]$
- $\alpha = \max_{P \in \mathcal{D}} \mathbb{E}_P[D(W_X||Q)]$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\alpha = \max_{x \in \mathcal{X}} D(W_x||Q)$
$\alpha \geq \max_{P \in \mathcal{D}} \mathbb{E}_P[D(W_X||Q)]$
$\alpha = \max_{P \in \mathcal{D}} \mathbb{E}_P[D(W_X||Q)]$

6) Let $\mathcal{X} = \{1, 2, 3\}$, $\mathcal{Y} = \{1, 2, 3, 4\}$. Let $W : \mathcal{X} \to \mathcal{Y}$ be a channel defined as follows: $W_1 = (\frac{1}{2}, \frac{1}{2}, 0, 0), W_2 = (0, 0, 1, 0), W_3 = (0, 0, 0, 1)$. Let Pbe the uniform distribution on $\mathcal{X}$. Which of the following are true? *1 point*
($C(W)$ denotes the capacity of channel W, defined as $C(W) = \max_R I(R; W)$, where $\max$ is over all probability distributions on $\mathcal{X}$.)

- $I(P; W) = H(P) - \mathbb{E}_P[H(W_X)] = \log 3 - \frac{1}{3}$
- $I(P; W) = H(\mathbb{E}_P[W_X]) - \mathbb{E}_P[H(W_X)] = \log 3$
- $C(W) \geq \log 3 - \frac{1}{3}$
- $C(W) \geq \log 3$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$I(P; W) = H(\mathbb{E}_P[W_X]) - \mathbb{E}_P[H(W_X)] = \log 3$
$C(W) \geq \log 3 - \frac{1}{3}$
$C(W) \geq \log 3$

7) Let $\mathcal{X} = \mathcal{Y} = \{1, 2, 3\}$. Define $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ as follows: $f(1,1) = f(2,2) = f(3,3) = 1; f(i,j) = 0$ for $i \neq j$. Define $g : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ as follows: $g(1,1) = g(2,2) = g(3,3) = 1; g(2,1) = g(2,3) = 1; g(1,2) = g(1,3) = g(3,1) = g(3,2) = 0$. Let $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be an arbitrary function. Which of the following are true? *1 point*

- $\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x,y) = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} f(x,y)$
- $\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} g(x,y) = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} g(x,y)$
- $\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} h(x,y) \geq \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} h(x,y)$
- $\min_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}} h(x,y) = \min_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} h(x,y)$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} g(x,y) = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} g(x,y)$
$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} h(x,y) \geq \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} h(x,y)$
$\min_{x \in \mathcal{X}} \min_{y \in \mathcal{Y}} h(x,y) = \min_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} h(x,y)$

8) Let P and Q be distributions on $\mathcal{X}$, and let $A \subset \mathcal{X}$. Define $f : \mathcal{X} \to \mathbb{R}$ as $f(x) = I_A(x) - Q(A)$, where $I_A(x) = 1$ if $x \in A$, and $I_A(x) = 0$ otherwise. Which of the following are true? *1 point*

- $\mathbb{E}_P[f(X)] = P(A) - Q(A)$
- $\mathbb{E}_Q[f(X)] = 0$
- $\mathbb{E}_P[f(X)] \leq d(P, Q)$
- $\mathbb{E}_Q[f(X)] \leq d(P, Q)$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\mathbb{E}_P[f(X)] = P(A) - Q(A)$
$\mathbb{E}_Q[f(X)] = 0$
$\mathbb{E}_P[f(X)] \leq d(P, Q)$
$\mathbb{E}_Q[f(X)] \leq d(P, Q)$

9) Let $\mathcal{X} = \{0, 1\}$. Let P, Q be distributions on $\mathcal{X}$, where $P = Q = (0.5 - \delta, 0.5 + \delta)$. Let $\mathcal{Z} = \{(0,0), (0,1), (1,0), (1,1)\}$ and let R be a distribution on $\mathcal{Z}$. Which of the following are true? *1 point*

- $R = (0.25 - \delta, 0.25, 0.25, 0.25 - \delta)$ is a coupling of P and Q.
- $R = (0.25, 0.25 - \delta, 0.25 - \delta, 0.25)$ is a coupling of P and Q.
- $R = (0.25, 0.25 - \delta, 0.25 - \delta, 0.25 + 2\delta)$ is a coupling of P and Q.
- $R = ((0.5 - \delta)^2, 0.25 - \delta^2, 0.25 - \delta^2, (0.5 + \delta)^2)$ is a coupling of P and Q.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$R = (0.25 - \delta, 0.25, 0.25, 0.25 - \delta)$ *is a coupling of P and Q.*
$R = (0.25, 0.25 - \delta, 0.25 - \delta, 0.25 + 2\delta)$ *is a coupling of P and Q.*
$R = ((0.5 - \delta)^2, 0.25 - \delta^2, 0.25 - \delta^2, (0.5 + \delta)^2)$ *is a coupling of P and Q.*

10) Let X be a random variable taking values in $\mathcal{X}$. Let $f : \mathcal{X} \to \{0,1\}^\ell$ and $\phi : \{0,1\}^\ell \to \mathcal{X}$ be deterministic functions, such that $\mathbb{P}(\hat{X} \neq X) \leq \epsilon$, where $\hat{X} = \phi(f(X))$. Which of the following are true? *1 point*

- $H(\hat{X}|X) \geq \epsilon \log|\mathcal{X}| + 1$
- $H(\hat{X}|X) \leq \epsilon \log|\mathcal{X}| + 1$
- $H(\hat{X}|X) \geq \ell$
- $I(X \wedge \hat{X}) \leq \ell$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$H(\hat{X}|X) \leq \epsilon \log|\mathcal{X}| + 1$
$I(X \wedge \hat{X}) \leq \ell$