Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis

Week 1 Assignment

March 7, 2017

1. What is the Fourier transform of the following function ?

$$f(t) = \begin{cases} e^{-|t|} & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) $\dfrac{2}{1+\Omega^2} - \dfrac{e^{-(1+j\Omega)}}{1+j\Omega} + \dfrac{e^{-(1-j\Omega)}}{1-j\Omega}$
   (b) $\dfrac{2j\Omega}{1+\Omega^2} - \dfrac{e^{-(1+j\Omega)}}{1+j\Omega} + \dfrac{e^{-(1-j\Omega)}}{1-j\Omega}$
   (c) $\dfrac{2j\Omega}{1+\Omega^2} + \dfrac{e^{-(1+j\Omega)}}{1+j\Omega} - \dfrac{e^{-(1-j\Omega)}}{1-j\Omega}$
   (d) $\dfrac{2}{1+\Omega^2} - \dfrac{e^{-(1+j\Omega)}}{1+j\Omega} - \dfrac{e^{-(1-j\Omega)}}{1-j\Omega}$

   Ans (d)

$$F(\Omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\Omega t} dt = \int_{-1}^{0} e^{t} e^{-j\Omega t} dt + \int_{0}^{1} e^{-t} e^{-j\Omega t} dt$$

$$= \frac{e^{(1-j\Omega)t}}{1-j\Omega}\Big|_{-1}^{0} - \frac{e^{-(1+j\Omega)t}}{1+j\Omega}\Big|_{0}^{1} = \frac{1}{1-j\Omega} - \frac{e^{-(1-j\Omega)}}{1-j\Omega} - \frac{e^{-(1+j\Omega)}}{1+j\Omega} + \frac{1}{1+j\Omega}$$

$$= \frac{2}{1+\Omega^2} - \frac{e^{-(1+j\Omega)}}{1+j\Omega} - \frac{e^{-(1-j\Omega)}}{1-j\Omega}$$

2. Let $x[n] = e^{-2n}u[n]$ be the input to a system. Which of the following impulse responses gives the bounded output for this input?

   (a) $h[n] = n$
   (b) $h[n] = 2^n u[n]$
   (c) $h[n] = u[n]$

(d) $h[n] = 0.5^n u[n]$

Ans (d)

A system is said to be BIBO stable if it's impulse response is absolutely summable, i.e.

$$\sum_{-\infty}^{\infty} |h[n]| < \infty \Rightarrow \text{BIBO stable system}$$

Out of all the options only $h[n] = 0.5^n u[n]$ is absolutely summable.

3. Given $x[n] = 2^{-n}u[n]$. Find l_2 norm of the sequence.

   (a) $\frac{2}{3}$
   (b) $\frac{2}{\sqrt{3}}$
   (c) 2
   (d) $\sqrt{2}$

   Ans (b)

   By definition, the l_p norm of a function $y[n]$ is defined as

$$\|y[n]\|_p = \left(\sum_{n=-\infty}^{\infty} |y[n]|^p \right)^{\frac{1}{p}}$$

   Therefore the l_2 norm of $x[n]$ is $= \|x[n]\|_2 = \left(\sum_{n=0}^{\infty} |\frac{1}{4}|^n \right)^{\frac{1}{2}} = \frac{2}{\sqrt{3}}$

4. Find the L$_4$-norm of the following function:

$$f(x) = \begin{cases} x & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) $\left(\frac{2}{5}\right)^{\frac{1}{4}}$
   (b) $\left(\frac{1}{5}\right)^{\frac{1}{4}}$
   (c) $\left(\frac{1}{4}\right)^{\frac{1}{4}}$

(d) $\left(\frac{1}{2}\right)^{\frac{1}{4}}$

Ans (a)

By definition, the L_p norm of a function $f(t)$ is defined as

$$\|f(t)\|_p = \left(\int_{-\infty}^{\infty} |f(t)|^p dt\right)^{\frac{1}{p}}$$

Therefore the L_4 norm of $f(x)$ is

$$\|f(x)\|_4 = \left(\int_{-\infty}^{\infty} |f(x)|^4 dx\right)^{\frac{1}{4}} = \left(\int_{-1}^{1} |x|^4 dx\right)^{\frac{1}{4}} = \left(\frac{2}{5}\right)^{\frac{1}{4}}$$

5. Which of the following function does **not** belong to $L_2(\mathbb{R})$? (Multiple options can be correct)

   (a)
$$f(t) = \begin{cases} \frac{1}{t} & \text{if } t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (b)
$$f(t) = \begin{cases} \frac{1}{t^2} & \text{if } t \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (c)
$$f(t) = \begin{cases} \frac{1}{\sqrt{t}} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

