Unit 6 - Week 5: Information and statistical inference-2

Course outline

Weekly Assignment 5

The due date for submitting this assignment has passed.
As per our records you have not submitted this assignment.

Due on 2020-10-21, 23:59 IST.

Note: All the questions below may have multiple correct answers. There is no negative marking for wrong choices. However, partial marking is considered only when none of the marked choices is wrong.

1) Let $P = (\frac{1}{2}, \frac{1}{2})$ and $Q = (\frac{1}{2}+\epsilon, \frac{1}{2}-\epsilon)$ be two distributions on alphabet $\mathcal{X}$ of cardinality 2. Here, $0 < \epsilon < 1/4$. Which of the following are true? (You may use the inequality $e^x \geq 1 + x$ for all x) *1 point*

- $D(P||Q) = (\frac{1}{2}+\epsilon)\log\frac{\frac{1}{2}+\epsilon}{\frac{1}{2}} + (\frac{1}{2}-\epsilon)\log\frac{\frac{1}{2}-\epsilon}{\frac{1}{2}}$
- $D(P||Q) = \frac{1}{2}\log\frac{1}{1-4\epsilon^2}$
- $D(P||Q) \leq 2\epsilon^2$
- $D(P||Q) \leq 4\epsilon^2$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$D(P||Q) = \frac{1}{2}\log\frac{1}{1-4\epsilon^2}$
$D(P||Q) \leq 4\epsilon^2$

2) Let $P = (1, 0)$ and $Q = (1-\epsilon, \epsilon)$ be two distributions on corresponding alphabet $\mathcal{X} = \{1, 2\}$. Suppose that $0 < \epsilon < 1/4$. Which of the following are true? *1 point*

- $D(P||Q) = \frac{1}{2}\log\frac{1}{1-4\epsilon^2}$
- $D(P||Q) \leq 2\epsilon^2$
- $D(P||Q) = \log\frac{1}{1-\epsilon}$
- $D(P||Q) \leq 2\epsilon$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$D(P||Q) = \log\frac{1}{1-\epsilon}$
$D(P||Q) \leq 2\epsilon$

3) Let $P \equiv \mathcal{N}(\mu_1, I_d)$ and $Q \equiv \mathcal{N}(\mu_{-1}, I_d)$, where $\mathcal{N}(\mu, C)$ denotes the d-dimensional Gaussian distribution with mean vector μ and covariance matrix C. Also, $\mu_1 = (1, 1, \ldots, 1), \mu_{-1} = (-1, -1, \ldots, -1)$, and I_d is the $d \times d$ identity matrix. Which of the following are true? *1 point*

- $D(P||Q) = 2$
- $D(P||Q) \leq 2\sqrt{d}$
- $D(P||Q) = d$
- $D(P||Q) = 2d$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$D(P||Q) = 2d$

4) Let $\mathcal{X}, \mathcal{Y}$ be finite alphabets, and let $f : \mathcal{X} \times \mathcal{Y} \to (0, \infty)$, $g : \mathcal{X} \to (0, \infty)$, and $h : \mathcal{Y} \to (0, \infty)$ be positive-valued functions. Define $x^* \triangleq \operatorname{argmax}_{x \in \mathcal{X}} f(x, y)\, h(y)$. Which of the following are true? *1 point*

- $x^* = \operatorname{argmax}_{x \in \mathcal{X}} f(x, y)$
- $x^* = \operatorname{argmax}_{x \in \mathcal{X}} e^{f(x,y)}\, h(y)$
- $x^* = \operatorname{argmax}_{x \in \mathcal{X}} e^{-f(x,y)}$
- $x^* = \operatorname{argmax}_{x \in \mathcal{X}} \frac{f(x,y)}{\sum_{x \in \mathcal{X}} g(x)}$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$x^* = \operatorname{argmax}_{x \in \mathcal{X}} f(x, y)$
$x^* = \operatorname{argmax}_{x \in \mathcal{X}} e^{f(x,y)}\, h(y)$
$x^* = \operatorname{argmax}_{x \in \mathcal{X}} \frac{f(x,y)}{\sum_{x \in \mathcal{X}} g(x)}$

5) We are given a coin with an unknown probability of Heads. We toss the coin n times independently and estimate the probability of Heads using the principle of maximum likelihood. *1 point*
Let $y = (y_1, \ldots, y_n) \in \{0, 1\}^n$ be the result of the n tosses (with the mapping `Heads'$\mapsto 1$, `Tails'$\mapsto 0$). Then, maximum likelihood estimate of the probability of Heads is given by

$$g_{\mathrm{ML}}(y) = \operatorname{argmax}_{x \in [0,1]} W_x^n(y)$$

where W_x^n denotes the n-fold product distribution of Bernoulli random variable with parameter x. Let n_y denote the number of times 1 appears in $y = (y_1, \ldots, y_n)$. Which of the following are true?
(Recall that $h(p) \triangleq -p\log p - (1-p)\log(1-p)$.)

