Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis

Solutions for Week 3 Assignment

March 16, 2017

1. When we decompose an image into different sub-bands, the highest energy is contained in

   (a) LH sub-band.
   (b) HL sub-band.
   (c) LL sub-band.
   (d) HH sub-band.

   Ans (c)

   As explained in the demonstration video, the LL sub-band contains the maximum or the highest energy.

2. The total number of sub-bands that we will obtain after performing level 4 decomposition of an image is

   (a) 13
   (b) 12
   (c) 14
   (d) 11

   Ans (a)

   Level 1, level 2 and level 3 decompositions will have 3 sub-bands each and level 4 will have 4 sub-bands which sums up to a total of 13 sub-bands.

3. Which of the following is true about down-sampling and up-sampling operations?

   (a) Down-sampling and up-sampling operations are linear and time invariant.
   (b) Down-sampling and up-sampling operations are non-linear but time invariant.

(c) Down-sampling and up-sampling operations are linear but time varying.

(d) Down-sampling and up-sampling operations are non-linear and time varying.

Ans (c)

Let us understand this with the help of an example. Let $x_1[n] = \{\underset{\uparrow}{1},\ 2,\ 3,\ 4\}$ and $x_2[n] = \{\underset{\uparrow}{5},\ 6,\ 7,\ 8\}$ be the two different signals. Let us also consider the situation of up-sampling by 2 (Later on this condition can be generalized to up-sampling by say M). Up-sampling by M operation inserts $(M-1)$ zeros between every two consecutive samples of the signal. Therefore Up-sampling by 2 will insert 1 zero between every two consecutive samples of the signal.
Now let us first up-sample both the signals and then add them. This will result in the output signal say, $x_{out1} = \{\underset{\uparrow}{1},\ 0,\ 2,\ 0,\ 3,\ 0,\ 4,\ 0\} + \{\underset{\uparrow}{5},\ 0,\ 6,\ 0,\ 7,\ 0,\ 8,\ 0\} = \{\underset{\uparrow}{6},\ 0,\ 8,\ 0,\ 10,\ 0,\ 12,\ 0\}$. Now, let us first add both the signals and then up-sample by 2. This will result in a signal say, $x_{out2} = \{\underset{\uparrow}{6},\ 8,\ 10,\ 12,\ \} = \{\underset{\uparrow}{6},\ 0,\ 8,\ 0,\ 10,\ 0,\ 12,\ 0\}$. It's clear that $x_{out1} = x_{out2}$.
Consider signal $x_{1,m}[n] = \alpha x_1[n] = \{\underset{\uparrow}{\alpha},\ 2\alpha,\ 3\alpha,\ 4\alpha\}$.
Up-sampling the signal $x_{1,m}$ by 2, we get $x_{1,m,up}[n] = \{\underset{\uparrow}{\alpha},\ 0,\ 2\alpha,\ 0,\ 3\alpha,\ 0,\ 4\alpha, 0\}$.
Now, first up-sampling the signal x_1 by 2, we get $x_{1,up}[n] = \{\underset{\uparrow}{1},\ 0,\ 2,\ 0,\ 3,\ 0,\ 4, 0\}$ and then multiplying by α, we get $x_{1,up,m}[n] = \{\underset{\uparrow}{\alpha},\ 0,\ 2\alpha,\ 0,\ 3\alpha,\ 0,\ 4\alpha, 0\}$.
Thus, $x_{1,m,up} = x_{1,up,m}$.
Therefore, up-sampling is **linear**.

Again consider the signal $x_1[n] = \{\underset{\uparrow}{1},\ 2,\ 3,\ 4\}$ and first perform shifting say, by 1 and call the shifted signal as $x_1[n+1] = \{1,\ \underset{\uparrow}{2},\ 3,\ 4\}$. Now perform up-sampling by 2 operation and we will get the output as $x_{out11}[n] = \{1,\ 0,\ \underset{\uparrow}{2},\ 0,\ 3,\ 0,\ 4,\ 0\}$. Now let us first up-sample the signal by 2 and then shift it by 1 to obtain $x_{out12}[n] = \{1,\ \underset{\uparrow}{0},\ 2,\ 0,\ 3,\ 0,\ 4,\ 0\}$. Clearly $x_{out11}[n] \neq x_{out12}[n]$ and therefore up-sampling operation is **time varying**.
Following the same line of arguments it can be shown that **down sampling operation is also linear and time varying**.

