

Solutions to Assignment 3

1 Saddle-Node bifurcation

- 1) a) Comparing $\dot{x} = 1 + rx + x^2$ with $\dot{x} = f(x)$, we obtain $f(x) = 1 + rx + x^2$. To understand the change in qualitative behaviour of system as r is varied, consider the variation of the equilibrium with change in r , obtained by equating $f(x)$ to 0. This yields,

$$x^* = \frac{-r \pm \sqrt{r^2 - 4}}{2}.$$

Clearly, when $r \in (-2, 2)$, there is no real equilibrium. At $r = \pm 2$, there is one equilibrium. When $r < -2$ and $r > 2$, there are two equilibria. That is, the equilibrium “splits” into two equilibria as r is appropriately varied. Figure 1.1 shows the vector field for $r < -2$, $r = 2$ and $r > 2$.

- b) As we can see from the above equations, the system has a single equilibrium when $r = \pm 2$, which splits into two equilibria as $|r|$ is increased. Hence, the critical value of r is $r_{cr} = \pm 2$.

2 Transcritical bifurcation

- 1) Comparing $\dot{x} = x(r - e^x)$ with $\dot{x} = f(x)$, we obtain $f(x) = x(r - e^x)$. When $r > 0$, there are two equilibria; namely, $x_1^* = 0$ and $x_2^* = \ln r$. At $r = 1$, the stability of the equilibria change. Hence, the critical value is $r_{cr} = 1$. The bifurcation diagram is as shown in Figure 2.1.

3 Pitchfork bifurcation

- 1) a) Qualitatively different vector fields for $\dot{x} = rx - 4x^3$ are as shown in Figure 3.1.
- b) Transcritical bifurcation takes place at $r_{cr} = 0$.

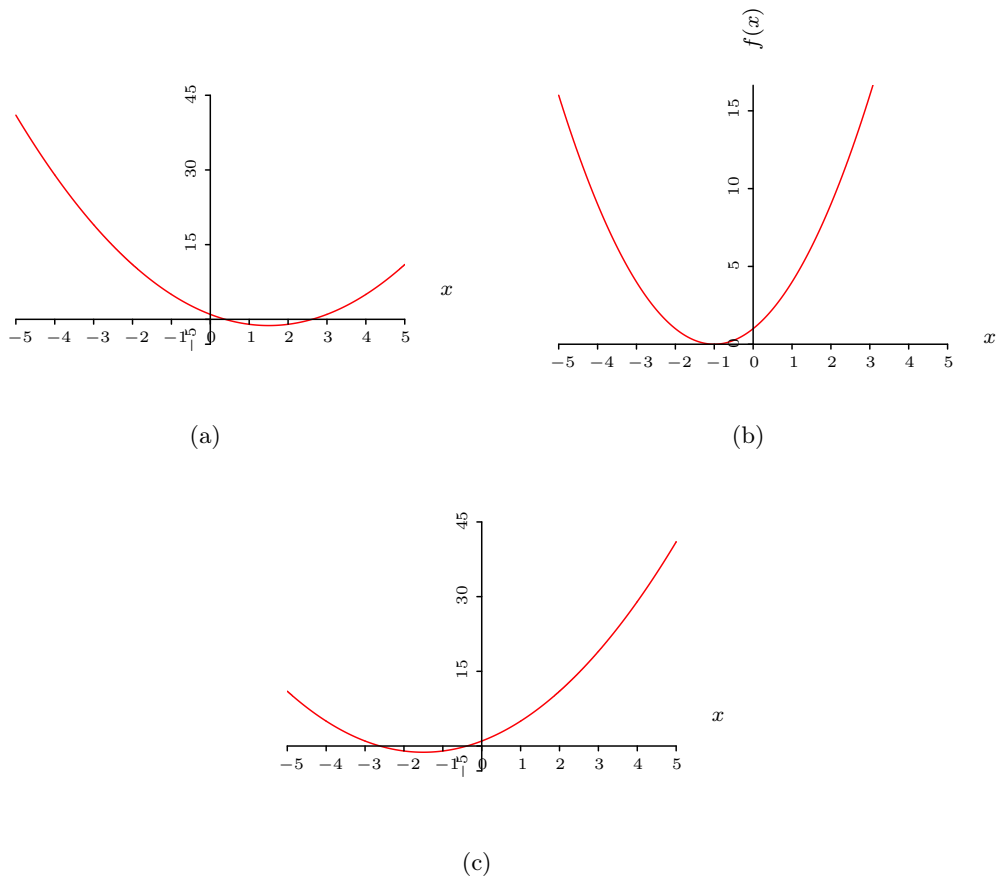


Figure 1.1: Plot of $f(x)$ vs x for $\dot{x} = 1 + rx + x^2$. (a) is for $r < -2$, (b) is for $r = 2$ and (c) is for $r > 2$.

