Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis

Solutions for Week 7 Assignment

April 21, 2017

1. Given $f(x) = x$ for $x \in [0,1]$ and 0 otherwise, $\|f\|_2, \|f\|_1 =$

   (a) $\frac{1}{\sqrt{3}}, \frac{1}{2}$

   (b) $\frac{1}{2}, \frac{1}{\sqrt{3}}$

   (c) $\frac{1}{3}, \frac{1}{\sqrt{3}}$

   (d) $\frac{1}{\sqrt{2}}, \frac{1}{2}$

   Solution: (a). By definition, $||f||_2 = \left(\int |f(x)|^2 dx\right)^{1/2} = \left(\int_0^1 x^2 dx\right)^{1/2} = \sqrt{\frac{1}{3}}$ and $||f||_1 = \int |f(x)| dx = \int_0^1 x dx = \frac{1}{2}$.

2. For $\phi(t) = sin(t)$, what is the value of the mean of the frequency distribution $\hat{\phi}(\omega)$?

   (a) 0

   (b) 1/2

   (c) 1/3

   (d) 2

   Solution: (a). For a real function $\phi(t)$, the fourier transform $\hat{\phi}(\omega)$ is magnitude symmetric.

3. The mean for the energy distribution corresponding to $\phi(t) = |t|$ for $t \in [0,1]$=

   (a) 1/2

   (b) 2/3

   (c) 3/4

   (d) 1/3

   Solution: (c) Here, $\|f\|_2 = \sqrt{\frac{1}{3}}$. So, $p_\phi(t) = 3t^2$. Hence, mean $= \int_0^1 t p_\phi(t) dx = \int_0^1 3t^3 dt = \frac{3}{4}$
   .

4. Find the variance for the energy distribution of the above function:

(a) 5/40
(b) 5/80
(c) 3/40
(d) 3/80

Solution: (d) variance = $\int_0^1 (t-0.75)^2 p_\phi(t)dx$ where $p_\phi(t) = 3t^2$.

5. The frequency variance for the function $\phi(t) = 1$ for t $\in$ [0,1], -1 for t $\in$ [1,2] and 0 otherwise =

   (a) 4
   (b) 4π
   (c) 2π
   (d) ∞

   Solution: (d). dϕ/dt is discontinuous. Hence the variance diverges.

6. The norm-squared and the frequency variance of the function $x(t) = \frac{1}{1+t^2}$ are respectively:

   (a) π, 0.5
   (b) $\pi/2$, 2
   (c) 2, 1
   (d) 4, 0,25

   Solution: (a). By duality, $\hat{x}(\omega) = \pi e^{-|\omega|}$. Hence, $||x||^2 = \frac{1}{2\pi}\|\hat{x}(\omega)\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\pi e^{-|\omega|}\right|^2 d\omega = \pi/2$. And frequency variance $= \int_{-\infty}^{\infty}\omega^2 e^{-2|\omega|}d\omega = 0.5$.

7. If the signal $\phi(t)$ is shifted by an amount t_0,

   (a) Mean in the time domain remains same and variance in the frequency domain remains unchanged
   (b) Mean in the time domain shifts by t_0 and variance in the time domain remains unchanged
   (c) Mean in the frequency domain shifts by $t_0 and variance change depends on the function$
   (d) Can't say since both of them depend on the function $\phi(t)$

   Solution: (b). This can be visualized using the center of mass notion: If the complete object shifts by t_0, the COM shifts by the same amount. As we have seen, a shift in the signal in time domain affects the variance neither in the time or the frequency domain.

8. Consider a well-defined function $\phi(t)$ and a transformation $\psi(t) = \phi(\alpha t)$. Which of the following is TRUE?

   (a) Variances of both the functions are the same.
   (b) Means of both the functions are the same.
   (c) Time-bandwidth products of both the functions are the same.

(d) Information insufficient to claim any of the above.

Solution (c). The means and variances are not invariant to scaling, although time-bandwidth product is invariant as proved in the lecture.

9. The energy of the function $\phi(t) = e^{-\alpha|t|}$, $\alpha > 0$ is:

   (a) $\frac{2}{\alpha}$
   (b) $\frac{1}{\alpha}$
   (c) $\frac{2}{\alpha^2}$
   (d) $\frac{1}{\alpha^2}$

   Solution: (b). Energy $= \int |\phi(t)|^2 dx = \int_{-\infty}^{\infty} e^{-2\alpha|t|} dt = \frac{1}{\alpha}$

10. The time center and frequency center of $\phi(t) = e^{-\alpha|t|}$ are respectively:

    (a) $\sqrt{\frac{1}{\alpha}}$, 0
    (b) $\sqrt{\frac{3}{\alpha}}, \alpha$
    (c) 0, α
    (d) 0, 0

    Solution: (d) Function is even, hence time center is 0. And for a real-time signal, frequency center is zero.

11. The time variance of the above function is:

    (a) $\frac{1}{2\alpha^2}$
    (b) $\frac{1}{4\alpha^2}$
    (c) $\frac{1}{6\alpha^2}$
    (d) $\frac{1}{3\alpha^2}$

    Solution: (a) $p_\phi(t) = \alpha e^{-2\alpha t}$. Hence, $\sigma_t^2 = \int_{-\infty}^{+\infty} t^2 p_\phi(t) dt = \frac{1}{2\alpha^2}$

12. The frequency variance of the above function is:

    (a) $4\alpha^2$
    (b) $2\alpha^2$
    (c) α^2
    (d) $3\alpha^2$

    Solution (c). The frequency variance is given by $\frac{Energy in d\phi/dt}{Energy in \phi(t)}$. Energy in d$\phi$/dt= $\int_{-\infty}^{+\infty} \alpha^2 e^{-2\alpha t} dt = \alpha$. Hence, frequency variance $= \alpha^2$.

13. The time-bandwidth product of the above function is:

(a) 0.4
(b) 0.5
(c) 1
(d) 0.75

Solution: (b). Time-bandwidth product = time-variance times freq-variance = 0.5.

14. The time-bandwidth product of y(t) = $e^{-|t|}e^{j\omega_0 t}$ is:

(a) 0.5
(b) 0.25
(c) 1
(d) 2

Solution: (a). The time-bandwidth product is invariant to modulation and time-scaling. Hence the answer is the same = 0.5.

15. Consider $h = f \star g$, where '$\star$' denotes the convolution operation. If $f(t) = e^{-t^2}$ and $g(t) = 2e^{-2t^2}$. The time-bandwidth product of $h(t)$ is:

(a) 1
(b) 2
(c) 0.25
(d) 0.5

Solution: (c). Convolution of two gaussians is a gaussian. And as we know, gaussian is the optimal function for the time-bandwidth product. Hence, the answer is 0.25.