4. Let ϕ be the Haar scaling function as discussed in the lectures. Let $m \in \mathbb{Z}$ be the scaling parameter and $n, k \in \mathbb{Z}$ be the translation parameters. Then what will the following function $\langle \phi(2^m t - n), \phi(2^m t - k) \rangle$ evaluate to ? ($\langle \cdot, \cdot \rangle$ represents the inner product of two functions and δ is the usual Kronecker delta function which takes the value 1 when the variables are equal and 0 otherwise)

   (a) 0

(b) $2^{-m}\delta_{n-k}$
(c) $2^{m}\delta_{n-k}$
(d) 1

Ans (b)

$$\begin{aligned}\langle\phi(2^m t-n),\phi(2^m t-k)\rangle &= \int_{-\infty}^{\infty}\phi(2^m t-n),\overline{\phi(2^m t-k)}dt\\ &= \frac{1}{2^m}\int_{-\infty}^{\infty}\phi(x-n),\phi(x-k)\rangle dx\\ &= 2^{-m}\delta_{n-k}\end{aligned}$$

Here we have used the fact that Haar scaling functions are orthogonal to integer translates i.e.

$$\int_{-\infty}^{\infty}\phi(x-n),\phi(x-k)\rangle dx = \begin{cases}0 & \text{if } n\neq k\\ 1 & \text{if } n=k\end{cases}$$

5. Let ψ be the Haar wavelet function as discussed in the lectures. Let $m\in\mathbb{Z}$ be the scaling parameter and $n,k\in\mathbb{Z}$ be the translation parameters. Then what will the following function $\langle\psi(2^m t-n),\psi(2^m t-k)\rangle$ evaluate to ? ($\langle\cdot,\cdot\rangle$ represents the inner product of two functions and δ is the usual Kronecker delta function which takes the value 1 when the variables are equal and 0 otherwise)

   (a) 0
   (b) $2^{-m}\delta_{n-k}$
   (c) $2^{m}\delta_{n-k}$
   (d) 1

   Ans (b)

$$\begin{aligned}\langle\psi(2^m t-n),\psi(2^m t-k)\rangle &= \int_{-\infty}^{\infty}\psi(2^m t-n),\overline{\psi(2^m t-k)}dt\\ &= \frac{1}{2^m}\int_{-\infty}^{\infty}\psi(x-n),\psi(x-k)\rangle dx\\ &= 2^{-m}\delta_{n-k}\end{aligned}$$

   Here we have used the fact that Haar wavelet functions are orthogonal to integer translates i.e.

$$\int_{-\infty}^{\infty}\psi(x-n),\psi(x-k)\rangle dx = \begin{cases}0 & \text{if } n\neq k\\ 1 & \text{if } n=k\end{cases}$$

6. Which of the following is correct dilation equation?
g(n) : analysis lowpass filter
h(n) : analysis highpass filter

(a) $\psi(t) = \sum_{n=-\infty}^{n=\infty} g(n)\phi(2t-n)$

(b) $\phi(t) = \sum_{n=-\infty}^{n=\infty} h(n)\phi(2t-n)$

(c) $\phi(t) = \sum_{n=-\infty}^{n=\infty} g(n)\phi(2t-2n)$

(d) $\psi(t) = \sum_{n=-\infty}^{n=\infty} h(n)\phi(2t-n)$

Solution: D

This is straight forward formula based question but the only exception is that g(n) here is analysis lowpass filter and h(n) is analysis highpass filter. Hence,

$$\phi(t) = \sum_{n=-\infty}^{n=\infty} g(n)\phi(2t-n)$$

$$\psi(t) = \sum_{n=-\infty}^{n=\infty} h(n)\phi(2t-n)$$

7. Let $\phi(t)$ be the scaling function for haar filter bank defined as follows:

$$\phi(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Calculate Fourier Transform of $y(t) = \phi(2t)$.

