Fundamentals of Wavelets, Filter Banks and Time Frequency Analysis

Week 4 Assignment

March 24, 2017

1. If $\mathbf{X(z)}$ is passed through

   a) An M-fold decimator followed by an M-fold expander.

   b) An M-fold expander followed by an M-fold decimator.

   The outputs from a) and b) will

   (a) be identical
   (b) be identical up to a sign
   (c) be such that b) is M times a)
   (d) None of the above

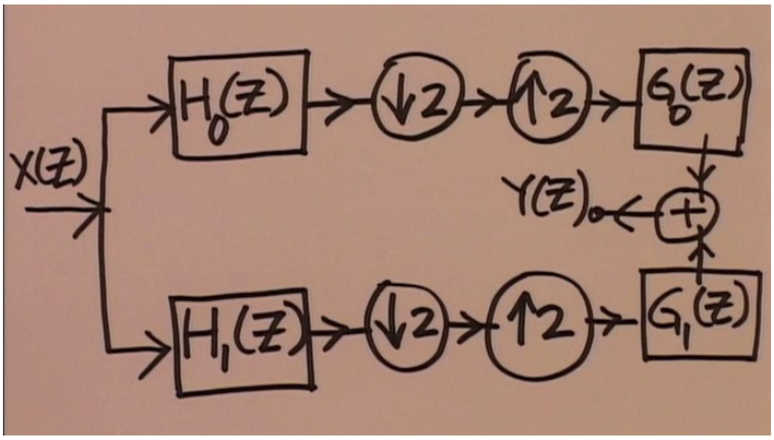
   Ans: D Solution: An expander followed by a decimator gives back the original signal, but a decimator followed by an expander contains aliased components. Hence they cannot be identical.

2. If $\mathbf{X}(z)$ is passed through a filter with Transfer Function $\mathbf{H}(z)$, followed by a two-fold decimator, followed by a filter with Transfer Function $\mathbf{G}(z)$, the output is ____.

   (a) $\frac{\mathbf{G}(z)}{2}[\mathbf{H}(z^{\frac{1}{2}})\mathbf{X}(z^{\frac{1}{2}}) + \mathbf{H}(z^{\frac{-1}{2}})\mathbf{X}(z^{\frac{-1}{2}})]$
   (b) $\frac{\mathbf{H}(z)}{2}[\mathbf{G}(z^{\frac{1}{2}})\mathbf{X}(z^{\frac{1}{2}}) + \mathbf{G}(z^{\frac{-1}{2}})\mathbf{X}(z^{\frac{-1}{2}})]$
   (c) $\frac{\mathbf{G}(z)}{2}[\mathbf{H}(z^{\frac{1}{2}})\mathbf{X}(z^{\frac{1}{2}}) + \mathbf{H}(-z^{\frac{1}{2}})\mathbf{X}(-z^{\frac{1}{2}})]$
   (d) $\frac{\mathbf{H}(z)}{2}[\mathbf{G}(z^{\frac{1}{2}})\mathbf{X}(z^{\frac{1}{2}}) + \mathbf{G}(-z^{\frac{1}{2}})\mathbf{X}(-z^{\frac{1}{2}})]$

   Ans: C Solution: The expression for the z-transform of the output when $\mathbf{X(z)}$ is passed through a twofold decimator is $\mathbf{X}(z^{\frac{1}{2}}) + \mathbf{X}(-z^{\frac{1}{2}})$. We can use this to derive the answer shown above.

3. Consider the standard two channel filter bank with analysis (synthesis) filters $\mathbf{H}_0(z)(\mathbf{G}_0(z))$ and $\mathbf{H}_1(z)(\mathbf{G}_1(z))$.



Give an expression for the z-transform of the signal in the upper branch just after the upsampler.

(a) $\frac{1}{2}[\mathbf{X}(z)\mathbf{H}_0(z^{-1}) + \mathbf{X}(z^{-1})\mathbf{H}_0(z)]$
(b) $\frac{1}{2}[\mathbf{X}(z)\mathbf{H}_0(z) + \mathbf{X}(z^{-1})\mathbf{H}_0(z^{-1})]$
(c) $\frac{1}{2}[\mathbf{X}(-z)\mathbf{H}_0(z) + \mathbf{X}(z)\mathbf{H}_0(-z)]$
(d) $\frac{1}{2}[\mathbf{X}(z)\mathbf{H}_0(z) + \mathbf{X}(-z)\mathbf{H}_0(-z)]$

Ans: D Solution: The expression for the z-transform of the output when $\mathbf{X(z)}$ is passed through a twofold decimator is $\mathbf{X(}z^{\frac{1}{2}}\mathbf{)} + \mathbf{X(}-z^{\frac{1}{2}}\mathbf{)}$. The expression for the z-transform of the output when $\mathbf{X(z)}$ is passed through a twofold expander is $\mathbf{X(}z^2\mathbf{)}$.