   (d)
$$f(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Ans (c),(d)

$$f(t) = \begin{cases} \frac{1}{\sqrt{t}} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

L_2 norm $= \left(\int_0^{\infty} \frac{1}{t}\right)^{\frac{1}{2}} = \left(ln\,(t)\Big|_0^{\infty}\right)^{\frac{1}{2}}$
This definite integral is not bounded, hence the function does not belong

to $L_2(\mathbb{R})$.

$$f(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

L_2 norm $= \left(\int\limits_0^{\infty} t^2 \right)^{\frac{1}{2}} = \left(\frac{t^3}{3} \Big|_0^{\infty} \right)^{\frac{1}{2}}$

This definite integral is not bounded, hence the function does not belong to $L_2(\mathbb{R})$.

Try to prove that (a), (b) have bounded L_2 norm.

6. Suppose we are given two functions $x_1(t)$ and $x_2(t)$ as

$$x_1(t) = \cos(10\pi t) \text{ and } x_2(t) = e^{-t^2}\cos(10t)$$

Then which of the signals represents a wavelet or a "small wave" as discussed in the lecture?

(a) $x_1(t)$

(b) $x_2(t)$

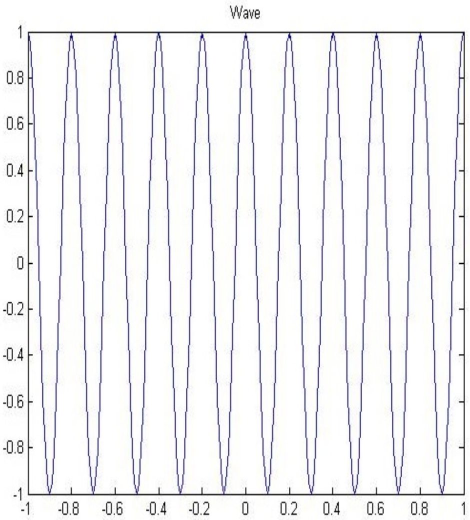
(c) Both $x_1(t)$ and $x_2(t)$

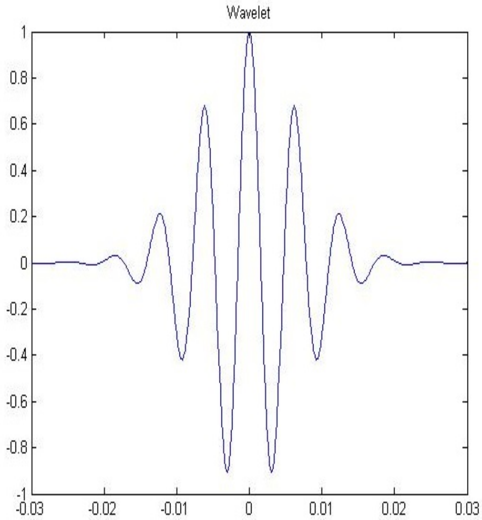
(d) None of these

Ans (b)

As told in lecture1, wavelets are waves that last for a finite time. They may be significant in a certain region of time and insignificant elsewhere or they might exist only for finite time duration.
It can be observed from the following figure that $x_2(t)$ decays very rapidly because of the decaying exponential multiple (in fact it's a Gaussian term) in it. So it does not exists forever and can be considered as a wavelet.

Note: In fact this signal $x_2(t)$ is called a Morlet wavelet.


Wave
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1


Wavelet
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.03 -0.02 -0.01 0 0.01 0.02 0.03

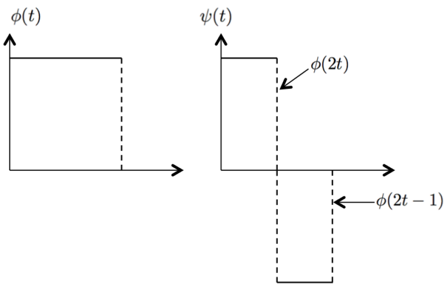
7. Let $\phi(t)$ and $\psi(t)$ be the Haar scaling and wavelet functions. Which one of the following relationship holds?

