Introduction to Nonlinear Dynamics
Indian Institute of Technology Madras
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Solutions to Assignment 2

1 Existence and uniqueness

1) Comparing $\dot{x} = -x^c$ with $\dot{x} = f(x)$, we obtain $f(x) = -x^c$.

   a) For $x \geq 0$, $x^c \geq 0 \, \forall c \in \mathbb{R}$. Therefore, for $x \geq 0$, $f(x) \leq 0 \, \forall c \in \mathbb{R}$. This implies that for any Real choice of c, $\mathrm{d}x/\mathrm{d}t$ is non-positive, and $x = 0$ would be a stable fixed point.

   b)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x^c$$
$$\Longrightarrow \frac{\mathrm{d}x}{x^c} = -\mathrm{d}t.$$

   Integrating and simplifying, we obtain $x^{(1-c)} = (1-c)(t_0 - t)$, where t_0 is the integration constant. Therefore, the time taken by the particle to travel from $x = 0$ to $x = 1$ is $T = t(1) - t(0)$. From the above relation between x and t, we obtain $T = 1/(1-c)$.

2 Impossibility of oscillations

1) To prove by contradiction, we assume that x is periodic with period $T > 0$, *i.e.*, $x(t+T) = x(t)$ and $x(t+s) \neq x(t), 0 < s < t$. As a result, f is periodic with period T as well. Therefore,

$$\begin{aligned}
\int f(x)dx &= 0, \\
\int_t^{t+T} f(x)\frac{dx}{dt}dt &= 0, \\
\int_t^{t+T} f^2(x)dt &= 0,
\end{aligned}$$

This implies that $f^2(x) = 0 \forall t$, which in turn implies that $f(x) = 0 \forall t$. Since $\dot{x} = f(x)$, x is forced to be a constant. A contradiction!

3 Potentials

1) a) The potential for the system $\dot{x} = x(1-x)$ is evaluated by solving $-\frac{\mathrm{d}V}{\mathrm{d}x} = x - x^2$. This yields $V(x) = -\frac{x^2}{2} + \frac{x^3}{3} + C$. For convenience, we consider $C = 0$. The potential function is given by Figure 3.1. From the figure, it is clear that, the potential function has a local minima at $x = 1$ and a local maxima at $x = 0$ respectively. Hence, $x = 0$ corresponds to an unstable equilibrium and $x = 1$ corresponds to a stable equilibrium.

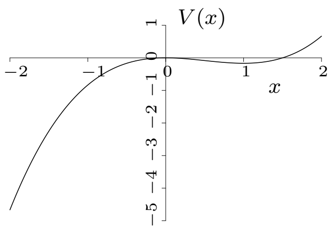

Figure 3.1: Plot of $V(x)$ vs x for $\dot{x} = x(1-x)$

b) The potential for the system $\dot{x} = 2 + \sin x$ is evaluated by solving $-\frac{\mathrm{d}V}{\mathrm{d}x} = 2 + \sin x$. This yields $V(x) = -2x + \cos x + C$. For convenience, we consider $C = 0$. The potential function is given by Figure 3.2. From the figure, it is clear that, the potential function does not have any local minima or maxima. Hence, the system does not have any equilibrium.

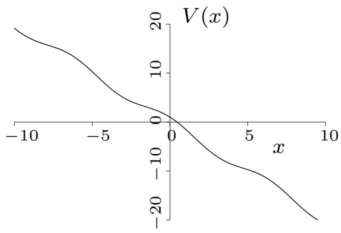

Figure 3.2: Plot of $V(x)$ vs x for $\dot{x} = 2 + \sin x$

c) The potential for the system $\dot{x} = r + x - x^3$ is evaluated by solving $-\frac{\mathrm{d}V}{\mathrm{d}x} = r + x - x^3$. This yields $V(x) = -rx - \frac{x^2}{2} + \frac{x^4}{4} + C$. For convenience, we consider $C = 0$. The potential function for this system for various values of r are given by Figure 3.3. From the figure, it is clear that, for both $r < 0$ and $r > 0$, the potential function has a local minima. Hence, the system has a stable equilibrium for both $r < 0$ and $r > 0$. For $r = 0$, it has two stable equilibria; one at $x = 1$ and the other at $x = -1$.

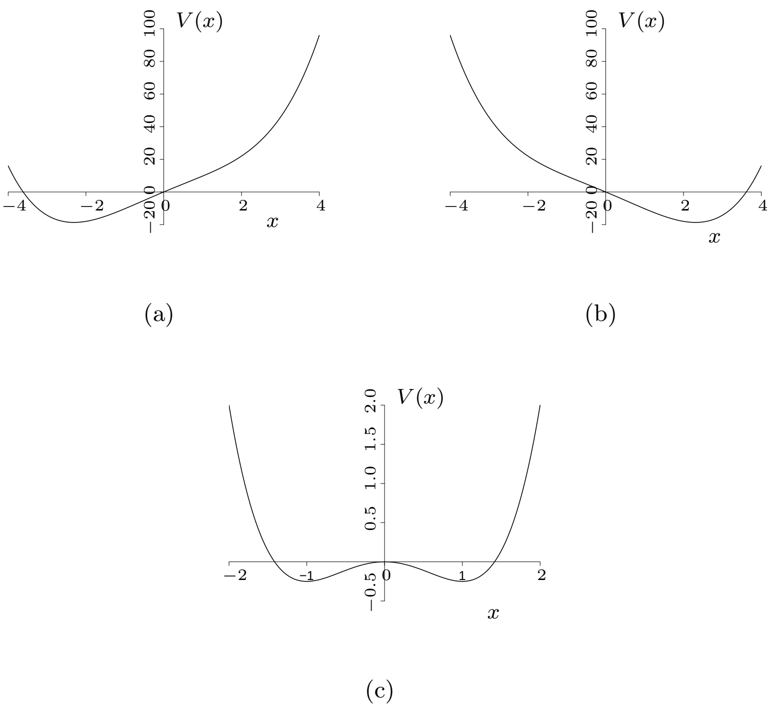

Figure 3.3: Plot of $V(x)$ vs x for $\dot{x} = r + x - x^3$. (a) is for $r < 0$, (b) is for $r > 0$, and (c) is for $r = 0$.