We can use these to derive the answer shown above.

4. Give the expression for the output if $\mathbf{X}(z)$ is passed through a 3-fold upsampler.

   (a) $\mathbf{X}(\frac{z}{3})$
   (b) $\mathbf{X}(3z)$
   (c) $\mathbf{X}(z^3)$
   (d) $\mathbf{X}(z^{-3})$

   Ans: C Solution: If y[n] = x[n/3] when n is a multiple of 3 , y[n] = 0 otherwise. $\mathbf{Y(z)} = \sum_{n=-\inf}^{\inf} x[n]z^{-3n} = \mathbf{X(}z^3\mathbf{)}$.

5. Give the expression for the output if $\mathbf{X}(z)$ is passed through a 3-fold downsampler. Where $\omega = e^{j\frac{2\pi}{3}}$

   (a) $\frac{1}{3}[\mathbf{X}(z) + \mathbf{X}(\omega z) + \mathbf{X}(\omega^2 z)]$
   (b) $\frac{1}{3}[\mathbf{X}(z^{\frac{1}{3}}) + \mathbf{X}(\omega z^{\frac{1}{3}}) + \mathbf{X}(\omega^2 z^{\frac{1}{3}})]$
   (c) $\frac{1}{3}[\mathbf{X}(z) + \mathbf{X}(3z) + \mathbf{X}(\frac{z}{3})]$
   (d) $\frac{1}{3}[\mathbf{X}(z) + \mathbf{X}(z\omega^{\frac{1}{3}}) + \mathbf{X}(z\omega^{\frac{2}{3}})]$

Ans: B Solution: Refer Proof of 4.1.4 from Multirate Systems and Filter Banks by P.P Vaidyanathan. The intuition one can use to solve this question is by using the 2-fold downsampler. In a 2-fold downsampler, the output is $\mathbf{X}(z^{\frac{1}{2}}) + \mathbf{X}(-z^{\frac{1}{2}})$, as we know. This can be interpreted as summing together expanded and shifted versions of the original spectra where the expansion is by a factor of 2 and the shift is by $\frac{\pi}{2}$. Hence there are 2 terms. Hence for a threefold downsampler, we would expect to sum expanded and shifted versions of the original spectra where the expansion is by a factor of 3 and the shift is by $\frac{\pi}{3}$. Hence there will be 3 terms. A shift by $\frac{\pi}{n}$ can be represented by multiplying the argument by the n-th root of unity. Hence the answer must be $\frac{1}{3}[\mathbf{X}(z^{\frac{1}{3}}) + \mathbf{X}(\omega z^{\frac{1}{3}}) + \mathbf{X}(\omega^2 z^{\frac{1}{3}})]$.

6. Fill in the blank:

   If $\mathbf{X}(z)$ is the z-transform of a complex exponential signal $exp[j\frac{4\pi}{5}n]$ then $\mathbf{X}(-z)$ is the z-transform of the signal ____.

   (a) $exp[j\frac{-\pi}{5}n]$
   (b) $exp[j\frac{\pi}{5}n]$
   (c) $exp[j\frac{-\pi}{5}n] + exp[j\frac{4\pi}{5}n]$
   (d) $exp[j\frac{-\pi}{5}n] - exp[j\frac{4\pi}{5}n]$

   Ans: $exp[j\frac{-\pi}{5}n]$ Solution: **X(-z)** $=$ $\mathbf{X}(e^{j\pi}\mathbf{z})$. Hence, when we put z $= e^{j\omega}$ to obtain the Frequency Response, we see that it corresponds to a shift along the axis by π. Hence a component at $\frac{4\pi}{5}$ goes to $\frac{-\pi}{5}$.

   One can also see this by seeing that each term in the Fourier series summation, namely: $x[n]e^{j\omega n + j\pi} = x[n]e^{j\pi}e^{j\omega n} = e^{\frac{9\pi}{5}}e^{j\omega}$.