   (a) $\psi(t) = \phi(t) - \phi(t-1)$
   (b) $\psi(t) = \phi(2t) + \phi(2t-1)$
   (c) $\psi(t) = \phi(2t) - \phi(2t-1)$
   (d) $\psi(t) = \phi(2t) - \phi(2t+1)$

   Ans (c)

   As it can be clearly seen from the figure below that the relation between $\phi(t)$ and $\psi(t)$ will be

$$\psi(t) = \phi\left(2t\right) - \phi\left(2t-1\right)$$



8. We know that any signal $x(t) \in L_2(\mathbb{R})$ (i.e. signal $x(t)$ having finite energy) can be expanded as $x(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \hat{X}(\Omega)e^{j\Omega t}d\Omega$, where $\hat{X}(\Omega)$ is the Fourier transform of signal $x(t)$. Then which of the following is true about time-frequency localization of the Fourier basis element?

   (a) The Fourier basis element has good time localization but poor frequency localization.
   (b) The Fourier basis element has poor time localization but good frequency localization.
   (c) Since we can expand any finite energy signal $x(t)$ into Fourier basis, it must have basis element exhibiting good time and frequency localizations.
   (d) None of the above.

   Ans (b)

   In question 2 we saw that the complex exponential $b(t) = e^{j\Omega t}$ is the basis for Fourier transform. It has a poor time localization because of the fact that $e^{j\Omega t}$ is evenly spread over all times from $t \in (-\infty, \infty)$. The Fourier transform of $e^{j\Omega_0 t}$ is frequency shifted Dirac delta function $\delta\left(\Omega - \Omega_0\right)$ which is just an impulse at the frequency location Ω_0. Therefore it has good frequency localization.

9. Suppose that we don't use Fourier basis as our basis elements and instead decide to use some other basis elements composed of shifted Dirac deltas $\delta(t-k)$, where $k \in \mathbb{Z}$. Then which of the following is true about time-frequency localization of this new basis element?

   (a) It has good time localization but poor frequency localization.
   (b) It has poor time localization but good frequency localization.
   (c) It exhibits good time as well as frequency localizations.
   (d) None of the above.

   Ans (a)

   Similar arguments to that of question 11 tells us that $\delta(t-k)$ has good time localization as it is just an impulse located at integer value k. The Fourier transform of $\delta(t-k)$ is $e^{-j\Omega k}$ and therefore the magnitude of its Fourier transform is 1 $\forall\ \Omega \in (-\infty, \infty)$. Therefore as it is spread at all frequencies, it has very poor frequency localization.

   Note: Question 11 and Question 12 gave a glance of the Uncertainty principle in time and frequency, i.e., the resolution in time domain can be increased at the compromise of resolution in frequency domain and vice-versa.

10. Which of the following pair of functions are orthogonal? (More than one may be correct)

   (a) $\phi(t), \psi(2t)$
   (b) $\phi(t), \psi(t)$
   (c) $\phi(t-m), \psi(t-n), m \neq n$
   (d) All of the above

   Ans (d)

   If the angle between $x(t)$ and $y(t)$ is θ then $\cos(\theta) = \dfrac{\langle x(t), y(t)\rangle}{\|x(t)\|_2 \|y(t)\|_2}$, where $\langle x(t), y(t)\rangle = \int\limits_{-\infty}^{\infty} x(t)\overline{y(t)}dt$. We know that $\langle \phi(t), \psi(2t)\rangle = 0$ and $\langle \phi(t), \psi(t)\rangle = 0$, so both options (a) and (b) are correct. Also we can show that $\langle \phi(t-m), \psi(t-n)\rangle = 0$ whenever $m \neq n$. Thus option (c) is also orthogonal function pair.

11. Which of the following is **not** a subspace of $\mathbb{R}^3$. (More than one may be correct):

   (a) $V = \{(a,b,c)|a,b,c \in \mathbb{R},\ a+b=p,\ p>0\}$
   (b) $V = \{(a,b,c)|a,b,c \in \mathbb{R},\ a+b=0\}$
   (c) $V = \{(a,a+1,0)|a \in \mathbb{R}\}$
   (d) $V = \{(a,2a,3a)|a \in \mathbb{R}\}$

Ans (a,c)
One of the properties of subspaces is that there should be a zero element in the subspace, such that $a + 0 = a, \forall a \in V$.
The zero element in $\mathbb{R}^3$ is $(0,0,0)$ and therefore for any subspace in $\mathbb{R}^3$ $(0,0,0)$ should be present in the subspace. Clearly among the four choices, options (a) and (c) does not contain $(0,0,0)$. We should have $p = 0$ in option (a) so that its a subspace and in option (c) no value of $a \in \mathbb{R}$ can make $(a, a+1, 0)$ a zero vector.

