

NPTEL COURSE ON CLASSICAL PHYSICS

V. Balakrishnan

*Department of Physics, Indian Institute of Technology Madras
Chennai 600 036, India*

Two-dimensional Autonomous Dynamical Systems

A two-dimensional (2D) autonomous dynamical system in continuous time is specified by a pair of real variables x and y whose time evolution is specified by two coupled, first-order, ordinary differential equations of the form

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y).$$

Here the overhead dots stand for time derivatives, and f and g are, in general, real-valued nonlinear functions of their arguments. It is convenient to combine the evolution equations for x and y into the single vector equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \text{where } \mathbf{x} = (x, y) \text{ and } \mathbf{f} = (f, g).$$

The dynamical behaviour of such a system is essentially governed by the equilibrium points (or **critical points**) of the system. These are the points in the (x, y) plane at which the vector field $\mathbf{f}(\mathbf{x})$ vanishes, namely, the roots of the simultaneous equations

$$f(x, y) = 0 \quad \text{and} \quad g(x, y) = 0.$$

Recall the classification of the critical points (CP's) in a 2D system, for which purpose we proceed as follows. If (\bar{x}, \bar{y}) is a CP, we first change variables to $u = x - \bar{x}$, $v = y - \bar{y}$, so that the CP is at the origin in terms of the variables (u, v) . We then linearize the system in the neighbourhood of the critical point to get

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \approx L \begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{where } L = \left[\frac{\partial(f, g)}{\partial(x, y)} \right]_{(\bar{x}, \bar{y})}$$

is the Jacobian matrix evaluated at the CP. Then:

- If both the eigenvalues of L are real and positive, the CP is an **unstable node**.
- If both the eigenvalues of L are real and negative, the CP is an **asymptotically stable node**.
- If they are real but differ in sign, the CP is a **saddle point**.
- If they are a complex conjugate pair with a positive real part, the CP is an **unstable spiral point**.
- If they are a complex conjugate pair with a negative real part, the CP is an **asymptotically stable spiral point**.
- If they are pure imaginary, the CP is a **stable centre**.
- Finally, if one or both eigenvalues of L vanish, the CP is a **degenerate** or **higher-order** one, and we must go beyond linearization. That is, linearization is not guaranteed to specify, in a unique manner, the precise nature of the flow in the neighbourhood of the CP. In fact, this difficulty crops up even

if the *real* part of the eigenvalues alone vanishes, leaving a pure imaginary part, as in the case of a centre. This is a general feature, valid even for higher-dimensional systems. Whenever the real part of any eigenvalue of the Jacobian matrix vanishes, we must examine the system more carefully because linearization is no longer a reliable guide to the actual flow.

1. To start with, consider the 1-dimensional system $\dot{x} = f(x)$ where $x, f \in \mathbb{R}$. (The phase space is just the x -axis.) Find the critical points and phase portrait for each of the following functions $f(x)$:

(a) x^3 (b) $\sin x$ (c) $\cos x^2$ (d) $\sin^2 x$ (e) $\sinh x$ (f) $(x^2 - 1)^2$.

2. Find the locations and nature of the critical points for the 2-dimensional dynamical systems given below, and sketch the phase portraits qualitatively.

(a) $\mathbf{f} = (2x, x + 2y)$.

(b) $\mathbf{f} = (x + y - 2, x - y)$.

(c) $\mathbf{f} = (-2x, x - 2y)$.

(d) $\mathbf{f} = (y, -x + x^2)$.

(e) $\mathbf{f} = (y, 4 - 3x - x^2)$.

In (d) and (e), we can regard x as the coordinate of a point mass moving in a potential $V(x)$. (Why?) Sketch $V(x)$ in these cases.

3. The Lotka-Volterra predator-prey model is given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} ax - \alpha xy \\ -by + \beta xy \end{pmatrix},$$

where $x, y \geq 0$, and a, α, b and β are positive constants.

(a) Find the critical points and their types.

(b) Show that $a \ln y + b \ln x - \alpha y - \beta x$ is a constant of the motion.

4. Consider the general linear 2D system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Show that $(0, 0)$ is a centre if and only if $a + d = 0$ and $ad - bc > 0$.

(b) Show that the phase trajectories are then given by $cx^2 + 2dxy - by^2 = \text{constant}$.

5. The relativistic linear harmonic oscillator is given by the equation of motion

$$\frac{d}{dt} \left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}} \right) + kx = 0, \quad \text{where } v = \dot{x} \quad \text{and} \quad k > 0.$$

Show that the first integral of the motion is

$$\frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} + \frac{1}{2} kx^2 = C,$$

where C is a constant whose value is fixed by the initial conditions. (For instance, if these are $x = a$ and $v = 0$ at $t = 0$, then $C = m_0 c^2 + \frac{1}{2} k x^2$.) What does C represent physically?

6. A Hamiltonian system: Recall that, in classical mechanics, a Hamiltonian system with n **degrees of freedom** is specified by n **generalized coordinates** (q_1, \dots, q_n) and n **canonically conjugate momenta** (p_1, \dots, p_n) . Together, these comprise $2n$ dynamical variables. The phase space of such a system is therefore always even dimensional. ‘Canonically conjugate’ means that the **Poisson bracket** relations

$$\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \text{and} \quad \{q_i, p_j\} = \delta_{ij}$$

hold good, where δ_{ij} is the Kronecker delta. The time evolution of the entire set of $2n$ variables is governed by a *single* scalar function $H(q_1, \dots, q_n, p_1, \dots, p_n)$ called the Hamiltonian of the system, according to Hamilton’s equations of motion, namely,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \text{where} \quad 1 \leq i \leq n.$$

The minus sign in the second set of equations is crucial. It is straightforward to verify that the total time derivative dH/dt vanishes identically, i. e., $H(q, p)$ itself is a constant of the motion.

Show that the 2-dimensional system

$$\dot{x} = -y + xy, \quad \dot{y} = x + \frac{1}{2}(x^2 - y^2)$$

is a Hamiltonian system specified by a certain Hamiltonian function $H(x, y)$, such that $\dot{x} = \partial H / \partial y$ and $\dot{y} = -\partial H / \partial x$. Is H unique? What if we had required that $\dot{x} = -\partial H / \partial y$, $\dot{y} = +\partial H / \partial x$? Can the system arise as the set of equations of motion of a particle moving in one dimension in some potential?

7. A gradient system: A gradient system is again one for which the time evolution of the entire set of dynamical variables is governed by a single scalar function $\phi(x_1, x_2, \dots, x_n)$, but in a manner that is somewhat simpler than the case of a Hamiltonian system. The dimensionality n of the system could be either even or odd. We have in this case

$$\dot{x}_i = \frac{\partial \phi}{\partial x_i}, \quad \text{where} \quad 1 \leq i \leq n.$$

The structure of gradient systems is nowhere near as intricate as that of Hamiltonian systems.

(a) Show that the 2D system

$$\dot{x} = -x + xy, \quad \dot{y} = -y + \frac{1}{2}(x^2 - y^2)$$

is a gradient system, i.e., there is a scalar function $\phi(x, y)$ such that $\dot{x} = \partial \phi / \partial x$, $\dot{y} = \partial \phi / \partial y$.

(b) Can this system also arise as a Hamiltonian system? Can a gradient system be a Hamiltonian system under any circumstances?

(c) For a 2D Hamiltonian system, show that the solution curves (i.e., the phase trajectories) are level curves $H(x, y) = \text{constant}$; while, for a 2D gradient system, the solution curves cross the level curves of $\phi(x, y)$ at right angles.

8. Oscillator with an inflection point: A particle of unit mass moving on the x -axis has the Hamiltonian (in suitable units)

$$H(x, p) = \frac{1}{2}p^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4.$$

- (a) Find the equilibrium points in the phase plane and classify them.
- (b) Sketch the phase portrait of the system.

9. Generalized nonlinear oscillator: The equation of motion of a nonlinear oscillator is found to be $\ddot{x} + g(x) = 0$, where x lies in the range $[-A, A]$, and $g(x)$ is a given, continuous, differentiable, odd function of x with $g'(x) > 0$.

- (a) Show that this is a Hamiltonian system (find H explicitly).
- (b) Determine the location, nature and stability of the critical point(s) of the system.
- (c) Find the time period of oscillation for motion with the initial conditions $x(0) = A$, $\dot{x}(0) = 0$.

10. Separatrix solution for the simple pendulum: A simple pendulum is a rigid rod of length l and negligible mass, suspended from one end with a bob of mass m at the other end. The Hamiltonian of the simple pendulum is given by

$$H(\theta, p_\theta) = \frac{p_\theta^2}{2ml^2} + mgl(1 - \cos \theta),$$

where θ is the angular displacement of the bob from the vertical. Let E denote the total energy of the bob (a constant of the motion). When $0 < E < 2mgl$, the motion of the bob is oscillatory (librational motion), while for $E > 2mgl$ the motion is rotational. $E = 2mgl$ corresponds to phase trajectories that are **separatrices** that separate the phase trajectories corresponding to librational and rotations motion.

- (a) Show that the time period of oscillation tends to infinity on a separatrix.
- (b) Show that the explicit solution $\theta(t)$ corresponding to motion on the separatrix on which $\theta(-\infty) = -\pi$, $\theta(0) = 0$, $\theta(\infty) = +\pi$ is given by

$$\theta(t) = 4 \tan^{-1}(e^{\omega_0 t}) - \pi,$$

where $\omega_0 = \sqrt{g/l}$. (Recall that this is also the frequency of small oscillations about the equilibrium position.)

- (c) Sketch $\theta(t)$ and the angular velocity $\dot{\theta}(t)$ as functions of t (where $-\infty < t < \infty$). The 'lump-like' shape of $\dot{\theta}(t)$ as a function of time is related to the concept of **instantons** in certain nonlinear field theories.

11. A family of isochronous oscillators: It is well known that the time period of a simple harmonic oscillator is independent of its amplitude, and hence independent of its total energy. Contrary to the naive belief that this property is unique to motion in a parabolic potential, it turns out that oscillatory motion in a whole family of potentials is also independent of the amplitude of oscillation (or the total energy of the oscillator).

A particle of unit mass moves on the x -axis in the force field

$$F(x) = -kx + \frac{a}{x^3}$$

where k and a are positive constants.

- Sketch the potential $V(x)$ as a function of x .
- Find the equilibrium points and sketch the phase portrait of the particle.
- Show that, if the energy E of the particle exceeds \sqrt{ka} , the particle oscillates either between the points x_1 and x_2 , or between the points $-x_2$ and $-x_1$, where

$$x_1 = \left(\frac{E - \sqrt{E^2 - ka}}{k} \right)^{1/2} \quad \text{and} \quad x_2 = \left(\frac{E + \sqrt{E^2 - ka}}{k} \right)^{1/2}.$$

(These are called the *turning points* of the motion.)