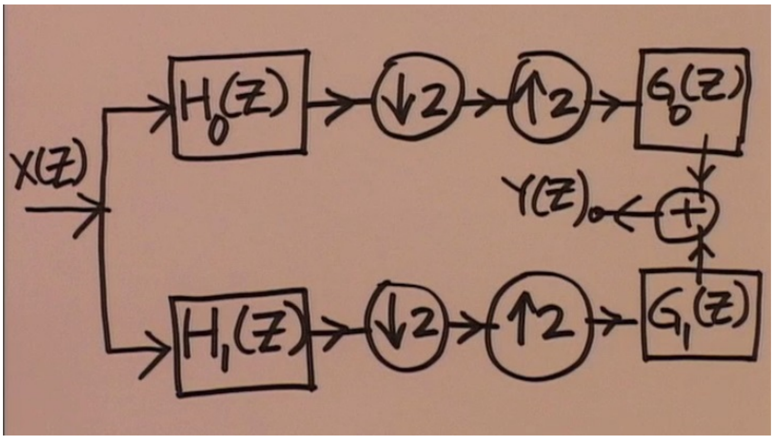
7. Fill in the blank:

   If $\mathbf{X}(z)$ is the z-transform of a highpass signal, then $\mathbf{X}(-z)$ is the z-transform of a ____ signal.

   (a) Highpass
   (b) Lowpass
   (c) Bandpass
   (d) Bandstop

   Ans: Lowpass Solution: **X(-z)** $=$ $\mathbf{X}(e^{j\pi}\mathbf{z})$. Hence, when we put z $= e^{j\omega}$ to obtain the Frequency Response, we see that it corresponds to a shift along the axis by π. Hence those components which were at high frequency get shifted to low frequencies and vice versa. Hence the highpass signal becomes lowpass.

8. In a general two channel filter bank, when the input is $\mathbf{X}(z)$, the expression of the output can be ____.



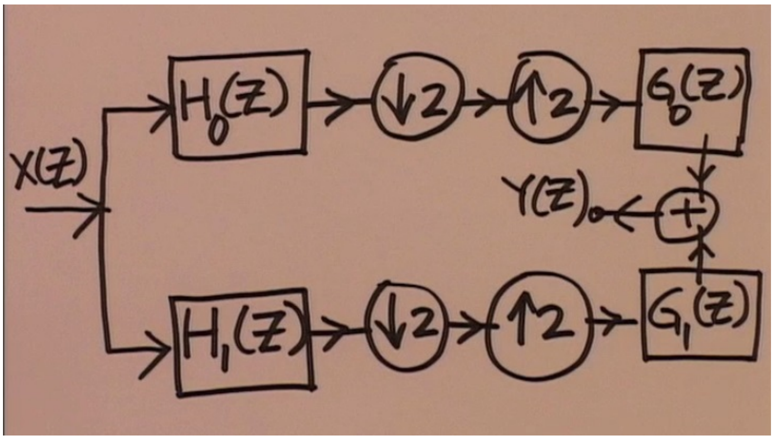
(a) $\mathbf{A}(z)\mathbf{X}(z)$

(b) $\mathbf{A}(z)\mathbf{X}(z) + \mathbf{B}(z)\mathbf{X}(-z)$

(c) $\mathbf{A}(z)\mathbf{X}(z) + \mathbf{B}(z)\mathbf{X}(-z) + \mathbf{C}(z)\mathbf{X}(z^{-1})$

Ans: **A(z)X(z)** + **B(z)X(-z)** Solution: Answer is in the lectures themselves. We see that an alias component corresponding to $\mathbf{X(}-z\mathbf{)}$ is created but no component corresponding to $\mathbf{X(}z^{-1}\mathbf{)}$ is.

9. Of the four filter banks presented below as $[\mathbf{H}_0(z)\mathbf{H}_1(z)\mathbf{G}_0(z)\mathbf{G}_1(z)]$, which is a Perfect Reconstruction Filter Bank (PRFB)?



(a) $1 + z^{-1}, 1 + z^{-1}, -z^{-1}, 1 - z^{-1}$

(b) $1 - z^{-1}, 1 + z^{-1}, -z^{-1}, 1 - z^{-1}$

(c) $1 - z^{-1}, 1 + z^{-1}, -1 + z^{-1}, 1 + z^{-1}$

(d) $1 + z^{-1}, 1 + z^{-1}, -z^{-1}, -z^{-1}$

Ans: $1 - z^{-1}, 1 + z^{-1}, -1 + z^{-1}, 1 + z^{-1}$ Solution: The following conditions must be satisfied.

$\mathbf{H}_0$**(-z)**$\mathbf{G}_0$**(z)** + $\mathbf{H}_1$**(-z)**$\mathbf{G}_1$**(z)** $= 0$

$\mathbf{H}_0(\mathbf{z})\mathbf{G}_0(\mathbf{z}) + \mathbf{H}_1(\mathbf{z})\mathbf{G}_1(\mathbf{z}) = cz^{-n_0}$

Hence we see that only C works.

