CS 746: Riemann Hypothesis and Its Applications
Solutions of Practice Problem Set 1

Question 1

Show that all the rational functions ($\frac{p(z)}{q(z)}$, for some polynomial $p(z)$ and $q(z)$) are analytic over a domain in which $q(z) \neq 0$ at every point.
Answer:

Claim 1. *If f and g are analytic over D, then so are $f+g$, $f-g$, $f.g$ and f/g (on all points where g does not become zero).*

Proof. Let, $f(z) = u_1(x,y) + iv_1(x,y)$ and $g(z) = u_2(x,y) + iv_2(x,y)$.
As f and g are analytic over D, so f and g are differentiable over D, i.e.,

a. $\frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial y}$ & $\frac{\partial u_1}{\partial y} = -\frac{\partial v_1}{\partial x}$, and

b. $\frac{\partial u_2}{\partial x} = \frac{\partial v_2}{\partial y}$ & $\frac{\partial u_2}{\partial y} = -\frac{\partial v_2}{\partial x}$.

Also f' and g' are continuous over D and so $(f+g)' = f' + g'$ is. As $\frac{\partial u_1}{\partial x}$, $\frac{\partial u_1}{\partial y}$, $\frac{\partial u_2}{\partial x}$ and $\frac{\partial u_2}{\partial y}$ are continuous, so are $\frac{\partial(u_1+u_2)}{\partial x}$ and $\frac{\partial(u_1+u_2)}{\partial y}$.
Now observe that,

i. $\frac{\partial(u_1+u_2)}{\partial x} = \frac{\partial(v_1+v_2)}{\partial y}$, and

ii. $\frac{\partial(u_1+u_2)}{\partial y} = -\frac{\partial(v_1+v_2)}{\partial x}$

Hence, $f+g$ is analytic over D.
Similarly, one can prove the remaining. □

The statement given in the question is just a corollary of the above claim.

Question 2

Show that $f(z) = e^{iz^2}$ is an entire function.
Answer:
Firstly we need to get the function into the form $f(z) = u(x, y) + iv(x, y)$. We do this using the definition of the exponential and Euler's equation.

$$f(z) = e^{iz^2} = e^{-2xy}e^{i(x^2-y^2)} = e^{-2xy}(\cos(x^2 - y^2) + i\sin(x^2 - y^2)).$$

Now using partial differentiation,

$$\frac{\partial u}{\partial x} = -e^{-2xy}\sin(x^2 - y^2) = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -e^{-2xy}\cos(x^2 - y^2) = -\frac{\partial v}{\partial x}.$$

So by Cauchy-Riemann equations, the function is analytic whenever these two equations are satisfied and continuous, which is for all x and y. Hence the function is entire.

Question 3

Study the analyticity of the following functions: e^z and $sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$.
Answer:
(a) Let us express e^z as $e^z = e^x(\cos y + i \sin y)$. Now by the argument similar to the answer of Question 2, it can easily be shown that e^z is entire.
(b) Similarly, e^{iz} and e^{-iz} are entire as well. Thus by Claim 1, $sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$ is also entire.

Question 4

Work out the relationship between absolute convergence and uniform convergence.
Answer:
For a series, absolute convergence implies uniform convergence and this is because $\sum_{n\geq 1} a_n \leq \sum_{n\geq 1} |a_n|$.

However the converse is not true. For example, let us consider the following alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

This series is alternating and $\frac{1}{n}$ is monotone decreasing to 0, so by Leibniz Alternating Series Test, the above series converges. It is clear that above series is not absolutely convergent. Let us now check the condition for uniform convergence for the above series.

$$\begin{aligned}
&\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} \\
&= 1 - \frac{1}{2} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} \\
&= (1 + \frac{1}{2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}) - 2(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}) \\
&= \sum_{k=1}^{2n} \frac{1}{k} - 2\sum_{k=1}^{n} \frac{1}{2k} \\
&= \sum_{k=1}^{n} \frac{1}{n+k} = \epsilon_{2n}
\end{aligned}$$

Observe the Riemann (lower) sum for integrating $f(x) = \frac{1}{x}$ is

$$\frac{1}{n}\sum_{k=1}^{n} f(1 + \frac{k}{n}) = \sum_{k=1}^{n} \frac{1}{n+k} = \epsilon_{2n}.$$

Thus we get the following,

$$\lim_{n\to\infty} \epsilon_{2n} = \int_1^2 \frac{1}{x} dx = \log 2.$$

This shows that the series under consideration is uniformly convergent.

Question 5

Let f be a power series with radius of convergence R, then show that for any z such that $|z| > R$, f is absolutely divergent.