- Show that, for any given $E > \sqrt{ka}$, the time period of oscillation of the particle about its equilibrium position is given by $T = \pi/\sqrt{k}$. This is independent of E (and the parameter a), and is one half the time period ($2\pi/\sqrt{k}$) of a particle of unit mass in the simple harmonic oscillator potential $\frac{1}{2}kx^2$.

12. LC circuit with nonlinear inductor: A lossless oscillatory LC circuit has an inductor with a ferromagnetic core. The net flux through the core is $\phi(I)$, a *nonlinear* function of the instantaneous current I in the circuit ($I \equiv \dot{q}$ where q is the instantaneous charge on C).

- Write down the equation of motion for q .
- Show that $q^2/(2C) + \int \phi'(I) I dI$ is a constant of the motion. (This is the total energy of the system.)
- Given that $\phi(I)$ is an *odd* function of I , what is the nature of the CP in the (q, I) plane? If $\phi(I) = \alpha I + \beta \tan^{-1}(\gamma I)$, where α, β and γ are positive constants, find the phase trajectories of the system.

13. Clock with escapement: The pendulum of a clock is energized by a spring-and-escapement mechanism that imparts an impact to the pendulum whenever it passes through its equilibrium position. The impact gives an instantaneous, constant increment Ω to the angular velocity of the pendulum. The equation of motion of the pendulum (in the absence of the driving force) is $\ddot{\theta} + \gamma \dot{\theta} + \omega^2 \theta = 0$ where γ is a friction constant, and $\gamma^2 \ll \omega^2$. Assume that the angular displacement of the pendulum is initially given by $\theta(t) = \theta_0 e^{-\gamma t/2} \sin \omega_0 t$, so that $\theta(0) = 0$ and $\dot{\theta}(0) = \omega_0 \theta_0$. The first time it receives an impact from the escapement mechanism is at $t = \pi/\omega_0$. The periodic impacts given by the driving force compensate for the damping due to friction, and enable the periodic motion to be maintained.

- What is the value of Ω required to maintain a steady periodic motion of the pendulum?
- Sketch the corresponding phase trajectory.

14. The damped simple harmonic oscillator: Consider the damped linear harmonic oscillator of unit mass, given by the system of equations

$$\dot{x} = v, \quad \dot{v} = -\omega_0^2 x - \gamma v,$$

where γ is a positive constant. It is clear that $(0, 0)$ is no longer a centre in the (x, v) phase plane, but rather an asymptotically stable spiral point, provided $\omega_0 > \frac{1}{2}\gamma$ (i.e., the oscillator is underdamped). The phase trajectories are inward spirals tending to the origin.

(a) Show that the phase trajectories are given by the equation

$$v^2 + \gamma x v + \omega_0^2 x^2 = C \exp \left\{ \frac{\gamma}{\omega} \tan^{-1} \left(\frac{v}{\omega x} + \frac{\gamma}{2\omega} \right) \right\},$$

where $\omega = (\omega_0^2 - \frac{1}{4}\gamma^2)^{1/2}$ and C is a constant. [Hint: Change variables from v to $u = v/x$.] The fact that a non-algebraic (or transcendental) function like the arctan function appears in the constant of the motion (COM) represented by the left-hand side of the equation above indicates that the COM is *not* an *isolating integral* in this case.

(b) What sort of critical point is the origin in the phase plane, in the critically damped ($\omega_0 = \frac{1}{2}\gamma$) case? Show that the phase trajectories are given, in this case, by

$$2v + \gamma x = C \exp \left\{ -\frac{\gamma x}{2v + \gamma x} \right\}.$$

(c) What sort of critical point is the origin in the phase plane, in the overdamped ($\omega_0 < \frac{1}{2}\gamma$) case? It is convenient, now, to define the shifted friction constant $\gamma_s = (\gamma^2 - 4\omega_0^2)^{1/2}$. Show that the phase trajectories are now given by

$$v^2 + \gamma x v + \omega_0^2 x^2 = C \left\{ \frac{2v + (\gamma - \gamma_s)x}{2v + (\gamma + \gamma_s)x} \right\}^{\gamma/\gamma_s}.$$

15. Generalized nonlinear oscillator with damping: The equation of motion of an undamped linear harmonic oscillator of unit mass, or that of any dynamical system modelled by the linear harmonic oscillator, is $\ddot{x} + \omega_0^2 x = 0$. The effects of dissipation and/or nonlinearities may be incorporated by modifying the equation of motion to read

$$\ddot{x} + f(x, \dot{x}) + \omega_0^2 x = 0.$$

(a) What is the condition that must be obeyed by the function $f(x, \dot{x})$, such that the motion continues to be a stable, periodic one with a given time period T ?

(b) Hence show that an LRC series circuit with a given initial amount of electric and magnetic energy *cannot* sustain periodic motion.

16. A 3-dimensional integrable system: the Euler top: Consider the free rotation of a rigid body about a fixed point, in the absence of any external torque. The dynamical variables are the components $(\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ of the angular velocity of the body in the body-fixed principal axis frame. They comprise a 3-dimensional dynamical system, satisfying Euler's equations

$$I_1 \dot{\omega}_1 = (I_2 - I_3) \omega_2 \omega_3, \quad I_2 \dot{\omega}_2 = (I_3 - I_1) \omega_3 \omega_1, \quad I_3 \dot{\omega}_3 = (I_1 - I_2) \omega_1 \omega_2.$$

The constants I_1 , I_2 and I_3 are the principal moments of inertia of the body. Without loss of generality, we may take them to satisfy the inequalities $I_1 > I_2 > I_3 > 0$. As the phase space is three-dimensional, the existence of two distinct, time-independent, smooth functions of ω_1 , ω_2 and ω_3 guarantees that the system is completely integrable. The Euler top is another important instance of a conservative, integrable dynamical system.

- (a) Let $M_i = I_i \omega_i$, where $i = 1, 2, 3$. Show that $M^2 = M_1^2 + M_2^2 + M_3^2$ is a constant of the motion.
- (b) Find the critical points of the system and the nature of their stability.
- (c) Find a constant of the motion *other* than M^2 . The existence of these two COMs suffices to determine (implicitly) the phase trajectories of the system. Can you sketch the phase trajectories?

Hamiltonian dynamics

1. We have seen that canonical transformations (CTs) are **symplectic transformations** in the following sense. Let $\mathbf{x} = (q, p)$ where q and p stand for the sets of generalized coordinates (q_1, \dots, q_n) and momenta (p_1, \dots, p_n) , respectively. Let $\boldsymbol{\xi} = (Q, P)$ where Q and P stand for the new generalized coordinates and momenta after a canonical transformation. Denote the $(2n \times 2n)$ Jacobian matrix of the transformation by $\partial\boldsymbol{\xi}/\partial\mathbf{x}$. Then the fact that the transformed variables also satisfy the canonical Poisson bracket relations can be summarized in the condition

$$\left(\frac{\partial\boldsymbol{\xi}}{\partial\mathbf{x}}\right)^T J \left(\frac{\partial\boldsymbol{\xi}}{\partial\mathbf{x}}\right) = J$$

where J (as defined in class) is the $(2n \times 2n)$ matrix

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

Here 0_n and I_n denote the $(n \times n)$ null matrix and unit matrix, respectively. In other words, the Jacobian of a CT is a *symplectic* matrix. (A matrix M is symplectic if it satisfies the condition $M^T J M = J$.) Recall that J has the properties

$$J^2 = -I \quad \text{and} \quad J^T = J^{-1} = -J,$$

where I stands for the $(2n \times 2n)$ unit matrix. We also have $\det J = 1$. Any symplectic matrix M is unimodular, i.e., $\det M = +1$.

- (a) If the CT is an infinitesimal one, the Jacobian is of the form $I + \epsilon G$. Show that the condition for a CT becomes the following condition on the infinitesimal generator of such a transformation:

$$G^T = J G J.$$

- (b) Use the above condition to show that the number of independent parameters (or generators) of the symplectic group $Sp(2n, \mathbb{R})$ is $n(2n + 1)$.
- (c) Note that J itself is a symplectic matrix. What is the explicit canonical transformation $(q, p) \rightarrow (Q, P)$ whose Jacobian is J ?
- (d) In the special case of a 1-freedom Hamiltonian system, the number of parameters of the symplectic group $Sp(2, \mathbb{R})$ is equal to 3. This is the same as the number of parameters of the special linear group $SL(2, \mathbb{R})$, the group of unimodular (2×2) matrices with real elements. In fact, the two groups are isomorphic to each other. Show that an arbitrary (2×2) matrix with real elements and unit determinant is a symplectic matrix.

Dynamical symmetry in a Hamiltonian system: Recall the **Liouville-Arnold criterion** for the integrability of an n -freedom Hamiltonian system:

- If n independent isolating integrals F_1, \dots, F_n exist, that are in involution with each other, the system is integrable by a canonical transformation to action-angle variables.

We may ask: What makes a Hamiltonian system integrable? What physical property of the system lies behind the existence of $(n - 1)$ isolating COMs other than H itself, all of them in involution with each other? The answer is that COMs are linked to certain dynamical symmetries possessed by the system.

- The dynamical symmetry group of a system is the group of transformations of its phase space variables under which the equations of motion of the system remain form-invariant, i.e., unchanged in form.
- This implies that *the solution space of the system remains invariant under this group of transformations*. Individual solutions may transform to other individual solutions, but the *set* of solutions remains unchanged.

Finding the dynamical symmetry group of an arbitrary dynamical system is, in general, a rather nontrivial task.

In the case of Hamiltonian systems, a more definite statement can be made. The transformations required must certainly be canonical transformations. Now, we have seen that the set of canonical transformations of an n -freedom system form the **symplectic group** $Sp(2n, \mathbb{R})$. (This is the group of all $(2n \times 2n)$ symplectic matrices with real elements.) But all canonical transformations may not leave a given Hamiltonian unchanged. Therefore,

- the dynamical symmetry group of a system with Hamiltonian H is the *subgroup* of $Sp(2n, \mathbb{R})$ that leaves H invariant.

It may also happen, of course, that *all* transformations (of the phase space variables) that leave H invariant are not canonical transformations, i.e., they may not all belong to the symplectic group. Thus:

- The dynamical symmetry group is, in general, the intersection of $Sp(2n, \mathbb{R})$ with the symmetry group of H .

These ideas are made clear by the following simple example of an integrable Hamiltonian system.