10. If all the filters in a Perfect Reconstruction Filter Bank (PRFB) $[\mathbf{Y}(z) = cz^{-n_0}\mathbf{X}(z)]$ are anti-causal, then:

    (a) $n_0 \in \mathbb{N}$
    (b) $n_0 \in \mathbb{Z}$
    (c) $n_0 \in \mathbb{Z} \setminus \mathbb{N}$
    (d) $n_0 \in \mathbb{R}$

    Ans: $n_0 \in \mathbb{Z} \setminus \mathbb{N}$ Solution: If all the filters are anti-causal, their z-transforms will only contain positive powers of z. Since neither the upsamplers nor the downsamplers can change the sign of the powers of z in the z-transform, the output transfer function of the singal component can only contain positive powers of z. Hence the answer is $n_0 \in \mathbb{Z} \setminus \mathbb{N}$.

11. The four filters in the Haar Filter Bank are localized in ____ but spread out in _____.

    (a) time, frequency
    (b) frequency, time
    (c) Both of the above
    (d) None of the above

    Ans: time, frequency Solution: Can be seen easily.

12. We know that the first filter on the second channel of the Haar filter bank $(\frac{1-z^{-1}}{2})$ reduces the degree of polynomials by 1. What does this filter do to $cos[\omega_0 n]$?

    (a) $-sin[\frac{\omega_0}{2}]sin[\omega_0 n - \frac{\omega_0}{2}]$
    (b) $-cos[\frac{\omega_0}{2}]sin[\omega_0 n - \frac{\omega_0}{2}]$
    (c) $sin[\omega_0 n - \frac{\omega_0}{2}]$
    (d) $sin[\omega_0 n]$

    Ans: $-sin[\frac{\omega_0}{2}]sin[\omega_0 n - \frac{\omega_0}{2}]$. Hence we get a sinusoid from a cosine. Solution: Output is $\frac{cos[\omega_0 n] - cos[\omega_0 n - \omega_0]}{2}$. Using $cos(C) - cos(D) = 2sin(\frac{C+D}{2})sin(\frac{D-C}{2})$, obtain the answer.

    Answer the next 4 questions on the basis of the filter bank shown here, given the following expressions for

    $\mathbf{H}_0(z) = 1 + z^{-1}$

    $\mathbf{H}_1(z) = 1 - z^{-1}$

    $\mathbf{G}_0(z) = z^{-1}$

13. Give an expression for $\mathbf{G}_1(z)$ that will ensure alias cancellation.

    (a) $\frac{(1-z^{-1})}{1+z^{-1}}$

(b) $\frac{-z^{-2}(1-z^{-1})}{1+z^{-1}}$

(c) $\frac{-z^{-1}(1-z^{-1})}{1+z^{-1}}$

(d) $\frac{z^{-1}(1+z^{2})}{1+z^{-1}}$

Ans: $\frac{-z^{-1}(1-z^{-1})}{1+z^{-1}}$ Solution: For alias cancellation, we need $\mathbf{H}_0\mathbf{(-z)}\mathbf{G}_0\mathbf{(z)} + \mathbf{H}_1\mathbf{(-z)}\mathbf{G}_1\mathbf{(z)} = 0$. Using this, we obtain the answer.

14. Give an expression for $\mathbf{G}_1(z)$ that will leave ONLY the aliasing term in the output.

(a) $\frac{(1-z^{-1})}{1+z^{-1}}$

(b) $\frac{-z^{-1}(1+z^{-1})}{1-z^{-1}}$

(c) $\frac{-z^{-1}(1+z^{-1})}{1+z^{-1}}$

(d) $\frac{z^{-1}(1+z^{-2})}{1+z^{-1}}$

Ans: $\frac{-z^{-1}(1+z^{-1})}{1-z^{-1}}$ Solution: For just the alias component, we need to cancel the non-aliased component. Hence, $\mathbf{H}_0\mathbf{(z)}\mathbf{G}_0\mathbf{(z)} + \mathbf{H}_1\mathbf{(z)}\mathbf{G}_1\mathbf{(z)} = 0$. Using this, we obtain the answer.

15. In the filter bank shown above, $\mathbf{H}_0(z)$ acts as a ____ filter.

(a) Highpass

(b) Lowpass

(c) Bandpass

(d) Bandstop

Ans: Low pass Solution: $1 + z^{-1}$ has a zero at $\omega = \pi$. We can also see that the magnitude response is a cosine. Hence it is low pass.

16. In the filter bank shown above, $\mathbf{H}_1(z)$ acts as a ____ filter.

(a) Highpass

(b) Lowpass

(c) Bandpass

(d) Bandstop

Ans: High pass Solution: $1 - z^{-1}$ has a zero at $\omega = 0$. We can also see that the magnitude response is a sine. Hence it is high pass.