(a) $\frac{e^{-j\frac{\Omega}{4}}}{2}\left(\frac{sin\frac{\Omega}{4}}{\frac{\Omega}{4}}\right)$

(b) $\frac{e^{-j\frac{\Omega}{2}}}{2}\left(\frac{sin\frac{\Omega}{2}}{\frac{\Omega}{2}}\right)$

(c) $\frac{e^{-j\frac{\Omega}{8}}}{4}\left(\frac{sin\frac{\Omega}{8}}{\frac{\Omega}{8}}\right)$

(d) None of these

Solution: A

Fourier transform of scaling function $\phi(t)$ for haar filter bank is derived in lecture which is as follows:

$$\hat{\phi}(\Omega) = \quad e^{-j\frac{\Omega}{2}} \left(\frac{sin\frac{\Omega}{2}}{\frac{\Omega}{2}} \right)$$

Using Fourier transform property, we get Fourier transform of $y(t) = \phi(2t)$ as follows:

$$\hat{y}(\Omega) = \quad \frac{e^{-j\frac{\Omega}{4}}}{2} \left(\frac{sin\frac{\Omega}{4}}{\frac{\Omega}{4}} \right)$$

8. Calculate $\hat{y}(\Omega)\Big|_{\Omega=0}$ for the Fourier transform calculated in above question.

   (a) $\frac{1}{4}$

   (b) $\frac{1}{2}$

   (c) 1

   (d) ∞

Solution: B

$$\hat{y}(\Omega)\Big|_{\Omega=0} = \lim_{\Omega \to 0} \frac{e^{-j\frac{\Omega}{4}}}{2} \left(\frac{sin\frac{\Omega}{4}}{\frac{\Omega}{4}} \right)$$

Also,

$$\lim_{x \to 0} \frac{sin(x)}{x} = 1$$

Using above limit formula, we get

$$\hat{y}(\Omega)\Big|_{\Omega=0} = 0.5$$

9. Following Fourier transform property is derived in class.
If $h(t)$ has Fourier transform $H(\Omega)$, then for $\alpha > 0$ Fourier transform of $h(\frac{t}{\alpha})$ is $\alpha H(\alpha\Omega)$.
Which of the following statement is true?

(a) Signal can be timelimited and bandlimited simultaneously.

(b) Signal can't be timelimited and bandlimited simultaneously.

(c) Nothing can be said

Solution: B

Using property of Fourier transform mentioned in question, we can tell that compression of signal in time domain results in expansion in of signal frequency domain and vice versa. For example,Consider dc(constant) signal which is not time limited but it has very compactly supported Fourier transform which is impulse(bandlimited signal). Also,bandlimited function such as low pass filter transfer function has infinite spread in time domain.

Hence we can say that Signal can't be timelimited and bandlimited simultaneously.

10. If we have following frequency domain equation,what would be equivalent equation in time domain?

$$\hat{\phi}(\Omega) = H(\frac{\Omega}{3})\hat{\phi}(\frac{\Omega}{4})$$

(a) $\phi(t) = h(3t) * \phi(4t)$

(b) $\phi(t) = 12 \times h(4t) * \phi(3t)$

(c) $\phi(t) = 12 \times h(3t) * \phi(4t)$

(d) None of these

Solution: C

Following Fourier transform property is discussed in lectures:

$$h(t) \xrightarrow{\mathbf{F}} H(\Omega) \implies h(\frac{t}{\alpha}) \xrightarrow{\mathbf{F}} \alpha H(\alpha\Omega) \quad ,\alpha > 0 \tag{1}$$

Where, **F** denotes Fourier transform operator. Using above property,

$$3h(3t) \xrightarrow{\mathbf{F}} H(\frac{\Omega}{3})$$

$$4\phi(4t) \xrightarrow{\mathbf{F}} \hat{\phi}(\frac{\Omega}{4})$$

Also, we know that product of two functions in time domain results in convolution of two functions in frequency domain. Combining all these, we get

$$\phi(t) = 12 \times h(3t) * \phi(4t)$$

where * denotes convolution.