2. The two-dimensional isotropic harmonic oscillator is given by the Hamiltonian (in units such that $m = 1, \omega = 1$)

$$H = \frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2).$$

Thus the number of degrees of freedom $n = 2$ in this case. The term ‘isotropic’ refers to the fact that the natural frequencies of the two oscillators are equal to each other. The potential then becomes a central potential (this becomes obvious on writing it in plane polar coordinates in the (q_1, q_2) plane), so that the system has circular symmetry (or ‘isotropy’). Consider the three functions of the dynamical variables given by

$$J_1 = \frac{1}{4}(q_1^2 + p_1^2 - q_2^2 - p_2^2), \quad J_2 = \frac{1}{2}(q_1 q_2 + p_1 p_2), \quad J_3 = \frac{1}{2}(q_1 p_2 - q_2 p_1).$$

- Verify that the J_i are constants of the motion.
- Verify that $\{J_i, J_j\} = \epsilon_{ijk} J_k$.

This last result shows that there is a deep connection between the two-dimensional (2D) isotropic harmonic oscillator and the angular momentum Lie algebra: the three constants of motion J_i in the former problem satisfy the same algebra as the orbital angular momentum components L_i . This is directly related to the dynamical symmetry group of the 2D isotropic harmonic oscillator, which turns out to be $SU(2)$, the group of unitary (2×2) matrices with unit determinant. The generators of $SU(2)$ obey the same Lie algebra as the J_i above. This is the algebra of the

generators of the rotation group in three dimensions, the usual angular momentum algebra.

What is the symmetry group of H itself? It is immediately obvious from the functional form of H for the 2D isotropic harmonic oscillator that the group of rotations in the four-dimensional phase space spanned by q_1, p_1, q_2, p_2 leaves H unchanged: the (hyper)surface $H = \text{constant}$ is obviously a sphere with centre at the origin in this space. The symmetry group that leaves H unchanged is therefore isomorphic to $SO(4)$, the group of rotations in a four-dimensional Euclidean space. But not all rotations in the phase space leave the *equations of motion* form-invariant. The transformations that leave Hamilton's equations unaltered in form belong to the symplectic group of canonical transformations, $Sp(4, \mathbb{R})$. The intersection of these two groups is the actual dynamical symmetry group of the system. In mathematical terms,

$$Sp(4, \mathbb{R}) \cap SO(4) \sim SU(2).$$

3. The three-dimensional isotropic harmonic oscillator is given by the Hamiltonian (again in units such that $m = 1, \omega = 1$)

$$H = \frac{1}{2} \sum_{i=1}^3 (p_i^2 + q_i^2).$$

- (a) Show that the quantities $T_{ij} = p_i p_j + q_i q_j$ (where $i, j = 1, 2, 3$) are constants of the motion.
- (b) Evaluate the Poisson brackets $\{T_{ij}, T_{kl}\}$ and $\{\Lambda_{ij}, T_{kl}\}$, where $\Lambda_{ij} = q_i p_j - q_j p_i$. (Λ_{ij} is essentially the orbital angular momentum, via the relation $\Lambda_{ij} = \epsilon_{ijk} L_k$.)

For completeness, I mention that the dynamical symmetry group of the 3D isotropic harmonic oscillator is $SU(3)$, the group of unitary unimodular (3×3) matrices. More generally, the dynamical symmetry group of the n -dimensional isotropic oscillator is the group $SU(n)$. The number of generators of $SU(n)$ is $n^2 - 1$.

4. The Kepler problem: The Hamiltonian of a particle of mass m moving in an inverse square force field is given by

$$H = \frac{\mathbf{p}^2}{2m} - \frac{k}{r},$$

where k is a constant and r is the distance of the particle from the centre of force, taken to be at the origin of coordinates.¹ We know that the orbital angular momentum components

$$L_i = \epsilon_{ijk} q_j p_k$$

are constants of the motion. This is in fact valid for *any* central force. In addition, the $1/r$ potential has the special property that there is *another* vector constant of the motion, namely, the **Laplace-Runge-Lenz vector** defined as

$$\mathbf{A} = (\mathbf{p} \times \mathbf{L}) - \frac{mk}{r} \mathbf{r}, \quad \text{or} \quad A_i = \epsilon_{ijk} p_j L_k - \frac{mk}{r} x_i.$$

¹A potential proportional to $1/r$ in three-dimensions is usually referred to as the **Coulomb potential**.

- (a) Verify that \mathbf{A} is a constant of the motion by showing directly that $d\mathbf{A}/dt$ vanishes identically.
- (b) The Hamiltonian H , the three components of \mathbf{L} , and the three components of \mathbf{A} comprise 7 time-independent constants of the motion. But the phase space is only 6-dimensional. It is obvious that there must be relations between the 7 COMs above. It is easy to see that $\mathbf{A} \cdot \mathbf{L} = 0$. Thus \mathbf{A} has no component along \mathbf{L} , i. e., it lies in the plane of the orbit. This eliminates one of the 7 COMs. Show also that

$$A^2 = 2mEL^2 + m^2k^2,$$

where E is the total energy of the particle, i. e., the numerical value of the COM represented by H . Hence in some sense there is really only one independent component of \mathbf{A} .

From this point onwards, let us consider the case $k > 0$, i. e., an attractive $1/r$ potential, and closed orbits (or periodic motion). In physical terms, the Laplace-Runge-Lenz vector of a particle describing an elliptical orbit under an attractive inverse square law force is a vector directed along the semi-major axis of the ellipse. Its magnitude $A = |\mathbf{A}|$ is proportional to the eccentricity of the ellipse: the exact relation is eccentricity = $A/(mk)$. The constancy of its direction implies that the orbit does not *precess* for a pure inverse square law force. A small perturbation of (or departure from) the inverse square law will generally cause a precession of the orbit, i.e., a relatively slow drift of the direction of the semi-major axis of the ellipse.

- (c) The components of \mathbf{L} and \mathbf{A} constitute an *algebra*: their Poisson brackets with each other turn out to be linear combinations of themselves. Show that

$$\{A_i, L_j\} = \epsilon_{ijk} A_k,$$

Similarly, show that

$$\{A_i, A_j\} = (2m|E|) \epsilon_{ijk} L_k,$$

where $E(< 0)$ is the conserved total energy of the particle in its closed orbit.

Here's how this is done. Use the canonical Poisson brackets

$$\{x_i, x_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij}$$

to show that

$$\{x_i, L_j\} = \epsilon_{ijk} x_k, \quad \{p_i, L_j\} = \epsilon_{ijk} p_k.$$

Hence verify that

$$\{L_i, L_j\} = \epsilon_{ijk} L_k,$$

which is the angular momentum algebra. Now consider the Poisson brackets $\{A_i, L_j\}$ and $\{A_i, A_j\}$. These involve Poisson brackets like $\{1/r, L_j\}$, and it is not immediately clear how this quantity can be evaluated. To avoid the problem, write

$$-\frac{k}{r} = H - \frac{\mathbf{p}^2}{2m}$$

and use the fact that L_j is a constant of the motion (that is, $\{L_j, H\} = 0$).

- (d) The result of the last part above suggest forming linear combinations of the components of \mathbf{L} and $\mathbf{A}/\sqrt{2m|E|}$ which satisfy simpler (or more easily recognized) Poisson bracket relations. Carry this out to show that the components of the two vectors

$$\mathbf{M} = \frac{1}{2} \left(\frac{\mathbf{A}}{\sqrt{2m|E|}} + \mathbf{L} \right) \quad \text{and} \quad \mathbf{N} = \frac{1}{2} \left(\frac{\mathbf{A}}{\sqrt{2m|E|}} - \mathbf{L} \right)$$

behave like the generators of two separate angular momentum algebras, i. e.,

$$\{M_i, M_j\} = \epsilon_{ijk} M_k, \quad \{N_i, N_j\} = \epsilon_{ijk} N_k, \quad \text{and} \quad \{M_i, N_j\} = 0.$$

In other words, we have two separate angular momentum (or $\mathfrak{so}(3)$) algebras! These relations can be shown to be precisely the Lie algebra satisfied by the generators of the group $SO(4)$, which is the group of unimodular, real, orthogonal (4×4) matrices (or the group of rotations in 4-dimensional Euclidean space). We conclude that the dynamical symmetry group of the Kepler problem in the case $E < 0$, i.e., bounded motion,² is $SO(4)$. (Actually, it is a bigger group than that. Since the set of orbits is invariant under reflection, the symmetry group is actually $O(4)$ rather than $SO(4)$. This is the group of all (4×4) orthogonal matrices with real elements.)

²For completeness, I mention that the dynamical symmetry group in the case of unbounded motion (or *scattering*) in a Coulomb potential is the pseudo-orthogonal group $SO(3,1)$, which happens to be isomorphic to the special Lorentz group of proper, homogeneous Lorentz transformations.

Quiz

1. Are the statements in quotation marks true or false?

(a) “The Lagrangian of a particle moving in a central potential $V(r)$ has two cyclic (or ignorable) coordinates.”

(b) In parts (b) to (f), we consider a system of N particles with a Lagrangian given by

$$L = \frac{1}{2} \sum_{i=1}^N m_i \dot{\mathbf{r}}_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^N V(|\mathbf{r}_i - \mathbf{r}_j|),$$

where V is a general function of its argument.

“The total angular momentum of the system is a constant of the motion.”

(c) “It is possible to make a Legendre transform to the Hamiltonian in this case.”

(d) “We can find N constants of the motion that are in involution with each other.”

(e) “The case $N = 2$ alone is integrable, but not $N \geq 3$, for a general V .”

(f) “There are three cyclic or ignorable coordinates in this system.”

(g) An n -dimensional dynamical system is given by $\dot{\mathbf{x}} = \nabla \phi(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$ and ∇ is the gradient operator.

“If $\nabla^2 \phi < 0$ everywhere, then the system is a conservative dynamical system.”

(h) Let $H(q, p)$ be the Hamiltonian of an autonomous system with 1 degree of freedom.

“If $A(q, p)$ is any function of the dynamical variables such that the Poisson bracket $\{A, H\} = 0$, then A is either a constant or a function of H itself.”

(i) “The critical points of an autonomous Hamiltonian system can only be saddle points and centers.”

(j) A particle moves on the x -axis in the potential $V(x) = K|x|^\alpha$ where K and α are positive numbers.

“All the phase trajectories of the particle (other than the critical point at the origin) are closed trajectories in the (x, p) plane.”

(k) Same system as in the preceding part:

“The time period of oscillatory motion of the particle is independent of its total energy in the cases $\alpha = 2$ and $\alpha = -1$.”

- (l) Consider an autonomous 4-dimensional dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^4$ and $\mathbf{f} \in \mathbb{R}^4$.