- $g_{\mathrm{ML}}(y) = \operatorname{argmax}_{x \in [0,1]} \binom{n}{n_y} x^{n_y}(1-x)^{n-n_y}$
- $g_{\mathrm{ML}}(y) = \operatorname{argmax}_{x \in [0,1]} x^{n_y}(1-x)^{n-n_y}$
- $g_{\mathrm{ML}}(y) = \frac{n_y}{n}$
- $g_{\mathrm{ML}}(y) = h(\frac{n_y}{n})$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$g_{\mathrm{ML}}(y) = \operatorname{argmax}_{x \in [0,1]} \binom{n}{n_y} x^{n_y}(1-x)^{n-n_y}$
$g_{\mathrm{ML}}(y) = \operatorname{argmax}_{x \in [0,1]} x^{n_y}(1-x)^{n-n_y}$
$g_{\mathrm{ML}}(y) = \frac{n_y}{n}$

6) Let $\mathcal{X} = \{0, 1\}$ and assume that $P_X(0) = 3/4, P_X(1) = 1/4$. Consider the binary hypothesis testing problem, where the hypotheses are *1 point*

$$\mathcal{H}_0 : Y \sim W_0.$$
$$\mathcal{H}_1 : Y \sim W_1$$

Let $P_e^*(P_X)$ be the minimum achievable (Bayesian) probability of error. Which of the following are true?

- $P_e^*(P_X)$ is achieved by the test $g(y) = \operatorname{argmax}_{x \in \mathcal{X}} W_x(y)$.
- $P_e^*(P_X)$ is achieved by the test $g(y) = \operatorname{argmax}_{x \in \mathcal{X}} P_X(x)W_x(y)$.
- $P_e^*(P_X) = \frac{1}{2}\left(1 - \max_{A \subset \mathcal{Y}} (W_0(A) - W_1(A))\right)$
- $P_e^*(P_X) = \frac{3}{4}\left(1 - \max_{A \subset \mathcal{Y}} (W_0(A) - \frac{1}{3}W_1(A))\right)$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$P_e^*(P_X)$ *is achieved by the test* $g(y) = \operatorname{argmax}_{x \in \mathcal{X}} P_X(x)W_x(y)$.
$P_e^*(P_X) = \frac{3}{4}\left(1 - \max_{A \subset \mathcal{Y}} (W_0(A) - \frac{1}{3}W_1(A))\right)$

7) Let X and Y be independent Ber(1/2) random variables defined on the alphabet $\{0, 1\} \times \{0, 1\}$. Which of the following are true? *1 point*

- $H(X|Y) = 0$
- $H(X|Y) = 1$
- $I(X \wedge Y) = 0$
- $I(X \wedge Y) = 1$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$H(X|Y) = 1$
$I(X \wedge Y) = 0$

8) Let $X \in \{0, 1\}$ and $Y \in \{0, 1\}$ be random variables with joint probability distribution P_{XY}. Moreover, suppose it is given that $D(P_{Y|X=0}||P_Y) = D(P_{Y|X=1}||P_Y) = 1/2$. Which of the following are true? *1 point*

- X and Y are independent random variables.
- $I(X \wedge Y) = 1/2$
- $H(Y) = \sum_{x,y} P_{XY}(x, y)\log\frac{1}{P_Y(y)}$
- $H(X|Y) = 0$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$I(X \wedge Y) = 1/2$
$H(Y) = \sum_{x,y} P_{XY}(x, y)\log\frac{1}{P_Y(y)}$

9) Let X and Y be Ber(1/2) random variables defined on the alphabet $\{0, 1\} \times \{0, 1\}$, such that $X = 1 - Y$. Which of the following are true? *1 point*

- $H(X|Y) = 0$
- $H(X|Y) = 1$
- $I(X \wedge Y) = 0$
- $I(X \wedge Y) = 1$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$H(X|Y) = 0$
$I(X \wedge Y) = 1$

10) Let $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ be random variables defined on a finite alphabets with joint probability distribution P_{XY}. Moreover, suppose we know the following: *1 point*

- X is distributed uniformly over $\mathcal{X}$.
- $\sum_{x \in \mathcal{X}} D(P_{Y|X=x}||P_Y) \leq 2\,|\mathcal{X}|$.
- There exists a function $g : \mathcal{Y} \to \mathcal{X}$ such that $P_{XY}(g(Y) \neq X) = 1/4$.

What can we infer about, $|\mathcal{X}|$, the cardinality of $\mathcal{X}$?

- $|\mathcal{X}| \leq 4$
- $|\mathcal{X}| \leq 16$
- $|\mathcal{X}| \geq 4$
- $|\mathcal{X}| \geq 16$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$|\mathcal{X}| \leq 16$