Introduction to Nonlinear Dynamics
Indian Institute of Technology Madras
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Solutions to Assignment 4 - Part 1

1 Examples and Definitions

1a) Consider the dynamical system $\dot{\theta} = 1 + 2\cos\theta$. Equating $f(\theta) = 0$, we can evaluate the fixed points of this system to be $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ for $0 \leq \theta \leq 2\pi$. In the intervals $\theta \in [0, \frac{2\pi}{3})$ and $\theta \in (\frac{4\pi}{3}, 2\pi]$, $f(\theta) > 0$. In the interval $\theta \in [\frac{2\pi}{3}, \frac{4\pi}{3}]$, $f(\theta) \leq 0$. Hence, it is evident that $\theta = \frac{2\pi}{3}$ is a stable fixed point and $\theta = \frac{4\pi}{3}$ is an unstable fixed point. The phase portrait is given by Fig. 1.1.

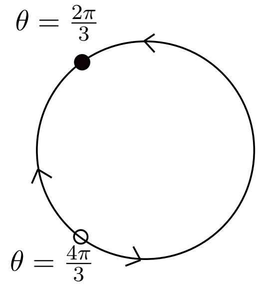


Figure 1.1: Phase portrait of $\dot{\theta} = 1 + 2\cos\theta$.

1b) Consider the dynamical system $\dot{\theta} = \sin\theta + \cos\theta$. Equating $f(\theta) = 0$, we can evaluate the fixed points of this system to be $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ for $0 \leq \theta \leq 2\pi$. In the intervals $\theta \in [0, \frac{3\pi}{4})$ and $\theta \in (\frac{7\pi}{4}, 2\pi]$, $f(\theta) > 0$. In the interval $\theta \in [\frac{3\pi}{4}, \frac{7\pi}{4}]$, $f(\theta) \leq 0$. Hence, it is evident that $\theta = \frac{3\pi}{4}$ is a stable fixed point and $\theta = \frac{7\pi}{4}$ is an unstable fixed point. The phase portrait is given by Fig. 1.2.

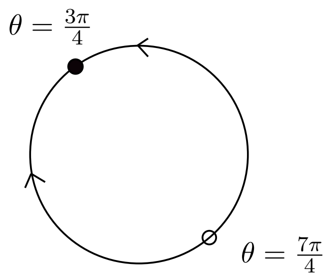


Figure 1.2: Phase portrait of $\dot{\theta} = \sin\theta + \cos\theta$.

2 Uniform Oscillator

1) The minute hand takes $T_1 = 1$ hour to finish one rotation and the hour hand takes $T_2 = 12$ hours. If θ_1 represents the position of the minute hand, and θ_2 that of the hour hand, we have

$$\begin{aligned} \dot{\theta}_1 &= \frac{2\pi}{T_1} \\ \dot{\theta}_2 &= \frac{2\pi}{T_2} \end{aligned}$$

If ϕ represents the phase difference between the two hands, we have

$$\phi = \theta_1 - \theta_2,$$

from which the rate of change of phase difference can be written as

$$\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2.$$

The two hands will overlap when the phase difference is 2π. The time required for the phase difference to reach 2π can be computed using the rate of change of phase difference as

$$\begin{aligned} T_{\text{lap}} &= \frac{2\pi}{\dot{\phi}} \\ &= \frac{2\pi}{\dot{\theta}_1 - \dot{\theta}_2} \\ &= \left(\frac{1}{T_1} - \frac{1}{T_2}\right)^{-1} \\ &= \frac{12}{11}. \end{aligned}$$

Therefore, it takes 12/11 hours for the two hands to overlap each other. If the hands first overlapped at 12 : 00, they would next overlap at 13 : 05 approximately.

3 Nonuniform Oscillator

1a) System in consideration is $\dot{\theta} = \mu \sin\theta - \sin 2\theta$. To obtain the fixed points, we equate $f(\theta)$ to zero. That is,

$$\begin{aligned} \mu \sin\theta - \sin 2\theta &= 0 \\ \implies \sin\theta\,(\mu - 2\cos\theta) &= 0 \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \sin\theta &= 0 & \qquad \mu - 2\cos\theta &= 0 \\ \implies \theta &= 0,\ \pi & \qquad \implies \cos\theta &= \frac{\mu}{2} \end{aligned}$$

Intuitively, we can see that $\mu = 0, -2, 2$ correspond to critical values of the parameter. We now plot the phase portraits to confirm this. The phase portraits are shown in Figure 3.1a-3.1e. The fixed points for each case are listed in Table 3.1.

Table 3.1

$\mu \geq 2$	$-\pi,\ 0,\ \pi$
$0 < \mu < 2$	$-\pi,\ 0,\ \pi,\ \cos^{-1}(\mu/2)$
$\mu = 0$	$-\pi, -\pi/2,\ 0,\ \pi/2,\ \pi$
$-2 < \mu < 0$	$-\pi,\ 0,\ \pi,\ \cos^{-1}(\mu/2)$
$\mu \leq -2$	$-\pi,\ 0,\ \pi$

It can be seen that fixed points are created and destroyed at the aforementioned critical values of μ, and hence the system undergoes a saddle-node bifurcation.

