Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis

Week 2 Assignment

March 10, 2017

1. Given 2 time limited signal, are they orthogonal?

$$x[n] = \{\underset{\uparrow}{3}, 5, -2, 2, -2\}$$

$$y[n] = \{\underset{\uparrow}{-1}, -7, 2, 17, -4\}$$

   (a) Yes
   (b) No

   Ans (a)
   The inner product of the x and y is equal to 0 and hence the signals are orthogonal to each other.

2. Find the angle between the below signals

$$x(t) = t^2$$

$$y(t) = t - 2t^2$$

   Assume both the signals to be defined for the interval [0,1].

   (a) 179.13 *degrees*
   (b) 110.78 *degrees*
   (c) 166.44 *degrees*
   (d) 156.71 *degrees*

   Ans (d)

$$\langle x, y\rangle = \int_0^1 t^2(t - 2t^2) = \frac{-3}{20}$$

$$\|x\|^2 = \int_0^1 t^4 = \frac{1}{5}$$

$$\|y\|^2 = \int_0^1 (t - 2t^2) = \frac{2}{15}$$

and then using $\theta = \cos^{-1} \frac{\langle x, y\rangle}{\|x\|\|y\|} = 156.7163 \; degrees$

3. How does the contours of l_1 and l_2 norm look like in 2 dimension? (Multiple choices can be correct.)

   (a) Diamond shape

   (b) Square

   (c) Circle

   (d) None of the Above

   Ans (a,c)
   $\|x\|_1 + \|y\|_1 = 1$ is a diamond shaped and $\|x\|_2 + \|y\|_2 = 1$ turns out to be a circle.

4. Consider a line L in $\mathbb{R}^2$ given by $y = 2x$ i.e

$$L = \{r(1, 2) : r \in \mathbb{R}\}$$

   Now let $P = (2, 1)$ and answer the following question:- What is the point on L closest to P?

   (a) $(\frac{2}{5}, \frac{7}{2})$

   (b) $(\frac{4}{5}, \frac{8}{5})$

   (c) $(\frac{7}{5}, \frac{14}{5})$

   (d) None of the Above

   Ans (b)
   Consider a point $u = (a, 2a)$ be in line L such that its distance from point P is the shortest. Therefore $P - u = (2 - a, 1 - 2a)$ is perpendicular to L. Therefore we have,

$$0 = \langle (a, 2a), (2 - a, 1 - 2a)\rangle = 2a - a^2 + 2a - 4a^2 = 4a - 5a^2$$

   Therefore we have $a = \frac{4}{5}$ and $u = (a, 2a) = (\frac{4}{5}, \frac{8}{5})$

5. In the question above, what is the distance of P from the line L?

   (a) $\frac{\sqrt{45}}{5}$

   (b) $\frac{7\sqrt{5}}{3}$

   (c) $\frac{2\sqrt{5}}{3}$

(d) $\frac{2\sqrt{5}}{5}$

Ans (a)
Here we need to find $\|P-u\| = (2,1)-(\frac{4}{5},\frac{8}{5}) = (\frac{6}{5},\frac{-3}{5})$. The magnitude of this gives us the distance: $\sqrt{\frac{36}{25}+\frac{9}{25}} = \frac{\sqrt{45}}{5}$

6. What are the l_0, l_1, l_2 and l_{inf} norms of x where x is defined as: $x = (1,1,1,\cdots)_{1\times n}$

   (a) $n, n, 1, 1$
   (b) $1, n, n, 1$
   (c) $n, n, \sqrt{n}, 1$
   (d) None of the above

   Ans (c)
   $\|x\|_0 = n$
   $\|x\|_1 = n$
   $\|x\|_2 = \sqrt{n}$
   $\|x\|_{inf} = 1$

7. A condition for a function $\psi(t)$ to be called as a wavelet is $\hat{\psi}(0) = 0$ where $\hat{\psi}(\omega) = \int_{-\infty}^{\infty} \psi(t)e^{-j\omega t}dt$, This is equivalent to saying:

   (a) $\int_{-\infty}^{\infty} |\psi(t)|dt = 0$
   (b) $\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 0$
   (c) $\int_{-\infty}^{\infty} \psi(t)dt = 0$
   (d) $\int_{-\infty}^{\infty} j\psi(t)dt = 0$ where $j = \sqrt{-1}$

   Ans (c)
   Here $\hat{\psi}(\omega) = \int_{-\infty}^{\infty} \psi(t)e^{-j\omega t}dt$
   Now if we substitute $\omega = 0$ we have
   $\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(t)dt$

8. Consider the function $x(t)$

$$x(t) = \begin{cases} 1-|t| & |t| \le 1 \\ 0 & elsewhere. \end{cases}$$

   Then which of the following is true

   a $x(t)$ is orthogonal to all its integer translates.

b $x(t)$ is orthogonal to all its real translates.

c $x(t)$ is orthogonal to all its integer translates which are dilated by a factor of 2.

d $x(t)$ is orthogonal to all its integer translates except when the translation is by $+1$ or -1.

Ans (d)

It is easy to see that the quantity $\int_{-\infty}^{\infty} x(t-m)x(t-n)dt$ where $m, n \in \mathbf{Z}$ is non-zero only if

a) $m = n = 0$

b) $m = \pm 1$ and $n = 0$ and vice versa.

9. Consider two functions which are square integrable $F_1(x) = \sum_{n=-\infty}^{\infty} a_n e^{jnx}$ and $F_2(x) = \sum_{n=-\infty}^{\infty} b_n e^{jnx}$ then,

(a) $\sum_{n=-\infty}^{\infty} a_n \bar{b_n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{F_1(x)F_2(x)}dx$.