11. As we have seen in lecture that scaling function $\phi(t)$ can be obtained by infinite convolution as follows:
$\phi(t) = h(2t) * h(4t) * h(8t) * .....$
where $h(t) = \sum_{n=-\infty}^{n=\infty} h(n)\delta(t-n)$ and $h[n] = \{1, 1\}$.
Now consider $\overline{\phi(t)} = h(2t) * h(4t) * h(8t) * .... * h(2^N t)$. Calculate the largest value of t for which $\overline{\phi(t)}$ is non zero.

    (a) $\dfrac{2^N - 1}{2^N}$

    (b) $\dfrac{1}{2^N}$

    (c) $\dfrac{2^N - 1}{2^N + 1}$

    (d) can't be determined

Solution: A

Consider convolution of two signals $x_1(t)$ and $x_2(t)$. Let $x_1(t)$ and $x_2(t)$ be non-zero in the interval $[0, a]$ and $[0, b]$ respectively.
If $y(t) = x_1(t) * x_2(t)$ then $y(t)$ will be non-zero in the interval $[0, a + b]$.

$h(2^N t)$ will have non-zero value only in the time interval $[0, 2^{-N}]$

Let $z_1(t) = h(2 * t) * h(4 * t)$ then $z_1(t)$ will be non-zero in the range $[0, \frac{1}{2} + \frac{1}{4}]$.
Let $z_2(t) = z_1(t) * h(8 * t)$ then $z_1(t)$ will be non-zero in the range $[0, \frac{1}{2} + \frac{1}{4} + \frac{1}{8}]$.

Extending above argument, $\overline{\phi(t)} = h(2t) * h(4t) * h(8t) * .... * h(2^N t)$ will be non-zero in the range $[0, \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... + \frac{1}{2^N}]$.

Using formula for geometric series, $S_n = a + ar + ar^2 + ... + ar^n = a\dfrac{r^n - 1}{r - 1}$,
We get $\overline{\phi(t)}$ is non-zero in the interval $[0, \dfrac{2^N - 1}{2^N}]$

12. In above question, let the length of the filter $h(t)$ be L. Then calculate the largest value of t for which $\overline{\phi(t)}$ is non zero as $N \to \infty$

(a) 1

(b) $L-1$

(c) $\frac{L}{2}$

(d) L

Solution: B

Since length of filter $h(n)$ is L, $h(t)$ will be non-zero in the interval $[0, L-1]$. Hence, $h(2^N t)$ will have non-zero value only in the time interval $[0, 2^{-N} * (L-1)]$.

Let $z_1(t) = h(2*t) * h(4*t)$, then $z_1(t)$ will be non-zero in the range $[0, \frac{L-1}{2} + \frac{L-1}{4}]$.
Let $z_2(t) = z_1(t) * h(8*t)$, then $z_2(t)$ will be non-zero in the range $[0, \frac{L-1}{2} + \frac{L-1}{4} + \frac{L-1}{8}]$.

Extending above argument, $\overline{\phi(t)} = h(2t)*h(4t)*h(8t)*....*h(2^N t)$ will be non-zero in the range $[0, \frac{L-1}{2} + \frac{L-1}{4} + \frac{L-1}{8} + \frac{L-1}{16} + ... + \frac{L-1}{2^N}]$.

Using formula for geometric series, $S_n = a + ar + ar^2 + ... + ar^n = a\frac{r^n - 1}{r-1}$,
We get $\overline{\phi(t)}$ is non-zero in the interval $[0, (L-1)\frac{2^N - 1}{2^N}]$
As $N \to \infty$, $\overline{\phi(t)}$ is non-zero in the interval $[0, L-1]$

13. Let $x(t) = \delta(t)$ be impulse(Dirac delta) function. what will be the signal $y(t)$ whose Fourier transform is given as follows?
$\hat{y}(\Omega) = \hat{x}(\frac{\Omega}{5})$.