“This system can have at most 3 functionally independent constants of the motion that do not have any explicit time-dependence.”

- (m) Same system as in the preceding part:

“This system can have at most 4 functionally independent constants of the motion, of which at least 1 must be explicitly time-dependent.”

- (n) “The product of all the eigenvalues of any rotation matrix in n -dimensional Euclidean space must be equal to $(-1)^n$.”

- (o) A system consists of N particles moving in space under k integrable (or holonomic) constraints. Some of the constraints may be time-dependent.

“The number of independent generalized coordinates $\{q_i\}$ of the system is $3(N - k)$.”

- (p) Same system as in the preceding part:

“The kinetic energy of the system is, in general, of the form

$$T = \sum_{i=1}^n \sum_{j=1}^n A_{ij} \dot{q}_i \dot{q}_j + \sum_{i=1}^n B_i \dot{q}_i + C$$

where n is the number of independent degrees of freedom, and A_{ij} , B_i and C are functions of $\{q_i\}$ and t .”

- (q) A particle moves in space under the potential $V(x) = K(x^4 + y^4 + z^4)$ where K is a positive constant.

“The angular momentum of the particle about the origin of coordinates is a constant of the motion.”

- (r) A particle moves in space under the potential $V(\mathbf{r})$.

“If $V(\mathbf{r})$ is invariant under the parity transformation $\mathbf{r} \rightarrow -\mathbf{r}$, this implies the existence of a constant of the motion, according to Noether’s Theorem.”

- (s) “The canonical momentum of a charged particle moving in an electromagnetic field is dependent on the gauge chosen for the electromagnetic potentials.”

(t) “The dynamical symmetry group of the n -dimensional isotropic harmonic oscillator is $SO(2n)$.”

(u) A system with two degrees of freedom has the Lagrangian

$$L = \frac{1}{2}(\dot{q}_1 - \dot{q}_2)^2 + V(q_1, q_2).$$

“It is not possible to make a Legendre transformation to the Hamiltonian in this case.”

(v) “The group of $(2n \times 2n)$ symplectic matrices with real elements is a subgroup of the group of $(2n \times 2n)$ orthogonal matrices with real elements.”

(w) Let $M = \partial(Q, P)/\partial(q, p)$ denote the Jacobian matrix of a canonical transformation.

“The transformation whose Jacobian matrix is given by M^T is also a canonical transformation.”

(x) Noether’s Theorem enables us to find a conserved quantity associated with a group of continuous transformations of the dynamical variables of a Lagrangian system.

“In order for the theorem to be applicable, the Lagrangian must be unchanged under the transformations belonging to the group.”

(y) “There is an analogue of the Laplace-Runge-Lenz vector, i.e., a vector constant of the motion, for the motion of a particle in a central potential of the form $V(r) = kr^n$ where n is any nonzero integer.”

(z) “Euler’s equations for the force-free motion of a rigid body describe a completely integrable system.”

2. Fill in the blanks in the following:

(a) The Cartesian coordinates and conjugate momenta of a particle moving in space satisfy the canonical Poisson bracket relations

$$\{x_i, x_j\} = 0, \{p_i, p_j\} = 0, \{x_i, p_j\} = \delta_{ij}, \quad \text{where } i, j = 1, 2, 3.$$

If $r^2 = x_1^2 + x_2^2 + x_3^2$ and $p^2 = p_1^2 + p_2^2 + p_3^2$ as usual, then $\{r^2, p^2\} = \dots$

(b) A bead of mass m is constrained to move without friction on a wire in the shape of a parabola located in a vertical plane (the yz -plane). The axis of the parabola is along the z -axis, and the equation to the parabola is $z = \frac{1}{2}ky^2$ where k is a positive constant. The wire is rotated about the z -axis at a constant angular speed ω . Let ρ denote the distance of the bead from the z -axis. After all the constraints are imposed, the momentum conjugate to ρ is $p = \dots$

- (c) Consider the two-dimensional dynamical system

$$\dot{x} = x + y - x(x^2 + y^2), \quad \dot{y} = -x + y - y(x^2 + y^2).$$

Using the Bendixson criterion, we may conclude that the largest circle centered at the origin, within which the system cannot have a closed trajectory, is of radius $r_{\max} = \dots$

- (d) In the Kepler problem with an attractive $1/r$ potential, let H , \mathbf{L} and \mathbf{A} denote the Hamiltonian, the orbital angular momentum and the Laplace-Runge-Lenz vector, respectively. A quantity formed from these that Poisson-commutes with every component of \mathbf{L} and \mathbf{A} is \dots
- (e) If ϵ_{ijk} denotes the Levi-Civita symbol in three dimensions, as usual, the $\epsilon_{ijk} \epsilon_{klm} \epsilon_{mni} = \dots$
- (f) The number of independent components of a tensor of rank k in n -dimensional Euclidean space is \dots .
- (g) In the preceding question, if the tensor is totally symmetric, the number of independent components is \dots .
- (h) The general form of a (2×2) unitary matrix with determinant $= +1$ is \dots .
- (i) The parameter space of the group of rotations in three-dimensional Euclidean space is \dots .
- (j) Parabolic coordinates (σ, τ) in a plane are defined in terms of the Cartesian coordinates by $x = \sigma\tau$, $y = \frac{1}{2}(\tau^2 - \sigma^2)$. The kinetic energy of a particle of mass m moving in the plane is, in these coordinates, $T = \dots$

Thermodynamics and Classical Statistical Physics

1. Some probability distributions:

- (a) A pair of (distinguishable) dice is tossed once. Each die can give a score of 1, 2, 3, 4, 5 and 6. Let s denote the total score of the pair of dice. What is the probability distribution of s , and what is its most probable value?
- (b) The normalized probability density of the x -component of the velocity of a molecule of mass m in a classical ideal gas in thermal equilibrium at a temperature T is given by the Maxwellian

$$\left(\frac{m}{2\pi k_B T}\right)^{1/2} \exp\left(-\frac{mv_x^2}{2k_B T}\right).$$

What is the probability density of the *magnitude* $|v_x|$ of the x -component of the velocity?

- (c) The y - and z -components v_y and v_z of the velocity of a molecule have exactly the same probability density functions as that of v_x given above. What is the normalized probability density function of the speed v of a molecule?

2. Specific heat of an ideal gas in a polytropic process: One mole of an ideal gas ($PV = RT$) undergoes a **polytropic process** defined by the condition $PV^n = \text{constant}$, where the index n is a positive number (not necessarily an integer).

- (a) Show that the specific heat of the gas corresponding to this process is given by

$$C_n = C_v \left(\frac{n - \gamma}{n - 1}\right),$$

where C_v is the specific heat at constant volume, and $\gamma = C_p/C_v$.

- (b) Sketch C_n versus n for $0 \leq n < \infty$. Interpret in physical terms what happens when (i) $n = 0$ (ii) $n = 1$ (iii) $n = \gamma$ (iv) $n \rightarrow \infty$. Observe that C_n can never take a value that lies in between C_v and C_p .
- (c) Can n be negative?

3. The chemical potential of a thermodynamic system may be defined as the change in the Helmholtz free energy F when one more particle is added to a system of N particles keeping all other thermodynamic variables unaltered, i.e.,

$$\mu = F(V, T, N + 1) - F(V, T, N).$$

- (a) The thermodynamic limit is defined as the limit in which $N \rightarrow \infty$ and $V \rightarrow \infty$ such that the number density $N/V \rightarrow n$, a finite quantity. Show that the chemical potential is given by

$$\mu = \left(\frac{\partial f}{\partial n}\right)_T, \text{ where } f = \lim \frac{F}{V}.$$

- (b) From the relation $P = -(\partial F/\partial V)_{T,N}$, show that $P = \mu n - f$.
- (c) Hence show that $\mu = G/N$, the Gibbs energy per particle.

- (d) We know that the internal energy U (a thermodynamic potential) is a function of S, V and N , for a single component system of N particles in a volume V . Show that the thermodynamic potential $\Phi(T, P, \mu)$ obtained by making a Legendre transform to the three corresponding conjugate variables T, P and μ is trivially zero. What is the physical reason why a nontrivial thermodynamic potential $\Phi(T, P, \mu)$ cannot exist?

4. Some thermodynamic identities:

- (a) Show that the difference $C_p - C_v$ is given in general, by

$$C_p - C_v = \frac{T\alpha_T^2 V}{\kappa_T},$$

where

$$\alpha_T = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_{P,N} \quad \text{and} \quad \kappa_T = -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_{T,N}$$

denote, respectively, the coefficient of thermal expansion and the isothermal compressibility.

- (b) Hence show that

$$\frac{\partial^2 F}{\partial T^2} = \frac{\partial^2 G}{\partial T^2} - \frac{(\partial^2 G / \partial T \partial P)^2}{(\partial^2 G / \partial P^2)}.$$

5. Number fluctuations in a classical ideal gas: Consider a classical ideal gas of N particles in a container of volume V . The particles move independently of each other, and each particle has an equal probability of being located in any volume element of the container. The probability that there are n particles in a sub-volume v at any instant of time is then given by the binomial distribution

$$P_N(n) = \binom{N}{n} \left(\frac{v}{V} \right)^n \left(1 - \frac{v}{V} \right)^{N-n}.$$

Here $\binom{N}{n}$ denotes the binomial coefficient ${}^N C_n$.

- (a) In the thermodynamic limit, $N \rightarrow \infty$ and $V \rightarrow \infty$ keeping $N/V = \rho$ finite. Use Stirling's formula, show that, in this limit, $P_N(n)$ approaches the Poisson distribution

$$P(n) = \frac{e^{-\bar{n}} (\bar{n})^n}{n!} \quad (n = 0, 1, 2, \dots)$$

where $\bar{n} = \rho v$.

- (b) The Poisson distribution has a number of striking properties. Show that the variance of n is equal to the mean value \bar{n} itself. Hence the relative fluctuation in n , given by the ratio of the standard deviation to the mean value, is equal to $1/\sqrt{\bar{n}}$, a characteristic property of a Poisson-distributed random variable.
- (c) Show that every higher cumulants κ_r ($r \geq 3$) is also equal to the mean value \bar{n} . Again, this is a characteristic property of the Poisson distribution.
- (d) Estimate \bar{n} for air at NTP (i. e., $T = 300$ K, $P = 1$ atmosphere), for a volume $v = 1$ m³.