- c) Since the cubic term is associated with a negative sign, it stabilizes the solution in the system in consideration. Hence, the resulting bifurcation is supercritical in nature.

4 Identify the bifurcation

- 1) a) For the system $\dot{x} = 5 - re^{-x^2}$, there is no equilibrium when $r < 0$. For $r \in [0, 5]$, it has one equilibrium which “splits” into two equilibria as r is increased beyond 5. Since there is a birth of two equilibria from one, the system undergoes a saddle-node bifurcation at $r_{cr} = 5$.
- b) The system $\dot{x} = r - 3x^2$ has no equilibrium for $r < 0$. At $r = 0$, there is one equilibrium which “splits” into two equilibria as r is increased beyond 0. Hence, this system undergoes a saddle-node bifurcation at $r_{cr} = 0$.

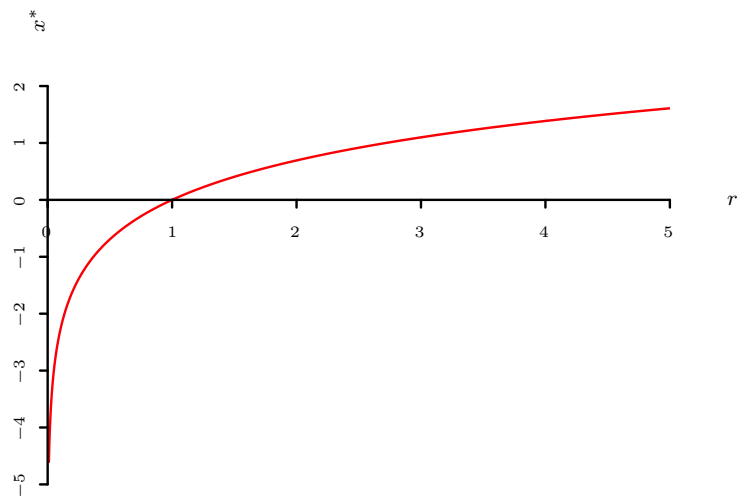
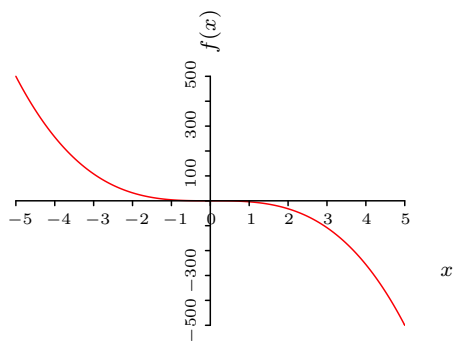
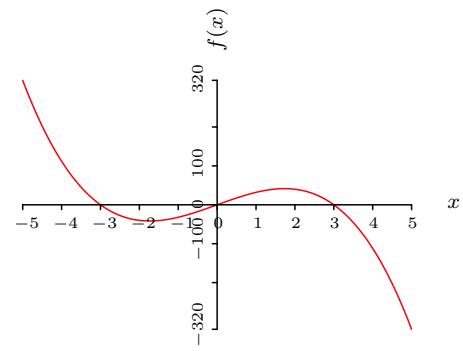


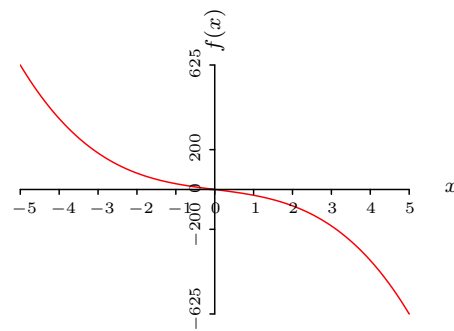
Figure 2.1: Bifurcation diagram for the system $\dot{x} = x(r - e^x)$.



(a)



(b)



(c)

Figure 3.1: Plot of $f(x)$ vs x for $\dot{x} = rx - 4x^3$. (a) is for $r = 0$, (b) is for $r > 0$ and (c) is for $r < 0$.