(a) $5\delta(t)$

(b) $\delta(t)$

(c) $\frac{1}{5}\delta(t)$

(d) $\delta(5t)$

Solution: B

Using Fourier transform scaling property,

$$\hat{y}(\Omega) = \hat{x}(\frac{\Omega}{5}) \implies y(t) = 5x(5t) = 5\delta(5t)$$

Impulse(dirac delta) function has following property:

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Thus,

$$y(t) = 5\delta(5t) = \delta(t)$$

14. Consider a signal $x(t)$ defined as follows.

$$x(t) = \begin{cases} 1 - |t|, & -1 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Let piece-wise constant representation of $x(t)$ on space V_m using haar MRA be denoted by $x_m(t)$. Then work out the energy in the error function as a function of m defined as follows: Error function = $x(t) - x_m(t)$

(a) $\frac{1}{6}2^{-2m}$

(b) $\frac{1}{3}2^{-m}$

(c) $\frac{1}{3}2^{-2m}$

(d) $\frac{1}{6}2^{-m}$

Solution: A

Let us consider $x_2(t)$ as piecewise constant approximation of function $x(t)$ on space V_2 shown in figure below:

As seen from figure,energy in error between $x(t)$ and $x_2(t)$ is same in all the time intervals $[n2^{-m}, (n+1)2^{-m}]$, $m = 2$ and $n \in \mathbb{Z}$.

Hence total energy in error function = total number of intervals of size 2^{-m} in time-interval $[-1, 1]$ $\times$ energy in error function in the interval $[0, 2^{-m}]$.

Total number of intervals of size 2^{-m} in $[-1, 1] = 2^{m+1}$.

Energy in the error function E

$$= 2^{m+1} \int_0^{2^{-m}} |x(t) - x_m(t)|^2 dt \tag{2}$$

This integral can be solved by directly substituting $x(t)$ and $x_m(t)$ but it will involve lots of computations so we use following method.

Figure 2 shows zoomed view of $x(t)$ and $x_m(t)$ for $m = 2$ and in the range $[0, 2^{-m}]$.

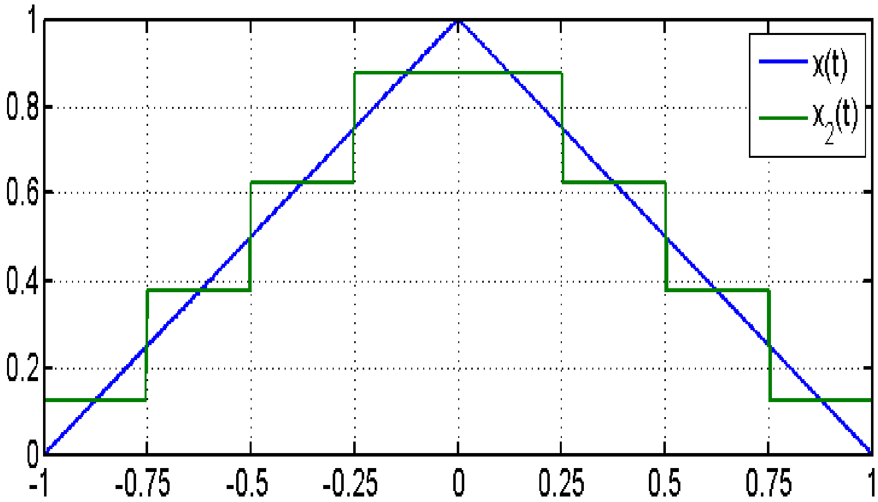

Figure 1

Now,Let us calculate $x_m(t)$ in the interval $[0, 2^{-m}]$.

$$\begin{aligned} x_m(t) &= \text{average of } x(t)\Big|_t = 0 \text{and} x(t)\Big|_t = 2^{-m} \qquad (3) \\ &= \frac{1 + 1 - 2^{-m}}{2} \\ &= 1 - 2^{-(m+1)} \end{aligned}$$

Since we are only interested in finding error energy between $x(t)$ and $x_m(t)$,we subtract 1 from $x(t)$ and $x_m(t)$ both and multiply both by -1.Doing so won't change the energy in error as shown in fig 3.

This leads to $x(t) = t$ and $x_m(t) = 2^{-(m+1)}$.