- (e) Now consider two neighbouring sub-volumes v_1 and v_2 . The probability that there are n_1 particles in v_1 is $e^{-\bar{n}_1}(\bar{n}_1)^{n_1}/n_1!$, where $\bar{n}_1 = \rho v_1$. Similarly, the probability that there are n_2 particles in v_2 is $e^{-\bar{n}_2}(\bar{n}_2)^{n_2}/n_2!$, where $\bar{n}_2 = \rho v_2$. Let $v = v_1 + v_2$. Show that the probability that there are n particles in v is again given by a Poisson distribution, with mean value $\bar{n} = \bar{n}_1 + \bar{n}_2$.

Extensivity of the internal energy: A homogeneous function f of degree r in the variables x_1, x_2, \dots, x_n satisfies the relation

$$f(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^r f(x_1, x_2, \dots, x_n)$$

for any λ . Homogeneity effectively reduces number of independent variables by one: for example, putting $\lambda = 1/x_1$ in the relationship above, we get

$$\frac{1}{x_1^r} f(x_1, x_2, \dots, x_n) = f\left(1, \frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\right).$$

In other words, the function f *must* have the form

$$f(x_1, x_2, \dots, x_n) = x_1^r \phi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \dots, \frac{x_n}{x_1}\right),$$

so that the ‘unknown’ part of the function depends only on the $(n - 1)$ ratios $x_2/x_1, \dots, x_n/x_1$. This sort of relation is called a **scaling relation**.

As we learn in elementary calculus, a homogeneous function of degree r satisfies Euler’s theorem, namely,

$$x_1 \frac{\partial f}{\partial x_1} + \dots + x_n \frac{\partial f}{\partial x_n} = r f.$$

Recall the application to the case of a single component, simple fluid. The laws of thermodynamics give

$$dU = T dS - P dV + \mu dN,$$

so that we can make the identifications

$$T = \left(\frac{\partial U}{\partial S}\right)_{V,N}, \quad P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}, \quad \mu = \left(\frac{\partial U}{\partial N}\right)_{S,V}.$$

Then the extensivity of the internal energy, i.e., the assumption that the internal energy U is a homogeneous function of degree 1 of the entropy S , the volume V and the number of particles N , leads immediately to

$$U = S \left(\frac{\partial U}{\partial S}\right)_{V,N} + V \left(\frac{\partial U}{\partial V}\right)_{S,N} + N \left(\frac{\partial U}{\partial N}\right)_{S,V} = TS - PV + \mu N.$$

This is the **Euler relation**. Since the Gibbs free energy is given by $G = U - TS + PV$, it follows that

$$G = \mu N.$$

Take the differentials of both sides of the Euler relation, and use the expression for dU that follows from the laws of thermodynamics. We get

$$SdT - VdP + Nd\mu = 0.$$

Therefore

$$d\mu = v dP - s dT,$$

where $v = V/N$ and $s = S/N$ denote, respectively, the specific volume and specific entropy. This is the **Gibbs-Duhem relation**. It implies that the chemical potential is a function of the intensive variables P and T , i.e., $\mu = \mu(P, T)$.

6. Generalized homogenous functions: A generalized homogeneous function $f(x_1, x_2, \dots, x_n)$ of n variables satisfies the relation

$$f(\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \dots, \lambda^{\alpha_n} x_n) = \lambda f(x_1, x_2, \dots, x_n)$$

for all λ . Here the exponents $\alpha_1, \dots, \alpha_n$ are in general different from each other. (It is obvious that when $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1/r$, you get an ordinary homogeneous function of degree r .) Once again, we can reduce the number of dependent variables by one, using the generalized homogeneity property. If we set $\lambda^{\alpha_1} = x_1^{-1}$ or $\lambda = x_1^{-1/\alpha_1}$, it follows that the generalized homogeneous function f must be of the form

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= x_1^{-1/\alpha_1} f\left(1, \frac{x_2}{x_1^{\alpha_2/\alpha_1}}, \dots, \frac{x_n}{x_1^{\alpha_n/\alpha_1}}\right) \\ &\equiv x_1^{-1/\alpha_1} \phi\left(\frac{x_2}{x_1^{\alpha_2/\alpha_1}}, \dots, \frac{x_n}{x_1^{\alpha_n/\alpha_1}}\right). \end{aligned}$$

Show that Euler's theorem is now generalized to read

$$\alpha_1 x_1 \frac{\partial f}{\partial x_1} + \alpha_2 x_2 \frac{\partial f}{\partial x_2} + \dots + \alpha_n x_n \frac{\partial f}{\partial x_n} = f(x_1, x_2, \dots, x_n).$$

In equilibrium statistical mechanics, generalized homogenous functions play a very important role in the understanding of scaling behavior in the context of phase transitions and critical phenomena.

7. Application to ideal gases: The equation of state of a given quantity of an ideal gas (classical or quantum!) can be written in the form $PV = aU$, where a is a constant which is equal to $\frac{2}{3}$ for a classical ideal gas, $\frac{1}{3}$ for a photon gas (or blackbody radiation), and so on.

(a) Given this equation of state, show that

$$U(T, V) = T \left(\frac{\partial U}{\partial T} \right)_V - \frac{V}{a} \left(\frac{\partial U}{\partial V} \right)_T.$$

(b) This relation implies that $U(T, V)$ is a generalized homogenous function. Show that U must be of the form

$$U(T, V) = V^{-a} \phi(TV^a) \quad \text{or, equivalently,} \quad U(T, V) = T \psi(VT^{1/a}).$$

If, further, we are also given that $U(V, T) = Vu(T)$, then the function U is fully determined. This is what happens in the case of blackbody radiation, for which $a = \frac{1}{3}$, so that it follows at once that $U = (\text{const.}) VT^4$. Thus, the Stefan-Boltzmann Law is derivable (except for the value of the multiplicative constant, of course) from purely thermodynamic considerations.

Quiz

1. Are the statements in quotation marks true or false?
 - (a) “The chemical potential of a thermodynamic system is an extensive quantity.”
 - (b) “The slope of the liquid-gas coexistence curve in the (T, P) plane (P plotted as a function of T) is always positive.”
 - (c) “The square of the mean of a random variable can never exceed the mean of its square.”
 - (d) Let $p_1(x)$ and $p_2(y)$ be the respective normalized probability density functions of two *independent* random variables x and y , where $-\infty < x, y < \infty$.

“The normalized probability density function of the random variable $z = xy$ is given by $\int_{-\infty}^{\infty} dx p_1(x) p_2(z - x)$.”
 - (e) “For the Maxwellian distribution of velocities of the molecules of a classical ideal gas, the mean speed of a molecule is equal to the r.m.s. speed of a molecule.”
 - (f) “The relation $C_p - C_v = T(\partial V/\partial T)_P (\partial P/\partial T)_V$ is valid for all gases, and not just for an ideal gas.”
 - (g) “In a system in thermal equilibrium at a temperature T , the mean value of any observable is equal to its most probable value.”
 - (h) Let F denote the Helmholtz free energy of a substance.

“ $(\partial^2 F/\partial T^2)_{V,N}$ must be negative definite.”
 - (i) When a biased coin is tossed, ‘heads’ appears with a probability p and ‘tails’ with a probability $q = 1 - p$. The coin is tossed repeatedly, till a ‘heads’ is obtained.

“The probability P_n that a ‘heads’ is obtained for the *first* time in the n^{th} toss is given by $P_n = (n - 1)q + p$.”
2. A system has N possible energy levels, given by $\varepsilon, 2\varepsilon, \dots, N\varepsilon$, where ε is a positive constant. The energy level $n\varepsilon$ is n -fold degenerate, i.e., for each allowed value of n , there are n different states corresponding to the same energy level $n\varepsilon$. The system is in thermal equilibrium in contact with a heat bath at temperature T . Fill in the blanks in the following:
 - (a) The total number of distinct states of the system is \dots
 - (b) The probability that the system is in its ground state is \dots

- (c) The probability that the system has its highest possible energy is \dots
- (d) If the temperature exceeds a certain value \tilde{T} , then the probability that the system has an energy 2ε actually *exceeds* the probability that it has an energy ε . This temperature \tilde{T} is equal to \dots
- (e) In the limit $T \rightarrow 0$, the probability that the system has energy 2ε is \dots
- (f) In the limit $T \rightarrow \infty$, the probability that the system is in a *particular one* of the states corresponding to energy $N\varepsilon$ is \dots

3. One mole of a Van der Waals gas obeying the equation of state

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT$$

undergoes an isothermal expansion from a volume V_1 to a volume V_2 , at a temperature T . Show that the change $\Delta U = U_2 - U_1$ in the internal energy of the gas is given by

$$\Delta U = a\left(\frac{1}{V_1} - \frac{1}{V_2}\right).$$

Special Relativity

Special Relativity is based on a general *principle* and a physical *postulate*.

The Principle of Relativity asserts that the laws of physical phenomena are unchanged in form for all mutually inertial observers—that is, in all frames of reference related to each other by **Lorentz transformations**. The term ‘Lorentz transformation’ is often used to mean a transformation to a frame of reference moving uniformly with respect to the original frame. This is a velocity transformation or **boost**. Lorentz transformations actually comprise boosts in all possible directions, *as well as rotations* of the spatial axes in all directions. More precisely: rotations and boosts constitute the set of homogeneous, proper Lorentz transformations, the so-called **special Lorentz transformations**. Such transformations comprise the **Lorentz group**, denoted by $SO(3, 1)$. *Inhomogeneous* Lorentz transformations include shifts (or **translations**) of the origin of the spacetime coordinates by constant amounts, over and above the set of rotations and boosts. Inhomogeneous Lorentz transformations also form a group, called the inhomogeneous Lorentz group or the **Poincaré group**. The principle of relativity stated above applies to this extended set of transformations. But we will not consider these here.³

The Postulate of Relativity says that there exists a fundamental limiting speed in nature, that is the same in all mutually inertial frames of reference. Light propagates in a vacuum with this limiting speed, denoted by c . So does any particle whose rest mass happens to be exactly zero.

The question arises as to what happens in different sets of mutually inertial frames of reference, which may be *accelerating* with respect to each other. Without going into details, I merely mention that, strictly speaking, the principle of (special) relativity stated above is only valid in ‘flat’ spacetime, i.e., spacetime in the absence of any curvature or gravitational fields. Gravitation enters the picture because of the **Principle of Equivalence** which says, broadly speaking, that any acceleration is equivalent to the effect of a gravitational field. The latter, in turn, is a manifestation of the curvature of spacetime. There is a specific criterion to determine whether any given region of spacetime is flat or not. It takes fairly intense gravitational fields to produce significant curvature in a region of spacetime, so that the latter may be taken to be flat to a good approximation even in the presence of mild gravitational fields. This is why special relativity, rather than general relativity, suffices to handle all situations except those involving the effects very high gravitational fields.