(b) $\sum_{n=-\infty}^{\infty} a_n \bar{b_n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_1(x)\overline{F_2(x)}dx$.

(c) $\sum_{n=-\infty}^{\infty} a_n \bar{b_n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F_1(x)||F_2(x)|dx$.

(d) $\sum_{n=-\infty}^{\infty} a_n \bar{b_n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F_1(x)F_2(x)|dx$.

Ans (b)

This is a very popular formula known as Parseval's relation for periodic signals. Start with right hand side.

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} F_1(x)\overline{F_2(x)}dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \{\sum_{n=-\infty}^{\infty} a_n e^{jnx}\}\overline{\{\sum_{m=-\infty}^{\infty} b_m e^{jmx}\}}dx \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} a_n \sum_{m=-\infty}^{\infty} \bar{b_m} \int_{-\pi}^{\pi} e^{j(n-m)x}dx. \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} a_n \sum_{m=-\infty}^{\infty} \bar{b_m} I
\end{aligned}$$

Now, the quantity $I = \int_{-\pi}^{\pi} e^{j(n-m)x}dx$ can easily seen to be $I = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$ so,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} F_1(x)\overline{F_2(x)}dx &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} a_n \bar{b_n} 2\pi \\
&= \sum_{n=-\infty}^{\infty} a_n \bar{b_n}
\end{aligned}$$

10. Which of the following is true with respect to Haar MRA

a $V_n = V_{-m} \bigoplus \{ \bigoplus_{i=-m}^{n-1} W_i \}$ where m and n are positive integers

b $V_n = V_{-m} \bigoplus \{ \bigoplus_{i=-m}^{n+1} W_i \}$ where m and n can be positive or negative integers.

c $V_n = V_{n+1} \bigoplus W_{n+1}$

d $V_n = W_{-m} \bigoplus \{ \bigoplus_{i=-m}^{n-1} V_i \}$ where m and n are positive integers

Ans (a)
Now at any level of resolution $n \in \mathbb{Z}_+$ we have the relation,

$$\begin{aligned} V_n &= V_{n-1} \bigoplus W_{n-1} \\ &= \{V_{n-2} \bigoplus W_{n-2}\} \bigoplus W_{n-1} \\ &= \{V_{n-3} \bigoplus W_{n-3}\} \bigoplus W_{n-2} \bigoplus W_{n-1} \\ &\vdots \\ &= \{V_{-m} \bigoplus W_{-m}\} \bigoplus \cdots \bigoplus W_{n-2} \bigoplus W_{n-1} \end{aligned}$$

for some $m \in \mathbb{Z}_+$, so $V_n = V_{-m} \bigoplus \{ \bigoplus_{i=-m}^{n-1} W_i \}$

11. A signal is processed by a causal filter with transfer function $G(z)$. For a distortion free output $G(z)$ must

(a) Provide zero phase shift for all frequency.
(b) Provides constant phase shift for all frequency.
(c) Provides linear phase shift that is proportional to frequency.
(d) Provides linear phase shift that is inversely proportional to frequency.

Ans-c

12. If the Z-transform of a sequence is $X(z) = \frac{0.5z}{z-2}$ it is given that the region of convergence of $X(z)$ includes the unit circle. The value of $x[0]$ is

(a) 0.5
(b) 0
(c) 0.25
(d) 0.05

Ans (b)
Now, $X(z) = \frac{0.5z}{z-2} = \frac{0.5}{1-2z^{-1}}$ now this transfer function has a pole at $z = 2$ and the ROC **includes the unit-circle** so the signal corresponding to this transfer function is a left- sided signal and is given by $x[n] = 0.5(2)^n u[-n-1]$ so it is clear that $x[0] = 0$

13. Let $h[n]$ be a signal of length N. What is the minimum value of N so that h[n] acts as a bandpass-filter?

    (a) 4
    (b) 2
    (c) 1
    (d) 3

    Ans (d)

    Note that the length 1 signal is simply an impulse function. For N = 2, $H(z) = \alpha + \beta z^{-1}$, we can only get low-pass and high-pass filters depending on the signs of α and β. For getting band-pass filter, we need to have a filter of length 3.

14. A system has input-output relation as $y(t) = e^{-|x(t)|}$ where $y(t)$ is output and $x(t)$ is the input, then $y(t)$ is bounded

    (a) Only when $x(t)$ is bounded.
    (b) Only when $x(t)$ is non-negative.
    (c) Even when $x(t)$ is bounded or unbounded.
    (d) None of the above.

    Ans (c)
    The given system has input-output relation as $y(t) = e^{-|x(t)|}$. Note that if the input $x(t)$ is bounded i.e if $|x(t)| \leq M \quad \forall t$ where M is a finite positive constant, then it is clear from the input-output relationship that $y(t) = e^{-|M|} < \infty$. Even if $x(t)$ is $+\infty$ or $-\infty$ the output $y(t) = 0$

15. Which of the following filters are magnitude complementary? (Multiple options can be correct)

    (a) $H_1(z) = \frac{-1 + z^{-1}}{2}, H_2(z) = \frac{1 - z^{-1}}{2}$
    (b) $H_1(z) = \frac{-1 + z^{-2}}{2}, H_2(z) = \frac{1 - z^{-1}}{2}$
    (c) $H_1(z) = \frac{1 + z^{-1}}{2}, H_2(z) = \frac{1 - 2z^{-1}}{2}$
    (d) $H_1(z) = \frac{1 + z^{-2}}{2}, H_2(z) = \frac{1 - z^{-2}}{2}$

    Ans (d)

    For magnitude complementarity, we should have $H_1(z) + H_2(z) = 1$.