12. Read the following to answer the questions that follows:

    For piece-wise constant representation of a function in an interval, we have used the average value of the function in the given interval. Suppose we devise a new scheme in which we take the average of the maximum and minimum of the function in that interval. For example, piece-wise constant representation of a function in the interval $[0,1]$ would be the average of maxima and minima of the function in the interval [0,1].

    Consider the following function:

$$f(t) = \begin{cases} t^3 & |t| < 1 \\ 0 & else \end{cases}$$

    (a) What will be the representation of the function in the space V_0 using the piece-wise constant representation scheme covered in the lecture (Let us call this scheme A i.e, taking the average of the function in an interval as it's piece-wise constant value in that interval)

        i. $\frac{-1}{2}, \frac{1}{2}$
        ii. $\frac{-1}{3}, \frac{1}{3}$
        iii. $\frac{-1}{4}, \frac{1}{4}$
        iv. $\frac{-1}{5}, \frac{1}{5}$

        Ans (iii)
        In the subspace V_m the piecewise constant representation is over the intervals of size 2^{-m}, hence in V_0 we have unit intervals in which we denote the function by a constant value.
        In an interval $]n, n+1[$, the constant value is given by $\int_n^{n+1} f(t)dt$.Therefore the value in interval $]0,1[$ is $\int_0^1 t^3 dt = \frac{1}{4}$.
        Similarly for the interval $]-1,0[$ the piece-wise constant value is $\frac{-1}{4}$.

    (b) Find the representation of a function in V_0 using the new scheme mentioned in the passage above.(Let us call this scheme B, i.e, taking the average of maximum and minimum value to be piece-wise constant value in a given interval)

i. $\frac{-1}{2}, \frac{1}{2}$
ii. $\frac{-1}{3}, \frac{1}{3}$
iii. $\frac{-1}{4}, \frac{1}{4}$
iv. $\frac{-1}{5}, \frac{1}{5}$

Ans (i)
The maximum and minimum value for the interval $[0, 1]$ are $0, 1$ respectively, which gives the piece-wise constant value $\frac{1}{2}$. Similarly, for interval $[-1, 0]$ piece-wise constant value is $\frac{-1}{2}$

(c) Find out the representation of the function in V_1 using the scheme B.

i. $\frac{-9}{16}, \frac{-1}{16}, \frac{1}{16}, \frac{9}{16}$
ii. $\frac{-9}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{9}{8}$
iii. $\frac{-9}{8}, \frac{-1}{16}, \frac{1}{16}, \frac{9}{8}$
iv. $\frac{-9}{16}, \frac{-1}{8}, \frac{1}{8}, \frac{9}{16}$

Ans (i)

Piece-wise constant value in $[0, \frac{1}{2}] = \dfrac{0 + \frac{1}{8}}{2} = \dfrac{1}{16}$
Piece-wise constant value in $[\frac{1}{2}, 1] = \dfrac{\frac{1}{8} + 1}{2} = \dfrac{9}{16}$
Piece-wise constant value in $[\frac{-1}{2}, 0] = \dfrac{\frac{-1}{8} + 0}{2} = \dfrac{-1}{16}$
Piece-wise constant value in $[-1, \frac{-1}{2}] = \dfrac{-1 + \frac{-1}{8}}{2} = \dfrac{-9}{16}$

(d) We define error between the function and its piece-wise constant representation as $e(t) = (f(t) - f'(t))^2$, where $f(t)$ is the original function and $f'(t)$ is its piece-wise constant representation at any given resolution. What is the average value of error using Scheme A and Scheme B respectively?

i. $\frac{9}{56}, \frac{1}{7}$
ii. $\frac{9}{112}, \frac{1}{7}$
iii. $\frac{9}{112}, \frac{2}{7}$
iv. $\frac{9}{56}, \frac{2}{7}$

Ans (ii)
First, let us calculate average error for scheme A.

$$\begin{aligned}
\overline{e(t)} &= \frac{1}{2}\int_{-1}^{1} e(t)dt \\
&= \frac{1}{2}\int_{-1}^{0}(t^3+\frac{1}{4})^2dt + \frac{1}{2}\int_{0}^{1}(t^3-\frac{1}{4})^2dt \\
&= \frac{9}{112}
\end{aligned}$$

Now let us calculate error for Scheme B.

$$\begin{aligned}
\overline{e(t)} &= \frac{1}{2}\int_{-1}^{1} e(t)dt \\
&= \frac{1}{2}\int_{-1}^{0}(t^3+\frac{1}{2})^2dt + \frac{1}{2}\int_{0}^{1}(t^3-\frac{1}{2})^2dt \\
&= \frac{1}{7}
\end{aligned}$$