Unit 3 - Week 2: Uncertainty, compression, and entropy

Course outline

Weekly Assignment 2

The due date for submitting this assignment has passed.
As per our records you have not submitted this assignment.

Due on 2020-09-30, 23:59 IST.

Note: *All the questions below may have multiple correct answers. There is no negative marking for wrong answers. However, partial marking is considered only when none of the choices marked is wrong.*

Let P be a probability distribution over a finite alphabet $\mathcal{X}$. Recall the following quantities (from Week 2 lectures) for Questions 1-5:

- $L_\epsilon(P) = \min\{\lceil \log|A| \rceil : P(A) \geq 1-\epsilon, A \subset \mathcal{X}\}$
- $\bar{L}(P) = \min\{\sum_{x\in\mathcal{X}} P(x)\,|e(x)| \;:\; e:\mathcal{X}\to\{0,1\}^* \text{ is a one-to-one function}\}$

1) Let $A \subset \mathcal{X}$ such that $P(A) \geq 1 - \frac{\epsilon}{2}$. Which of the following are true? *1 point*

- $\lceil \log|A| \rceil \geq L_\epsilon(P)$
- $\lceil \log|A| \rceil < L_\epsilon(P)$
- It can never be that $\lceil \log|A| \rceil = L_\epsilon(P)$.
- $\lceil \log|A| \rceil \geq L_{\epsilon/2}(P)$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\lceil \log|A| \rceil \geq L_\epsilon(P)$
$\lceil \log|A| \rceil \geq L_{\epsilon/2}(P)$

2) Let $e:\mathcal{X}\to\{0,1\}^*$ be an arbitrary one-to-one encoding function (encoder). Which of the following are true? *1 point*
($\mathbb{E}$ is the expectation operator, $|s|$ denotes the length of binary string s)

- $\bar{L}(P) \leq \mathbb{E}[\,|e(X)|\,]$
- $\bar{L}(P) > \mathbb{E}[\,|e(X)|\,]$
- $\bar{L}(P) \leq \lceil \log|\mathcal{X}| \rceil$
- $\bar{L}(P) > \lceil \log|\mathcal{X}| \rceil$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\bar{L}(P) \leq \mathbb{E}[\,|e(X)|\,]$
$\bar{L}(P) \leq \lceil \log|\mathcal{X}| \rceil$

3) It is not difficult to see that any $x \in \mathcal{X}$ can be uniquely associated with a $\lceil \log|\mathcal{X}| \rceil$-bit binary string. Let $A_0 \subset \mathcal{X}$ with $P(A_0) \geq 1-\epsilon$. Consider an encoder $e:\mathcal{X}\to\{0,1\}^*$ that assigns a $\lceil \log|A_0| \rceil$-bit string to each $x \in A_0$, and a $\lceil \log|\mathcal{X}| \rceil$-bit string to each $x \notin A_0$. What can we say about $\mathbb{E}[\,|e(X)|\,]$? *1 point*

- $\mathbb{E}[\,|e(X)|\,] \geq (1-\epsilon)\lceil \log|A_0| \rceil + \epsilon\lceil \log|\mathcal{X}| \rceil$
- $\mathbb{E}[\,|e(X)|\,] \leq (1-\epsilon)\lceil \log|A_0| \rceil + \epsilon\lceil \log|\mathcal{X}| \rceil$
- $\mathbb{E}[\,|e(X)|\,] < L_\epsilon(P)$
- $\mathbb{E}[\,|e(X)|\,] \geq \bar{L}(P)$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\mathbb{E}[\,|e(X)|\,] \leq (1-\epsilon)\lceil \log|A_0| \rceil + \epsilon\lceil \log|\mathcal{X}| \rceil$
$\mathbb{E}[\,|e(X)|\,] \geq \bar{L}(P)$

4) For $\lambda > 0$, define $A_\lambda = \{x \in \mathcal{X} : P(x) \geq 2^{-\lambda}\}$. For each $x \in \mathcal{X}$, let $z(x) = -\log P(x)$. So, A_λ can also be written as $A_\lambda = \{x \in \mathcal{X} : z(x) \leq \lambda\}$. In the Week 2 lectures, we saw that $H(P) \triangleq \mathbb{E}[z(X)]$ is called the Shannon entropy of P. Which of the following are true? ($\mathrm{var}(Z)$ denotes the variance of random variable Z) *1 point*

- $L_\epsilon(P) \leq \lambda$ for every $\lambda > 0$
- If λ is such that $P(A_\lambda) \geq 1-\epsilon$, then $L_\epsilon(P) \leq \lambda$.
- For $\lambda = \frac{H(P)}{\epsilon}$, $P(A_\lambda) \geq 1-\epsilon$.
- For $\lambda = H(P) + \sqrt{\frac{\mathrm{var}(z(X))}{\epsilon}}$, $P(A_\lambda) \geq 1-\epsilon$

No, the answer is incorrect.
Score: 0
Accepted Answers:
If λ is such that $P(A_\lambda) \geq 1-\epsilon$, then $L_\epsilon(P) \leq \lambda$.
For $\lambda = \frac{H(P)}{\epsilon}$, $P(A_\lambda) \geq 1-\epsilon$.
For $\lambda = H(P) + \sqrt{\frac{\mathrm{var}(z(X))}{\epsilon}}$, $P(A_\lambda) \geq 1-\epsilon$

5) Recall from lectures the optimal encoding scheme for $\bar{L}(P)$: Arrange the symbols in decreasing order of their probabilities, and assign binary sequences of increasing lengths to the symbols. Let (i) denote the symbol with the i -th highest probability. Denote by ℓ_i the length of the binary sequence assigned to (i). For instance, $\ell_1 = 1, \ell_2 = 1, \ell_3 = 2, \ldots$ and so on. Which of the following are true for ℓ_i? *1 point*

