Solutions to Assignment 4 - Part 2

1 Nonuniform Oscillator

1b) System in consideration is $\dot{\theta} = \mu + \cos\theta + \cos 2\theta$. To obtain the fixed points, we equate $f(\theta)$ to zero. That is,

$$\mu + \cos\theta + \cos 2\theta = 0$$
$$\implies 2\cos^2\theta + \cos\theta + \mu - 1 = 0$$
$$\implies \cos^2\theta + \frac{1}{2}\cos\theta + \frac{\mu-1}{2} = 0$$
$$\implies \cos\theta = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4\left(\frac{\mu-1}{2}\right)}}{2}$$

Note that, in general, $\cos\theta \in \mathbb{C}$. For $\cos\theta \in [-1, 1] \subset \mathbb{R}$, we require

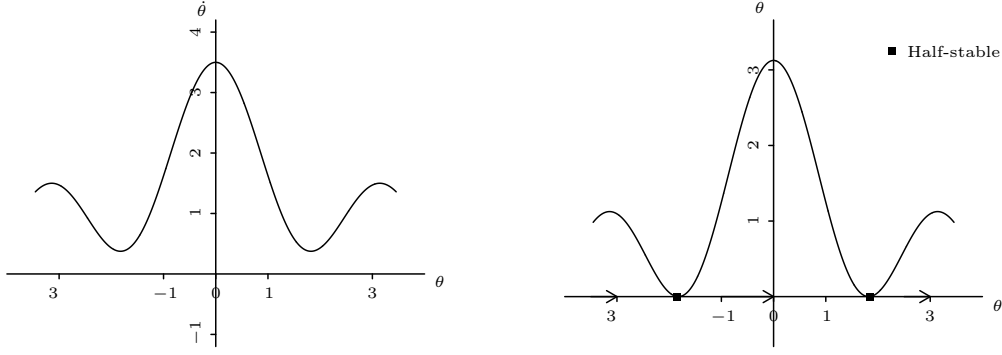$$\sqrt{\frac{1}{4} - 4\left(\frac{\mu-1}{2}\right)} \geq 0.$$

This yields $\mu \leq 9/8$. For $\mu > 9/8$, there exists no real value for $\cos\theta$, and hence there are no fixed points. This case is shown in Figure 1.1a.

When $\mu = 9/8$, $\cos\theta = -1/4$. There exist two half-stable fixed points as shown in Figure 1.1b. The case where $\mu < 9/8$ is yet to be considered.

We know that $-1 \leq \cos\theta \leq 1$. We now use these bounds on $\cos\theta$ to obtain limits on the value of μ. Recall that,

$$\cos\theta = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4\left(\frac{\mu-1}{2}\right)}}{2}.$$

(a) $\mu > 9/8$, no fixed points.

(b) $\mu = 9/8$, 2 half-stable fixed points.

Figure 1.1: Phase portraits for $\dot{\theta} = \mu + \cos\theta + \cos 2\theta$, for various values of parameter μ.

The smaller of the two roots must be > -1. This gives us

$$-1 \leq \frac{-\frac{1}{2} - \sqrt{\frac{1}{4} - 4\left(\frac{\mu-1}{2}\right)}}{2},$$

which gives $\mu \geq 0$.

The larger of the two roots must be < 1. This gives us

$$\frac{-\frac{1}{2} + \sqrt{\frac{1}{4} - 4\left(\frac{\mu-1}{2}\right)}}{2} \leq 1,$$

which on simplification yields $\mu \geq -2$. We now know that $\mu = 0$ and $\mu = -2$, could be critical points of μ. Recall that, the upper bound for μ is $9/8$.

We now consider the regime where $0 < \mu < 9/8$. In this range of μ, there exist two possible values of $\cos\theta$ given by the $\pm$ roots of the quadratic equation, and this gives 4 fixed points since the inverse cosine of each root yields 2 fixed points. The phase portrait for this regime is shown in Figure 1.2a, it can be seen that there are 2 stable and 2 unstable fixed points.

When $\mu = 0$, the negative root of the quadratic equation gives $\cos\theta = -1$, which is a boundary condition. This value of $\cos\theta$ yields two fixed points. However, the positive root of the quadratic equation also exists in this case, and this root gives rise to two fixed points. Therefore, the number of fixed points remains the same, but the existence of the boundary condition highlights the possibility of a change in stability of the equilibrium. This indeed occurs as seen in the phase portrait shown in Figure 1.2b.
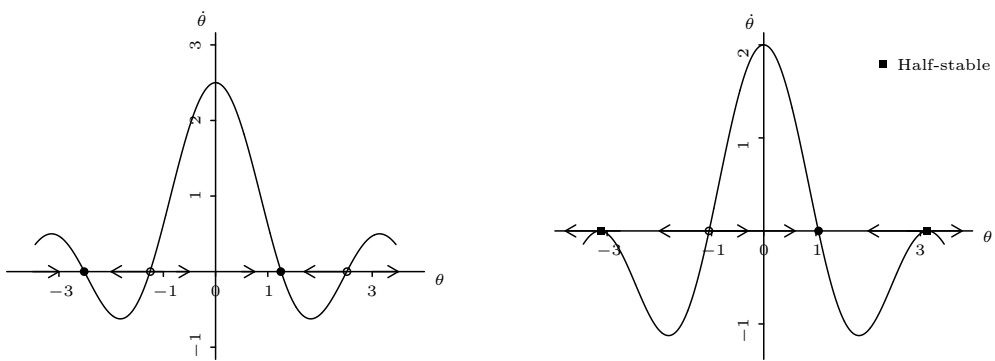
(a) $0 < \mu < 9/8$, 2 stable and 2 unstable fixed points.

(b) $\mu = 0$, 1 stable, 1 unstable and 2 half-stable fixed points.

Figure 1.2: Phase portraits for $\dot{\theta} = \mu + \cos\theta + \cos 2\theta$, for various values of parameter μ.