Boost formulas: Consider a frame of reference S , and another frame of reference S' moving at a uniform velocity \mathbf{v} with respect to it. Let (\mathbf{r}, t) and (\mathbf{r}', t') be the respective spacetime coordinates in S and S' . We assume, for simplicity, that the origins and the Cartesian axes of the two frames coincide at $t = 0$. You are no doubt familiar with the Lorentz transformation formulas in the special case when $\mathbf{v} = v \mathbf{e}_x$. But it is quite simple to write down the transformation rules for a boost velocity \mathbf{v} in any *arbitrary* direction.

Resolve the coordinate vector \mathbf{r} (in S) into components along \mathbf{v} and transverse to it:

$$\mathbf{r} = \mathbf{r}_{\parallel} + \mathbf{r}_{\perp} = \frac{\mathbf{r} \cdot \mathbf{v}}{v^2} \mathbf{v} + \left(\mathbf{r} - \frac{\mathbf{r} \cdot \mathbf{v}}{v^2} \mathbf{v} \right).$$

The transverse component \mathbf{r}_{\perp} is not affected by the boost, as you might expect.

³There are also *improper* transformations such as **parity** (under which $\mathbf{r} \rightarrow -\mathbf{r}$) and **time reversal** (under which $t \rightarrow -t$).

The longitudinal component undergoes the customary transformation. Let

$$\gamma_v = 1/\sqrt{1 - (v/c)^2},$$

as usual. Then the Lorentz transformation formulas corresponding to the boost are

$$ct' = \gamma_v \left(ct - \frac{\mathbf{r} \cdot \mathbf{v}}{c} \right),$$

$$\mathbf{r}' = \gamma_v \left(\frac{\mathbf{r} \cdot \mathbf{v}}{v^2} \mathbf{v} - \mathbf{v}t \right) + \left(\mathbf{r} - \frac{\mathbf{r} \cdot \mathbf{v}}{v^2} \mathbf{v} \right).$$

In three-dimensional Euclidean space, the square of the distance to any point, $r^2 = x_i x_i$, is preserved under rotations of the coordinate axes about the origin. In the same way, what is preserved under Lorentz transformations is the square of the **spacetime interval** from the origin to any point in spacetime, $c^2 t^2 - r^2$. The surface $r^2 = \text{constant}$ is a sphere in space. The hypersurface $c^2 t^2 - r^2 = \text{constant}$ is a *hyperboloid* in spacetime.

1. Given the boost formulas above, verify that $c^2 t'^2 - r'^2 = c^2 t^2 - r^2$.

2. Collinear boosts and the velocity addition rule: The usual special case is the one in which the frame S' moves with a velocity $\mathbf{v} = v \mathbf{e}_x$ along the x -axis of S . The boost formulas then reduce to the familiar ones for the spacetime coordinates in S' , namely,

$$ct' = \gamma_v \left(ct - \frac{xv}{c} \right), \quad x' = \gamma_v (x - vt), \quad y' = y, \quad z' = z.$$

Now suppose a third frame of reference S'' is moving at a uniform velocity $u \mathbf{e}_x$ with respect to S' . The spacetime coordinates in this frame are therefore given by

$$ct'' = \gamma_u \left(ct' - \frac{x'u}{c} \right), \quad x'' = \gamma_u (x' - ut'), \quad y'' = y', \quad z'' = z',$$

where $\gamma_u = 1/\sqrt{1 - (u/c)^2}$. Put in the expressions for the primed variables in terms of the unprimed ones, to show that the spacetime coordinates in S'' are related to those of the original frame S by a *single* boost $w \mathbf{e}_x$, as follows:

$$ct'' = \gamma_w \left(ct - \frac{xw}{c} \right), \quad x'' = \gamma_w (x - wt), \quad y'' = y, \quad z'' = z,$$

where

$$\gamma_w = 1/\sqrt{1 - (w/c)^2}, \quad \text{with} \quad w = \frac{v + u}{1 + (vu/c^2)}.$$

Hence the resultant of two successive boosts in the same direction is again a boost in the same direction. However, the resultant boost is *nonlinear* in the individual boosts, and explicitly involves the fundamental velocity c . The expression for w differs strikingly from that in the nonrelativistic (or Newtonian) case, in which w is simply equal to $u + v$.

3. The rapidity: Although velocities (along the same direction) do not simply add up according to the relativistic law of addition of velocities, there does exist a certain function of the velocity that obeys an additive rule. Define the **rapidity** ξ_v corresponding to a velocity v as

$$\xi_v = \tanh^{-1}(v/c).$$

Show that the relativistic law of addition of velocities derived above is just

$$\xi_w = \xi_v + \xi_u .$$

Thus, when the velocities are collinear, *rapidities*, rather than velocities, add up. For $v \ll c$, the rapidity $\xi_v \simeq v/c$ to leading order. As $v \rightarrow c$, $\xi_v \rightarrow \infty$. Note also that the quantities γ_v and ξ_v are related according to

$$\gamma_v = 1/\sqrt{1 - (v/c)^2} = \cosh \xi_v .$$

Boosts in different directions: Surprisingly enough, two boosts in *different* directions do not combine into a single resultant boost in some direction! Instead, two successive boosts in different directions are equivalent to a single boost *together* with a rotation. This rotation is called a **Wigner rotation**, and is responsible for the phenomenon of **Thomas precession**.

Why don't two boost velocity vectors \mathbf{v} and \mathbf{u} just add up to produce a resultant boost velocity $(\mathbf{v} + \mathbf{u})$, modulated by some 'correction factor' involving c ? A physical way of understanding the reason why is as follows. Consider a boost from a frame S to a frame S' by a boost \mathbf{v} . The expression for the new spatial coordinates \mathbf{r}' indicates the way in which the components of *any* three-vector transform under a boost. It shows that the component of the vector *along* the direction of the boost, and the part *normal* to the boost, transform in *different* ways. Now, when a boost \mathbf{v} is followed by a boost \mathbf{u} , the latter acts on not only the original coordinate \mathbf{r} , but also on the original boost velocity vector \mathbf{v} , because \mathbf{r}' involves both \mathbf{r} and \mathbf{v} . The part of \mathbf{v} that is directed along \mathbf{u} and the part that is normal to \mathbf{u} get transformed in different ways. This produces a kind of 'twist', whose effect shows up as a *rotation* of the axes. As a consequence, while the set of all possible rotations constitutes a *sub-group* of the group of Lorentz transformations, the set of all possible boosts does not.

Lorentz scalars and four-vectors: The spacetime coordinates $(ct, \mathbf{r}) \equiv (x_0, \mathbf{r})$ form a **four-vector**, which I will denote by \underline{x} . (Obviously, all the four components of a four-vector must have the same physical dimensions; hence the replacement of t by ct .) As in the case of three-vectors in Euclidean space, any other set of four quantities (a_0, \mathbf{a}) constitutes a four-vector \underline{a} if it transforms, under *Lorentz* transformations, exactly as the spacetime coordinate \underline{x} does. The component a_0 is the *time-like* component of the four-vector, while the Cartesian components of \mathbf{a} are its *space-like* components. Similarly, if E and \mathbf{p} are the energy and linear momentum of a particle, respectively, then $\underline{p} = (E/c, \mathbf{p})$ is the four-momentum of the particle. Other four-vectors of relevance to us in the context of electromagnetism are the *four-vector current density*

$$\underline{j} = (j_0, \mathbf{j}) = (c\rho, \mathbf{j})$$

that combines the charge density ρ and the current density \mathbf{j} ; and the *four-vector potential*

$$\underline{A} = (A_0, \mathbf{A}) = (\phi/c, \mathbf{A})$$

that combines the scalar potential ϕ and the vector potential \mathbf{A} .

A Lorentz scalar is a quantity that remains invariant under Lorentz transformations. A crucial feature of special relativity is incorporated in the way the scalar product of two four-vectors $\underline{a} = (a_0, \mathbf{a})$ and $\underline{b} = (b_0, \mathbf{b})$ is *defined* so as to produce a Lorentz scalar. Recall that the square of the interval from the origin to any point

in spacetime, $c^2t^2 - r^2$, is preserved under Lorentz transformations. This means that the scalar product of $\underline{x} = (ct, \mathbf{r})$ with itself must be defined as

$$\underline{x} \cdot \underline{x} = c^2t^2 - r^2,$$

where $r = |\mathbf{r}|$. Similarly, the ‘square’ of the four-momentum \underline{p} of a particle must be a Lorentz scalar. For a free particle, it is given by

$$\underline{p} \cdot \underline{p} = (E^2/c^2) - p^2 = m^2c^2,$$

where $p = |\mathbf{p}|$. The constant m is, as you know, the **rest-mass** of the particle. For physical particles, $m \geq 0$. More generally, the scalar product of two four-vectors \underline{a} and \underline{b} is *defined* as

$$\underline{a} \cdot \underline{b} = a_0 b_0 - \mathbf{a} \cdot \mathbf{b}.$$

The relative minus sign between the squares of the time-like and space-like components is all-important. (This sign will emerge automatically if we define an appropriate metric tensor, and introduce contravariant and covariant indices. I shall not do so here.)

The *four-dimensional gradient operator* is defined as

$$\underline{\partial} = (\partial_0, -\nabla) = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right).$$

The minus sign in the space-like components in the definition above ensures that the four-divergence of \underline{x} is correctly given by

$$\underline{\partial} \cdot \underline{x} = \partial_0 x_0 - (-\nabla \cdot \mathbf{r}) = \partial_0 x_0 + \nabla \cdot \mathbf{r} = 1 + 3 = 4,$$

which is the dimensionality of spacetime. The equation of continuity is, in this notation,

$$\partial \rho / \partial t + \nabla \cdot \mathbf{j} = \underline{\partial} \cdot \mathbf{j} = 0.$$

Hence the equation of continuity is the statement that the four-divergence of the four-current density is zero. But $\underline{\partial} \cdot \mathbf{j}$ is a Lorentz scalar, so that it remains equal to zero in *all* mutually inertial frames of reference. This is as it should be, because the physical content of the equation of continuity, namely, the conservation of electric charge, must remain valid for all mutually inertial observers.

As you know, from the gradient operator ∇ we can construct the Laplacian $\nabla \cdot \nabla = \nabla^2$, which is a scalar operator in the sense that it is invariant under rotations of the spatial coordinate axes. The relativistic analog of the Laplacian is the **d’Alembertian** (or box operator, or wave operator), defined as

$$\square = \underline{\partial} \cdot \underline{\partial} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2.$$

By construction, the box operator is a Lorentz scalar. This property is crucial to the relativistic invariance of Maxwell’s equations of electromagnetism, as we will see shortly.