- $2^1 + 2^2 + \cdots + 2^{\ell_i - 1} < i$
- $2^1 + 2^2 + \cdots + 2^{\ell_i} \geq i$
- $\ell_i = \lceil \log(\frac{i}{2}+1) \rceil$
- $\ell_i = \lfloor \log(\frac{i}{2}+1) \rfloor$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$2^1 + 2^2 + \cdots + 2^{\ell_i - 1} < i$
$2^1 + 2^2 + \cdots + 2^{\ell_i} \geq i$
$\ell_i = \lceil \log(\frac{i}{2}+1) \rceil$

6) Let X be a random variable defined on $\{0,1\}$ and Y be a random variable defined on $\{-1,1\}$. Let P_X and P_Y be the probability distributions of X and Y, respectively.
If $P_X(0) = 0.25, P_X(1) = 0.75$ and $P_Y(-1) = 0.75, P_Y(1) = 0.25$, which of the following are true? *1 point*

- $H(P_Y) > H(P_X)$
- $H(P_Y) < H(P_X)$
- $H(P_Y) = H(P_X)$
- Can't compare because X and Y are defined on different alphabets.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$H(P_Y) = H(P_X)$

7) Let X be a binary random variable with $P_X(0) = (1-p), P_X(1) = p$.
Note that $H(P_X) = h(p)$, where $h(p) \triangleq -p\log p - (1-p)\log(1-p)$. Which of the following are true? (Take $0\log 0 = 0$.) *1 point*

- $h(p)$ is a strictly increasing function for $p \in [0,1]$.
- $h(p) = h(1-p)$ for $p \in [0,1]$.
- $h(p) \leq h(0.5)$ for every $p \in [0,1]$.
- $h(1) = 0$.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$h(p) = h(1-p)$ for $p \in [0,1]$.
$h(p) \leq h(0.5)$ for every $p \in [0,1]$.
$h(1) = 0$.

8) A coin with probability of Heads p is tossed repeatedly until the first Heads appears. Let $P = (p_1, p_2, \ldots)$, where p_i denotes the probability that the first Heads appears at the i -th toss. What is $H(P)$? (Choose all that apply) *1 point*

- $H(P) = \infty$.
- $H(P) = -\log p - \left(\frac{1}{p} - 1\right)\log(1-p)$.
- $H(P) = h(p)$, where $h(p)$ is the entropy of a $\mathrm{Ber}(p)$ random variable.
- $H(P) = \frac{h(p)}{p}$, where $h(p)$ is the entropy of a $\mathrm{Ber}(p)$ random variable.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$H(P) = -\log p - \left(\frac{1}{p} - 1\right)\log(1-p)$.
$H(P) = \frac{h(p)}{p}$, where $h(p)$ is the entropy of a $\mathrm{Ber}(p)$ random variable.

9) Suppose $\mathcal{X} = \{1, \ldots, n\}$. Let $\mathcal{F}$ be the collection of all functions from $\mathcal{X}$ to $\{0,1\}^\ell$, and let $F:\mathcal{X}\to\{0,1\}^\ell$ be a random function sampled uniformly from $\mathcal{F}$ (i.e. F is a random hash). Let $i, j \in \mathcal{X}$ be such that $i \neq j$. Which of the following are true?($\mathbb{P}$ denotes probability) *1 point*

- $F(i)$ and $F(j)$ are independent random variables..
- $F(1), \ldots, F(n)$ are mutually independent random variables..
- $\mathbb{P}(F(i) = s) = \frac{1}{2^\ell}$ for every $s \in \{0,1\}^\ell$.
- $\mathbb{P}(F(i) = s | F(j) = s') = \frac{1}{2^\ell}$ for every $s, s' \in \{0,1\}^\ell$.

No, the answer is incorrect.
Score: 0
Accepted Answers:
$F(i)$ and $F(j)$ are independent random variables..
$F(1), \ldots, F(n)$ are mutually independent random variables..
$\mathbb{P}(F(i) = s) = \frac{1}{2^\ell}$ for every $s \in \{0,1\}^\ell$.
$\mathbb{P}(F(i) = s | F(j) = s') = \frac{1}{2^\ell}$ for every $s, s' \in \{0,1\}^\ell$.

10) Suppose $Y_0, Y_1, \ldots, Y_r$ are independent random variables, where each Y_i is chosen uniformly at random from $\{1, \ldots, m\}$. Let $\mathcal{D}$ denote the event that $Y_0 \notin \{Y_1, \ldots, Y_r\}$, i.e., Y_0 is distinct from $Y_1, \ldots, Y_r$. Let $\mathcal{C} = \mathcal{D}^c$, the complement of $\mathcal{D}$.
Which of the following are true?($\mathbb{P}$ denotes probability) *1 point*

- $\mathbb{P}(\mathcal{C}) = 1 - \left(1 - \frac{1}{m}\right)^r$.
- $\mathbb{P}(\mathcal{C}) \leq \frac{r}{m}$
- $\mathbb{P}(\mathcal{C}) > \frac{r}{m}$
- $\mathbb{P}(\mathcal{C}) \leq \sum_{i=1}^{r} \mathbb{P}(Y_0 = Y_i)$

No, the answer is incorrect.
Score: 0
Accepted Answers:
$\mathbb{P}(\mathcal{C}) = 1 - \left(1 - \frac{1}{m}\right)^r$.
$\mathbb{P}(\mathcal{C}) \leq \frac{r}{m}$
$\mathbb{P}(\mathcal{C}) \leq \sum_{i=1}^{r} \mathbb{P}(Y_0 = Y_i)$