CHAPTER 1

Representation of Signals

1.1 Introduction

The process of (electronic) communication involves the generation, transmission and reception of various types of signals. The communication process becomes fairly difficult, because:

a) the transmitted signals may have to travel long distances (there by undergoing severe attenuation) before they can reach the destination *i.e., the receiver.*

b) of imperfections of the channel over which the signals have to travel

c) of interference due to other signals sharing the same channel and

d) of *noise* at the receiver input[1].

In quite a few situations, the desired signal strength at the receiver input may not be significantly stronger than the disturbance component present at that point in the communication chain. (But for the above causes, the process of communication would have been quite easy, if not trivial). In order to come up with appropriate signal processing techniques, which enable us to extract the desired signal from a distorted and noisy version of the transmitted signal, we must clearly understand the nature and properties of the desired and undesired signals present at various stages of a communication system. In this lesson, we begin our study of this aspect of communication theory.

---

[1] Complete statistical characterization of the noise will be given in chapter 3, namely, Random Signals and Noise.

Signals physically exist in the time domain and are usually expressed as a function of the time parameter[1]. Because of this feature, it is not too difficult, at least in the majority of the situations of interest to us, to visualize the signal behavior in the **Time Domain.** In fact, it may even be possible to view the signals on an oscilloscope. But equally important is the characterization of the signals in the **Frequency Domain** or Spectral Domain. That is, we characterize the signal in terms of its various frequency components (or its **spectrum**). Fourier analysis (Fourier Series and Fourier Transform) helps us in arriving at the spectral description of the pertinent signals.

1.2 Periodic Signals and Fourier Series

Signals can be classified in various ways such as:

a) Power or Energy
b) Deterministic or Random
c) Real or Complex
d) Periodic or Aperiodic etc.

Our immediate concern is with periodic signals. In this section we shall develop the spectral description of these signals.

1.2.1 Periodic signals

Def. 1.1: A signal $x_p(t)$ is said to be periodic if

$$x_p(t) = x_p(t + T), \quad (1.1)$$

for all t and some T. ♦

(♦ denotes the end of definition, example, etc.)

[1] We will not discuss the multi-dimensional signals such as picture signals, video signals, etc.

Let T_0 be the smallest value of T for which this is possible. We call T_0 as the **period** of $x_p(t)$.

Fig. 1.1 shows a few examples of periodic signals.