Classical Electromagnetism

The sources for the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$ are the charge density $\rho(\mathbf{r}, t)$ and the current density $\mathbf{j}(\mathbf{r}, t)$. The homogeneous pair of Maxwell's equations for the electromagnetic (EM) fields in free space are

$$\nabla \cdot \mathbf{B} = 0, \quad (\nabla \times \mathbf{E}) + \partial \mathbf{B} / \partial t = 0,$$

while the inhomogeneous pair of equations (in which the sources of the electromagnetic fields are explicitly present) are

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0, \quad (\nabla \times \mathbf{B}) - \mu_0 \epsilon_0 \partial \mathbf{E} / \partial t = \mu_0 \mathbf{j}.$$

Here ϵ_0 and μ_0 denote, respectively, the permittivity and permeability of free space. The combination $(\mu_0 \epsilon_0)^{-1/2} = c$, the speed of EM waves in free space. Note that the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0,$$

is built into the Maxwell equations. The conservation of electric charge follows from the equation of continuity.

The homogeneous Maxwell equations apply to all EM fields, regardless of the sources ρ and \mathbf{j} . They imply at once that \mathbf{B} and \mathbf{E} can be written in the form

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t) \quad \text{and} \quad \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) - \nabla \phi(\mathbf{r}, t),$$

respectively, in terms of the the **vector potential** \mathbf{A} and the **scalar potential** ϕ . Inserting these into the inhomogeneous Maxwell equation, we get

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) + \nabla^2 \phi = -\frac{\rho}{\epsilon_0},$$

and

$$\nabla \left(\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} \right) + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j}.$$

Note that the equations for \mathbf{A} and ϕ are *coupled* to each other. Solving then is made easier by exploiting a certain arbitrariness in the potentials called **gauge freedom**.

Gauge freedom and gauge invariance: It is easy to see from the expressions for \mathbf{E} and \mathbf{B} in terms of the potentials that the physical EM fields do not get affected if \mathbf{A} and ϕ are replaced by \mathbf{A}' and ϕ' , respectively, where

$$\mathbf{A}' = \mathbf{A} + \nabla \chi \quad \text{and} \quad \phi' = \phi - \frac{\partial \chi}{\partial t}.$$

These comprise a **gauge transformation** of the EM potentials. The fact that \mathbf{E} and \mathbf{B} remain unaltered is called the **gauge invariance** of the EM fields. The arbitrariness in \mathbf{A} and ϕ implied by the foregoing is called gauge freedom. The choice of any specific function χ 'fixes the gauge'. Maxwell's field equations are obviously gauge invariant, because they involve the fields \mathbf{E} and \mathbf{B} directly, rather than the potentials. All physical or measurable quantities pertaining to the EM fields, such as the energy of the field, its momentum, angular momentum, and so on, *must* be expressible in terms of these fields (rather than the potentials alone), and must therefore be gauge invariant as well.

These statements imply that the potentials themselves are auxiliary mathematical quantities rather than physical observables, at least in *classical* electrodynamics.⁴ Gauge freedom enables us to work in specific gauges that simplify the problem of solving for the EM fields.

1. Coulomb and Lorenz gauges:

- (a) In the so-called Coulomb gauge, the vector potential is solenoidal: the gauge is specified by setting $\text{div } \mathbf{A} \equiv 0$. Show that it is *always* possible to impose this condition, along the following lines. Suppose $\nabla \cdot \mathbf{A}$ is not zero, but is equal to some function $f(\mathbf{r}, t)$. A gauge transformation in which the gauge function $\chi(\mathbf{r}, t)$ is chosen to be the solution of Poisson's equation, with $-f(\mathbf{r}, t)$ as the source term, will then ensure that the transformed potential \mathbf{A}' is solenoidal.
- (b) The Lorenz gauge is specified by imposing the condition

$$\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

on the scalar and vector potentials. Show that it is *always* possible to transform to the Lorenz gauge, along the following lines. Suppose the quantity $(1/c^2) \partial \phi / \partial t + \nabla \cdot \mathbf{A}$ is not zero, but is equal to some function $g(\mathbf{r}, t)$. A gauge transformation in which $\chi(\mathbf{r}, t)$ is chosen to be the solution of the wave equation, with $-g(\mathbf{r}, t)$ as the source term, will then ensure that the new potentials \mathbf{A}' and ϕ' satisfy the Lorenz gauge condition.

In the Coulomb gauge, finding ϕ reduces to solving Poisson's equation with $-\rho/\epsilon_0$ as the source term. Subsequently, the solution of an inhomogeneous wave equation determines \mathbf{A} , and thence the EM fields. In the Lorenz gauge, finding \mathbf{A} reduces to solving the wave equation with $\mu_0 \mathbf{j}$ as the source term. Subsequently, the solution of an inhomogeneous wave equation determines ϕ , and thence the EM fields. Both the equations involved are among the standard equations of mathematical physics, and a variety of techniques have been developed to solve them under different initial and boundary conditions.

2. Special cases: electrostatics and magnetostatics:

- (a) Electrostatics corresponds to the special case in which there is a static charge density $\rho(\mathbf{r})$, and no current density. There is no magnetic field present, and the electric field is irrotational:

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \rho(\mathbf{r})/\epsilon_0 \quad \text{and} \quad \nabla \times \mathbf{E}(\mathbf{r}) = 0.$$

Hence \mathbf{E} can be written as $-\nabla\phi(\mathbf{r})$, the potential satisfying $\nabla^2 \phi = -\rho/\epsilon_0$. Show that the solution satisfying the 'natural' boundary condition $\phi \rightarrow 0$ as $r \rightarrow \infty$ leads to the fundamental result

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

for the electrostatic field.

More complicated boundary conditions lead to more intricate solutions. The complexity in electrostatics arises essentially because of the boundary conditions.

⁴The situation is more complicated in quantum physics, and I shall not digress into this matter here.

- (b) Magnetostatics, too, involves no time dependence. The static magnetic field $\mathbf{B}(\mathbf{r})$ is induced by a steady current density $\mathbf{j}(\mathbf{r})$. The field equations are

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \quad \text{and} \quad \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{j}(\mathbf{r}).$$

Set $\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$, use the standard identity for $\nabla \times (\nabla \times \mathbf{A})$ and work in the Coulomb gauge to get

$$\nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{j}(\mathbf{r}).$$

Hence show that the solution corresponding to natural boundary conditions is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}.$$

You will recognize this result as the general form of the **Biot-Savart Law** for the magnetostatic field due to a steady current density.

Relativistic invariance of electromagnetism: Recall that the four-dimensional gradient operator is $\underline{\partial} = ((1/c)\partial/\partial t, -\nabla)$, while the four-vector potential is $\underline{\mathbf{A}} = (\phi/c, \mathbf{A})$. The Lorenz gauge condition is

$$\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = \underline{\partial} \cdot \underline{\mathbf{A}} = 0.$$

That is, the Lorenz gauge condition is just the requirement that the four-divergence of the four-vector potential be zero. The analogy with the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ is now obvious. The Lorenz gauge is the relativistic generalization of the Coulomb gauge. The great advantage of the Lorenz gauge is that *it remains unchanged under Lorentz transformations*, because $\underline{\partial} \cdot \underline{\mathbf{A}}$ is a Lorentz scalar.⁵

Recall, further, that in the Lorenz gauge, the vector and scalar potentials of EM satisfy the wave equation with \mathbf{j} and ρ as the respective sources. But $(c\rho, \mathbf{j})$ is just the four-current density. Therefore the two wave equations can be combined into the single compact equation

$$\square \underline{\mathbf{A}} = \mathbf{j}, \quad \text{with the gauge condition } \underline{\partial} \cdot \underline{\mathbf{A}} = 0.$$

Thus, Maxwell's equations in free space reduce to the wave equation for the four-vector potential in the Lorenz gauge. Both the gauge condition and the wave equation are *manifestly covariant*, i.e., they are form-invariant under Lorentz transformations.

Lorentz transformation properties of \mathbf{E} and \mathbf{B} : It is clear that electric and magnetic fields are frame-dependent, i.e., they transform as one goes from one intertial frame to another. Their transformation properties under rotations of the spatial coordinate axes are obvious, because both \mathbf{E} and \mathbf{B} are three-vectors under such rotations. But under boosts, the components of the electric and magnetic fields get mixed up with each other.⁶ What are the transformation rules for EM fields?

⁵Many other Lorentz-invariant gauges are possible, of course: for instance, the gauge in which $\underline{\mathbf{x}} \cdot \underline{\mathbf{A}} = 0$. But the most useful one, by far, is the Lorenz gauge.

⁶For instance, a static charge in one frame looks like a current in a moving frame; this means an electrostatic field in the first frame could appear as an electric field as well as a magnetic field in the moving frame.

This question is most directly answered by noting, first, that the general expressions $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t - \nabla\phi$, though rather different-looking, are really very close analogs when expressed in terms of four-vectors: they represent different components of the four-dimensional ‘curl’ of the four-vector potential. As such, they can be combined in a single **electromagnetic field tensor**, which is an antisymmetric tensor of rank 2 in $(3 + 1)$ dimensional spacetime. The Lorentz transformation properties of such a tensor are manifest, and the manner in which \mathbf{E} and \mathbf{B} themselves transform can be read off from these. Here I merely quote the result.

Let \mathbf{E}, \mathbf{B} be the electric and magnetic fields as measured in a frame of reference S , and let \mathbf{E}', \mathbf{B}' be the same fields measured in a frame S' that is boosted with a velocity \mathbf{v} with respect to S . The fields in S' are related to the ones in S as follows. Let the subscripts \parallel and \perp , respectively, denote the components of the fields respectively parallel and perpendicular to the direction of the boost \mathbf{v} . Let $\gamma_v = 1/\sqrt{1 - (v/c)^2}$, as usual. Then,

$$E'_{\parallel} = E_{\parallel}, \quad \mathbf{E}'_{\perp} = \gamma_v [\mathbf{E}_{\perp} + (\mathbf{v} \times \mathbf{B}_{\perp})]$$

and

$$B'_{\parallel} = B_{\parallel}, \quad \mathbf{B}'_{\perp} = \gamma_v \left[\mathbf{B}_{\perp} - \frac{(\mathbf{v} \times \mathbf{E}_{\perp})}{c^2} \right].$$

Several noteworthy points follow from these relations.

- (i) The components of the EM fields *along* the direction of the boost are unaffected by the boost.
- (ii) It is the transverse components \mathbf{E}_{\perp} and \mathbf{B}_{\perp} that get mixed up with each other as a consequence of the boost.
- (iii) For sufficiently small boosts, such that v^2/c^2 (note the *square*) is negligible compared to unity, we have

$$\mathbf{E}' \simeq \mathbf{E} + (\mathbf{v} \times \mathbf{B}) \quad \text{and} \quad \mathbf{B}' \simeq \mathbf{B} - \frac{(\mathbf{v} \times \mathbf{E})}{c^2}.$$

These relations suggest how the Lorentz force on a moving charge arises—or, from another point of view, how the magnetic field itself is a natural consequence of charges in motion.