4 Linear Systems

1) The given system is

$$\begin{aligned} \dot{x} &= 4x - y, \\ \dot{y} &= 2x + y. \end{aligned}$$

a) The above system can be re-written in matrix form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

It is now in the form $\dot{\mathbf{x}} = A\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}.$$

b) In order to find the characteristic polynomial, we need to evaluate

$$\det(A - \lambda I).$$

That is,

$$\det\left(\begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix}\right).$$

This simplifies to

$$(4-\lambda) \times (1-\lambda) + 2,$$
$$\lambda^2 - 5\lambda + 6.$$

Therefore, $f(\lambda) = \lambda^2 - 5\lambda + 6$ is the required characteristic polynomial.

c) To find the eigenvalues, we need to solve the equation $f(\lambda) = \lambda^2 - 5\lambda + 6 = 0$. This yields $\lambda_1 = 2$ and $\lambda_2 = 3$ as the eigenvalues.

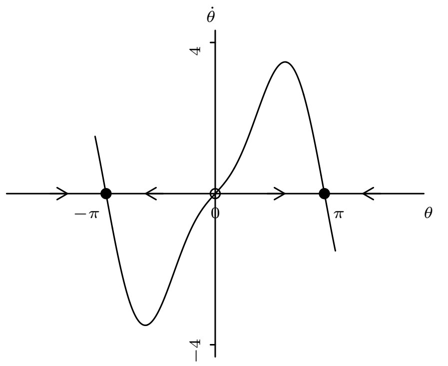
To obtain the eigenvector corresponding to an eigenvalue λ, we need to solve the equation $A\mathbf{x} = \lambda\mathbf{x}$. This computation yields the eigenvector $\mathbf{v}_1$ corresponding to the eigenvalue $\lambda_1 = 2$ and the eigenvector $\mathbf{v}_2$ corresponding to the eigenvalue $\lambda_2 = 3$, where

$$\mathbf{v}_1 = \begin{bmatrix} a \\ 2a \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} b \\ b \end{bmatrix},$$

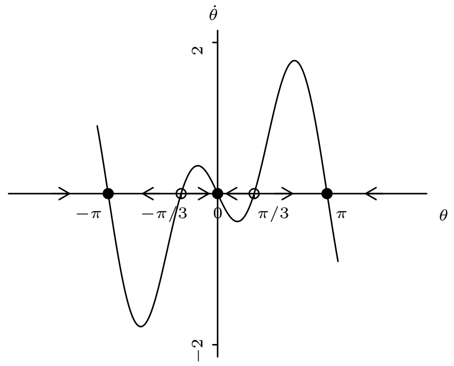
where $a, b \in \mathbb{R}$. For the particular value (for simplicity) of $a = b = 1$, we obtain

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

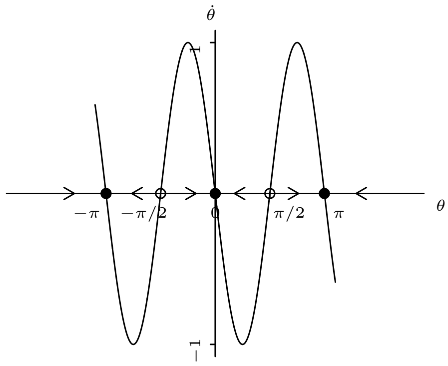
d) For the matrix A, we compute $\Delta = \det(A)$ and $\tau = \operatorname{trace}(A)$. We obtain $\Delta = 6$ and $\tau = 5$. The quantity $\tau^2 - 4\Delta$ evaluates to 1. Since $\tau^2 - 4\Delta > 0$, the origin is a node. Since $\tau > 0$, the origin is unstable. Hence, the origin is an unstable node for the given system.



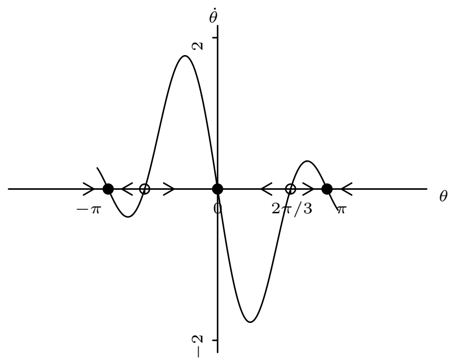
(a) Regime: $\mu \geq 2$. Value: $\mu = 3$.



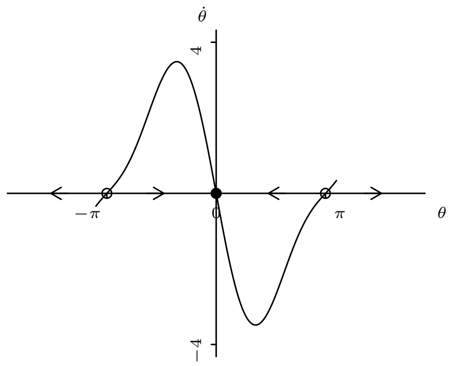
(b) Regime: $0 < \mu < 2$. Value: $\mu = 1$.



(c) Regime: $\mu = 0$.



(d) Regime: $-2 < \mu < 0$. Value: $\mu = -1$



(e) Regime: $\mu \leq -2$. Value: $\mu = -3$.

Figure 3.1: Phase portraits for $\dot{\theta} = \mu \sin\theta - \sin 2\theta$, for various values of parameter μ.