Using this approach,Energy in error function in interval $[0, 2^{-m}]$ is given by following integral:

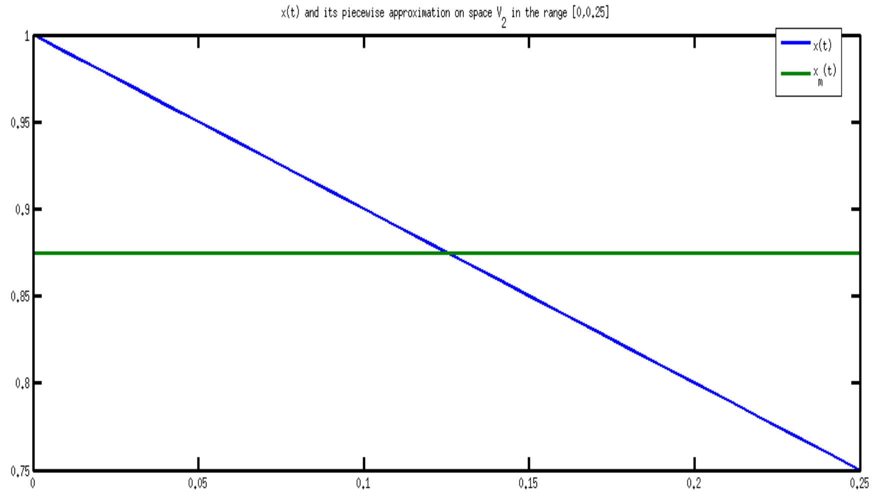

Figure 2

$$\begin{aligned}
\hat{E} &= \int_0^{2^{-m}} (t - 2^{-(m+1)})^2 dt \qquad (4) \\
&= \left. \frac{(t - 2^{-(m+1)})^3}{3} \right|_0^{2^{-m}} \\
&= \frac{\left(2^{-m} - 2^{-(m+1)}\right)^3}{3} + \frac{2^{-3(m+1)}}{3} \\
&= 2^{-3m} \left(\frac{(1 - \frac{1}{2})^3}{3} + \frac{1}{24} \right) \\
&= \frac{1}{12} 2^{-3m}
\end{aligned}$$

From eq (2), Total energy in the error signal E is given by

$$\begin{aligned}
E &= \hat{E} \times 2^{m+1} \qquad (5) \\
&= \frac{1}{12} 2^{-3m} 2^{m+1} \\
&= \frac{2^{-2m}}{6}
\end{aligned}$$

15. Let $x[n]$ be a discrete time signal given by

$$x_1[n] = \{\underset{\uparrow}{-1}, 3, 5, 23, 1, 0\}$$

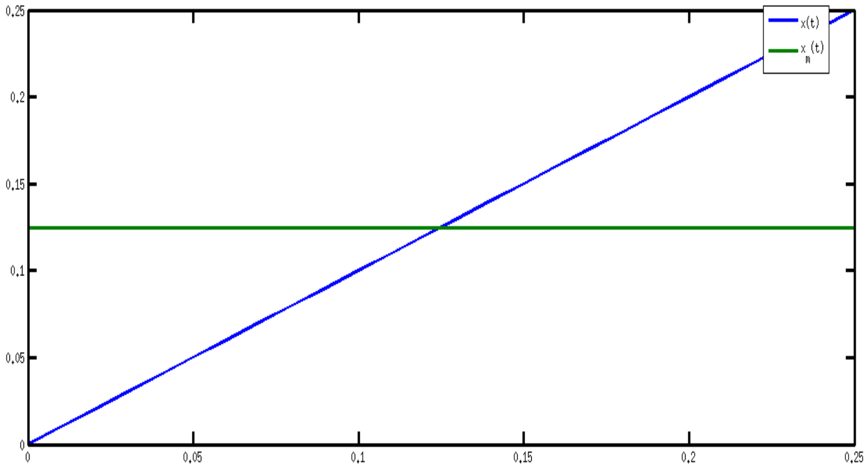

Figure 3

$$x_2[n] = \{\underset{\uparrow}{-1}, 0, 5, 0, 1, 0\}$$

If the above signals are subjected to down-sampling by 2 operation, then which signal will suffer aliasing and which signal won't ?

(a) $x_1[n]$ will be not get aliased whereas $x_2[n]$ will suffer from aliasing.

(b) Both signals $x_1[n]$ and $x_2[n]$ will not get aliased.

(c) $x_1[n]$ will be aliased and $x_2[n]$ will not get aliased.

(d) Both signals $x_1[n]$ and $x_2[n]$ will get aliased.