3. Lorentz invariants of the EM fields: Certain combinations of the electric and magnetic fields remain invariant, i.e., are scalars, under Lorentz transformations. From the expressions for \mathbf{E}' and \mathbf{B}' given above, show that

$$(i) \quad \mathbf{E} \cdot \mathbf{B} = \mathbf{E}' \cdot \mathbf{B}' \quad (ii) \quad \mathbf{E}^2 - c^2 \mathbf{B}^2 = \mathbf{E}'^2 - c^2 \mathbf{B}'^2.$$

Since $\mathbf{E} \cdot \mathbf{B}$ and $\mathbf{E}^2 - c^2 \mathbf{B}^2$ are also scalars under rotations of the spatial coordinate axes, it follows that these combinations are invariant under (proper) Lorentz transformations.⁷

Thus, if $\mathbf{E} \cdot \mathbf{B} = 0$ in one frame of reference, it remains so for all frames obtained from it by Lorentz transformations. It follows that transverse electromagnetic waves remain transverse electromagnetic waves for all mutually inertial observers. That is, light remains light in all inertial frames. This is only to be expected, given that our starting point was the postulate of relativity!

⁷The quantity $\mathbf{E}^2 - c^2 \mathbf{B}^2$ turns out to be essentially the Lagrangian density of the EM field.

More generally, since $\mathbf{E} \cdot \mathbf{B} = \mathbf{E}' \cdot \mathbf{B}'$, we have $EB \cos \theta = E'B' \cos \theta'$, where θ and θ' are the angles between the electric and magnetic fields in the frame S and the boosted frame S', respectively. It follows immediately that $\cos \theta$ and $\cos \theta'$ must have the same sign. That is, if \mathbf{E} and \mathbf{B} make an acute (respectively, right and obtuse) angle with each other in a given frame, they continue to make such an angle in any Lorentz-transformed frame.

4. Let \mathbf{E} and \mathbf{B} be constant, uniform fields making an arbitrary acute angle with each other, in a frame of reference S.

- (a) Show that it is always possible to find a boosted frame S' such that \mathbf{E}' is parallel to \mathbf{B}' .
- (b) Find an expression for the boost velocity required to go from S to S'.

Energy density and the Poynting vector: The **energy density** of the EM field is given by

$$W = \frac{1}{2} \epsilon_0 (\mathbf{E}^2 + c^2 \mathbf{B}^2).$$

W is *not* a Lorentz scalar, in contrast to the Lagrangian density of the EM field. The energy flux density of the EM field (i.e., the energy crossing unit area per unit time) is given by the **Poynting vector**

$$\mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}).$$

5. In view of their physical meanings, we may expect W and \mathbf{S} to be related by a continuity equation. Using Maxwell's equations, show that

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -(\mathbf{E} \cdot \mathbf{j}).$$

The right-hand side of the equation above is just the rate of **Ohmic dissipation**, as one would expect. In the absence of sources ($\rho = 0$, $\mathbf{j} = 0$), i.e., for a pure radiation field, the quantity $\partial W / \partial t + \nabla \cdot \mathbf{S} = 0$ in all mutually inertial frames of reference.

Test

1. Are the statements in quotation marks true or false?
 - (a) “The Lagrangian formalism is not applicable to a system of particles subject to velocity-dependent forces.”
 - (b) “The slope of the sublimation curve of a substance in the P versus T plot can never be negative. ”
 - (c) Let $\langle x^n \rangle$ denote the n^{th} moment of a random variable x .
“ $\langle x^4 \rangle$ is always greater than or equal to $3\langle x^2 \rangle^2$.”
 - (d) “The time evolution of a classical Hamiltonian system can be regarded as a sequence of infinitesimal canonical transformations.”
 - (e) “The special linear group $SL(2n, \mathbb{R})$ is a subgroup of the symplectic group $Sp(2n, \mathbb{R})$.”
 - (f) “The specific heat of a classical ideal gas can be made to take on any value between C_v and C_p by subjecting it to a process of the form $PV^n = \text{constant}$, with a suitable value of the index n .”
 - (g) “Boosts (or velocity transformations from one inertial frame to another) do not constitute a group, but boosts together with rotations of the coordinate axes do constitute a group.”
 - (h) Let G denote the Gibbs free energy of a substance.
“ $(\partial^2 G / \partial P^2)_{T,N}$ must be negative definite.”

2. Fill in the blanks in the following.

- (a) The Lagrangian of a particle with a single degree of freedom, q , is given by $L(q, \dot{q}, t)$. The Euler-Lagrange equation of motion of the particle is then *cdots*
- (b) The Hamiltonian of a particle of mass m moving in an attractive inverse square field of force is given by $H = \mathbf{p}^2 / (2m) - k/r$, where k is a positive constant.
 - (i) The time-independent constants of the motion are *cdots*
 - (ii) The value of the total energy when the trajectory of the particle in space is a parabola is $E = \dots$
- (c) The critical point of the dynamical system

$$\dot{x} = -3x, \quad \dot{y} = -y + 2z, \quad \dot{z} = -2y - z$$

located at $(x, y, z) = \dots$ is a stable/unstable/asymptotically stable/higher-order critical point (select one).

- (d) A particle of mass m moving in the (q_1, q_2) plane has the Lagrangian

$$L = \frac{\dot{q}_1^2 + \dot{q}_2^2}{2m} - V\left(\sqrt{q_1^2 + q_2^2}\right).$$

L is invariant under the continuous group of rotations about the origin in the (q_1, q_2) plane. It then follows from Noether's Theorem that the quantity \dots is a constant of the motion.

- (e) Under a parity transformation $\mathbf{r} \rightarrow -\mathbf{r}$ in three-dimensional space, the transformation properties of the charge density ρ and the scalar product $\mathbf{E} \cdot \mathbf{B}$ are given by $\rho \rightarrow \dots$ and $\mathbf{E} \cdot \mathbf{B} \rightarrow \dots$.

- (f) Let u denote the rapidity of a relativistic particle moving along the x -axis in a frame of reference S . Let S' be a frame of reference moving with a velocity v with respect to S , along the x -axis of S . The rapidity of the particle in the frame of reference S' is then $u' = \dots$

3. Let $H(q_1, \dots, q_n, p_1, \dots, p_n)$ be the Hamiltonian of a system with n degrees of freedom. When the system is in thermal equilibrium in contact with a heat bath at temperature T , the average value of any physical quantity $A(q, p)$ is given by

$$\langle A \rangle = \frac{1}{Z} \int d^n p \int d^n q A e^{-\beta H}, \text{ where } Z = \int d^n p \int d^n q e^{-\beta H} \text{ and } \beta = \frac{1}{k_B T}.$$

- (a) Obtain a formula for the variance of H in terms of derivatives of Z with respect to β .
- (b) Assuming that $H \rightarrow \infty$ when any $p_i \rightarrow \pm\infty$, show that $\langle p_j \dot{q}_j \rangle = k_B T$ for each $j (= 1, 2, \dots, n)$.

4. Consider the logistic map of the unit interval, $x_{n+1} = f(x_n) = \mu x_n(1 - x_n)$, where $x_0 \in [0, 1]$, $n = 0, 1, \dots$, and μ is a positive constant. The Lyapunov exponent corresponding to an initial value x_0 is defined as

$$\lambda(x_0, \mu) = \lim_{n \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \frac{1}{n} \ln \left| \frac{f^{(n)}(x_0 + \epsilon) - f^{(n)}(x_0)}{\epsilon} \right|,$$

where $f^{(n)}(x)$ denotes the n^{th} iterate of the map. (The order in which the limits are taken is important.)

- (a) Find the Lyapunov exponent for a general value of μ when (i) $0 < \mu < 1$ and (ii) $1 < \mu < 3$.
- (b) Why are these values of λ independent of x_0 ?
- (c) Express x_n as an explicit function of x_0 in the case $\mu = 2$.
- (d) What happens to the map at $\mu = 3$?

5. A particle of rest mass m and charge e moves along the x -axis in a constant, uniform electric field $\mathbf{E} = E \mathbf{e}_x$. At $t = 0$, the particle starts from rest from the point $x = 0$. The equation of motion of the particle is

$$\frac{d}{dt} \left(\frac{mv}{\sqrt{1 - (v^2/c^2)}} \right) = eE,$$

where $v = \dot{x} = dx/dt$. Set $(mc)/(eE) = \tau$ (note that τ represents a natural time scale in the problem).

- Solve the equation of motion to find its velocity v as a function of t .
 - Schematically sketch v as a function of t .
 - Hence find the position x of the particle as a function of t .
6. The normalized probability density of any Cartesian component v of the velocity of a molecule of mass m in a classical ideal gas in thermal equilibrium at temperature T is given by

$$p(v) = \left(\frac{m}{2\pi k_B T} \right)^{1/2} \exp \left(-\frac{mv^2}{2k_B T} \right), \quad (-\infty < v < \infty).$$

- Write down the normalized probability density $\rho(\varepsilon)$ of the energy of a molecule.
- Find the normalized probability density $q(u)$ of the *relative* velocity $u = v_1 - v_2$ between two molecules whose velocities are v_1 and v_2 , respectively. You may need the integral

$$\int_{-\infty}^{\infty} \exp(-ax^2 + bx) = \sqrt{(\pi/a)} \exp[(b^2/(4a)],$$

where $a > 0$ and b is arbitrary.

7. A $(2n \times 2n)$ matrix with real elements is an element of the symplectic group $Sp(2n, \mathbb{R})$ if it satisfies the relation

$$M^T J M = J \quad \text{where} \quad J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix},$$

0_n and I_n being the $(n \times n)$ null matrix and unit matrix, respectively.

- Show that M^T is also symplectic if M is symplectic.
- Given that a general symplectic matrix M can be written as $M = \exp(aT)$ where a is a real parameter and T is a $(2n \times 2n)$ matrix, obtain the condition that T must satisfy, by considering a to be an infinitesimal quantity.
- Hence find the number r of independent generators of the group $Sp(2n, \mathbb{R})$.

- (d) A canonical transformation from the $2n$ dynamical variables (q_i, p_i) to a new set of $2n$ dynamical variables (Q_i, P_i) leaves the canonical Poisson bracket relations unchanged. What is the connection between such a transformation and a $(2n \times 2n)$ symplectic matrix?