Answer:

Let $f(z) = \sum_{k\geq 0} a_k z^k$, and $|z| = r$, $R < s < r$.

then the sequence $\{|a_k| s^k\}$ does not converge to 0.

Therefore, $|a_k| s^k \geq \epsilon > 0$, which implies

$$|a^k| r^k = |a_k| s^k (r/s)^k \geq \epsilon (r/s)^k$$

and this completes the proof.

Question 6

Show that given the absolutely convergent series

$$A = \sum_{n=0}^{\infty} \alpha_n, \ B = \sum_{n=0}^{\infty} \beta_n$$

we have the absolutely convergent series

$$AB = \sum_{n=0}^{\infty} \gamma_n, \ \gamma_n = \sum_{j=0}^{n} \alpha_j \beta_{n-j}.$$

Answer:
Let, $A_n = \sum_{k=0}^{n} |\alpha_k|$, $B_n = \sum_{k=0}^{n} |\beta_k|$ and $C_n = \sum_{k=0}^{n} |\gamma_k|$. Then,

$$\begin{aligned}
C_n &= |\alpha_0\beta_0| + |\alpha_0\beta_1 + \alpha_1\beta_0| + \cdots + |\alpha_0\beta_n + \alpha_1\beta_{n-1} + \cdots + \alpha_n\beta_0| \\
&\leq |\alpha_0||\beta_0| + (|\alpha_0||\beta_1| + |\alpha_1||\beta_0|) + \cdots + (|\alpha_0||\beta_n| + |\alpha_1||\beta_{n-1}| + \cdots + |\alpha_n||\beta_0|) \\
&= |\alpha_0|B_n + |\alpha_1|B_{n-1} + \cdots + |\alpha_n|B_0 \\
&\leq |\alpha_0|B_n + |\alpha_1|B_n + \cdots + |\alpha_n|B_n \\
&= (|\alpha_0| + |\alpha_1| + \cdots + |\alpha_n|)B_n = A_nB_n \leq \alpha\beta
\end{aligned}$$

where $\alpha = \lim A_n$, $\beta = \lim B_n$. Hence, $\{C_n\}$ is bounded. Note that $\{C_n\}$ is increasing and thus $\{C_n\}$ is a convergent sequence.

Question 7

If D be a domain bounded by a contour C for which Cauchy's theorem is valid and f is continuous on C and regular (analytic and single-valued) in D, then show that $|f| \leq M$ on C implies $|f| \leq M$ in D and if $|f| = M$ in D, then f is a constant.

Answer:
(a) Let $z_0 \in D$, n a positive integer. Then

$$\begin{aligned}
|f(z_0)|^n &= \Big| \frac{1}{2\pi i} \int_C \frac{\{f(z)\}^n dz}{z - z_0} \Big| \\
&\leq \frac{l_c M^n}{2\pi\delta}
\end{aligned}$$

where l_c is the length of C, δ is the distance of z_0 from C. As $n \to \infty$, $|f(z)| \leq M$.

(b) If $|f(z_0)| = M$, then f is a constant. Applying Cauchy's integral formula to $\frac{d}{dz}[\{f(z)\}^n]$, we get

$$
\begin{aligned}
|n\{f(z_0)\}^{n-1}.f'(z_0)| &= \left|\frac{1}{2\pi i}\int_C \frac{f^n dz}{(z-z_0)^2}\right| \\
&\leq \frac{l_c M^n}{2\pi\delta^2}
\end{aligned}
$$

so that

$$|f'(z_0)| \leq \frac{l_c M}{2\pi\delta^2}\frac{1}{n} \to 0, \quad \text{as } n \to \infty.$$

Hence, $|f'(z_0)| = 0$.

(c) If $|f(z_0)| = M$ and $|f'(z_0)| = 0$, then $f^{''}(z_0) = 0$, for

$$\frac{d^2}{dz^2}[\{f(z)\}^n] = n(n-1)\{f(z)\}^{n-2}\{f'(z)\}^2 + n\{f(z)\}^{n-1}f^{''}(z).$$

At z_0, we have

$$\frac{d^2}{dz^2}[\{f(z)\}^n] = nf^{n-1}(z_0)f^{''}(z_0),$$

so that

$$
\begin{aligned}
|nM^{n-1}f^{''}(z_0)| &= \left|\frac{2!}{2\pi i}\int_C \frac{\{f(z)\}^n dz}{(z-z_0)^3}\right| \\
&\leq \frac{2! l_c}{2\pi\delta^3}M^n,
\end{aligned}
$$

and letting $n \to \infty$, we see that $f^{''}(z_0) = 0$. By a similar argument, we prove that all derivatives of f vanish at z_0 (an arbitrary point of D). Thus f is a constant.