Ans (c) When a signal is subjected to a downsampling operation then it **May** result in aliasing in the spectrum of the signal, but this is not necessary. When signal $x_2[n]$ passes through a downsampler by 2 then the output sequence obtained is $\{\underset{\uparrow}{-1}, 5, 1\}$ and when this signal is again passed through an upsampler by 2 then we would get our original sequence $x_2[n] = \{\underset{\uparrow}{-1}, 0, 5, 0, 1, 0\}$. Therefore, no aliasing was done to the signal on passing through the downsampler by 2.
This is certainly not the case with the signal $x_1[n]$, and thus it suffers from aliasing problem.

16. If a signal $x[n]$ is passed through a M-point downsampler, then what is the z-transform of the output signal ?

(a) $\dfrac{1}{M}\sum\limits_{k=0}^{M-1} X(z^{\frac{-1}{M}} e^{\frac{j2\pi k}{M}})$ where $k = 0, 1, 2, 3, \cdots, M-1$

(b) $\dfrac{1}{M}\sum\limits_{k=0}^{M} X(z^{\frac{-1}{M}} e^{\frac{j2\pi k}{M}})$ where $k = 0, 1, 2, 3, \cdots, M$

(c) $\frac{1}{M}\sum_{k=0}^{M-1} X(z^{\frac{1}{M}} e^{\frac{j2\pi k}{M}})$ where $k = 0, 1, 2, 3, \cdots, M-1$

(d) $\frac{1}{M}\sum_{k=0}^{M-1} X(z^{\frac{1}{M}} e^{\frac{-j2\pi k}{M}})$ where $k = 0, 1, 2, 3, \cdots, M-1$

Ans (d)
Following the same steps as discussed in the lecture we can obtain the z-transform of the output signal. We can break down the process in two steps, wherein the first step involves "multiplication of the signal by killing sequence (a window)" and in the second step we have an upsampling process in the opposite direction(i.e its an upsampling process as seen from the output towards input side and we know that the upsampling process is an invertible process).
The killing sequence for the downsampling by M must have a repeating sequence of $\{\underset{\uparrow}{1}, 0, 0, \cdots, 0(M-1\text{times})\}$. The periodic killing sequence can be represented in the inverse DFT transform form as

$$\begin{aligned} P_M[n] &= \frac{1}{M}\sum_{k=0}^{M-1} B(k)(e^{\frac{j2\pi nk}{M}}) \text{ where } k = 0, 1, 2, 3, \cdots, M-1 \\ &= \frac{1}{M}\sum_{k=0}^{M-1} e^{\frac{j2\pi nk}{M}} \text{ where } k = 0, 1, 2, 3, \cdots, M-1 \end{aligned}$$

STEP 1: The z-transform of the product of input sequence $x[n]$ and $p_M[n]$ is therefore,

$$\begin{aligned} \widehat{X}_M(Z) &= \sum_{n=-\infty}^{\infty} [\frac{1}{M} x[n] \sum_{k=0}^{M-1} e^{\frac{j2\pi nk}{M}}] z^{-n} \text{ where } k = 0, 1, 2, 3, \cdots, M-1 \\ &= \frac{1}{M}\sum_{k=0}^{M-1} [\sum_{n=-\infty}^{\infty} x[n] e^{\frac{j2\pi nk}{M}} z^{-n}] \text{ where } k = 0, 1, 2, 3, \cdots, M-1 \\ &= \frac{1}{M}\sum_{k=0}^{M-1} [\sum_{n=-\infty}^{\infty} x[n] (e^{\frac{-j2\pi k}{M}} z)^{-n}] \text{ where } k = 0, 1, 2, 3, \cdots, M-1 \\ &= \frac{1}{M}\sum_{k=0}^{M-1} X(z e^{\frac{-j2\pi k}{M}}) \text{ where } k = 0, 1, 2, 3, \cdots, M-1 \end{aligned}$$

STEP 2: In this step the above output signal is passed through an upsampler by M by in the reverse direction (i.e. its an upsampling process as seen from the output towards input side). Therefore the result so obtained after this process is

$$X_{out}(Z) = \widehat{X}_M(Z^{\frac{1}{M}}) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{\frac{1}{M}} e^{\frac{-j2\pi k}{M}}) \text{ where } k = 0, 1, 2, 3, \cdots, M-1$$