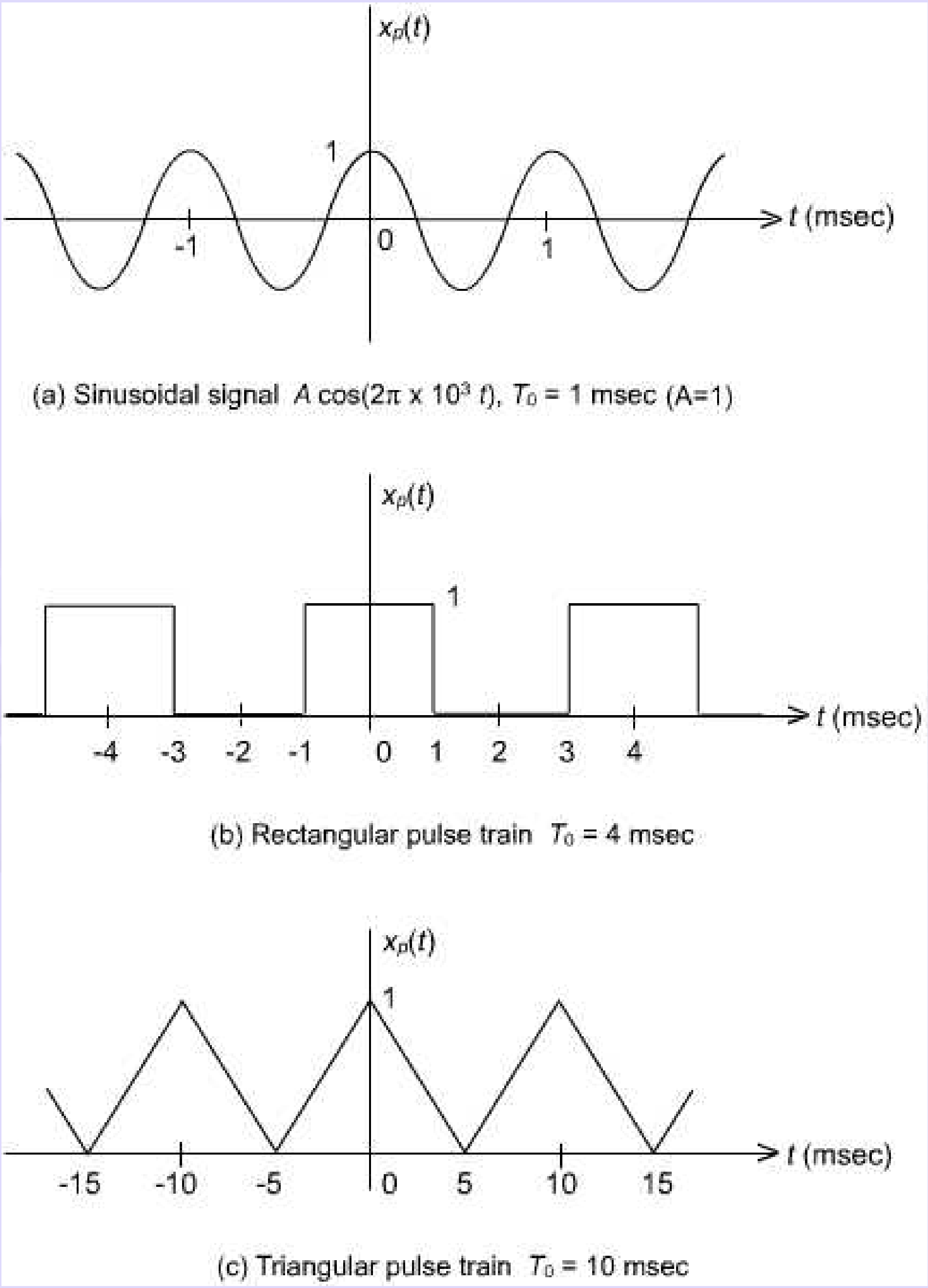


Fig. 1.1: Some examples of periodic signals

The basic building block of Fourier analysis is the complex exponential, namely, $A\, e^{j(2\pi f t + \varphi)}$ or $A \exp\left[j\left(2\pi f t + \varphi\right)\right]$, where

A : Amplitude (in Volts or Amperes)

f : Cyclical frequency (in Hz)

φ : Phase angle at $t = 0$ (either in radians or degrees)

Both A and f are real and non-negative. As the radian frequency, ω (in units of radians/ sec), is equal to $2\pi f$, the complex exponential can also written as $A\, e^{j(\omega t + \varphi)}$. We use subscripts on A, f (or ω) and φ to denote the specific values of these parameters.

Fourier analysis uses $\cos \omega t$ $\left[\text{or } \sin \omega t = \cos\left(\omega t - \frac{\pi}{2}\right)\right]$ in the representation of real signals. From Euler's relation, we have, $e^{j\omega t} = \cos \omega t + j \sin \omega t$.

As $\cos \omega t$ is the $\text{Re}\left[e^{j\omega t}\right]$, where $\text{Re}[x]$ denotes the real part of x, we have

$$\cos \omega t = \frac{e^{j\omega t} + \left(e^{j\omega t}\right)^*}{2} \quad (* \text{denotes the complex conjugate})$$

$$= \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

The term $e^{-j\omega t}$ or $e^{-j2\pi f t}$ is referred to as the complex exponential at the **negative frequency** $-\omega$ (or $-f$).

1.2.2 Fourier series

Let $x_p(t)$ be a periodic signal with period T_0. Then $f_0 = \frac{1}{T_0}$ is called the fundamental frequency and $n f_0$ is called the n^{th} harmonic, where n is an integer (for $n = 0$, we have the DC component and for the DC singal,

T_0 is not defined; $n = 1$ results in the fundamental). Fourier series decomposes $x_p(t)$ in to DC, fundamental and its various higher harmonics, namely,

$$x_p(t) = \sum_{n = -\infty}^{\infty} x_n e^{j 2 \pi n f_0 t} \tag{1.2a}$$

The coefficients $\{x_n\}$ constitute the Fourier series and are related to $x_p(t)$ as

$$x_n = \frac{1}{T_0} \int_{T_0} x_p(t) e^{-j 2 \pi n f_0 t} dt \tag{1.2b}$$

where $\int_{T_0}$ denotes the integral over any one period of $x_p(t)$. Most often, we use the interval $\left(-\frac{T_0}{2}, \frac{T_0}{2}\right)$ or $(0, T_0)$. Eq. 1.2(a) is referred to as the *Exponential form* of the Fourier series.

The coefficients $\{x_n\}$ are in general complex; hence

$$x_n = |x_n| e^{j \varphi_n} \tag{1.3}$$

where $|x_n|$ denotes the magnitude of the complex number and φ_n, the argument (or the angle). Using Eq. 1.3 in Eq. 1.2(a), we have,

$$x_p(t) = \sum_{n = -\infty}^{\infty} |x_n| e^{j(2 \pi n f_0 t + \varphi_n)}$$

Eq. 1.2(a) states that $x_p(t)$, in general, is composed of the frequency components at DC, fundamental and its higher harmonics. $|x_n|$ is the magnitude of the component in $x_p(t)$ at frequency $n f_0$ and φ_n, its phase. The plot of $|x_n|$ vs. n (or $n f_0$) is called the *magnitude spectrum*, and φ_n vs. n (or $n f_0$) is called the *phase spectrum*. It is important to note that the spectrum of a periodic signal exists **only at** discrete frequencies, namely, at $n f_0$, $n = 0, \pm 1, \pm 2, \cdots$, etc.

Let $x_p(t)$ be real; then

$$x_{-n} = \frac{1}{T_0} \int_{T_0} x_p(t) e^{j 2 \pi n f_0 t} \, dt$$

$$= x_n^*$$

That is, for a real periodic signal, we have the two symmetry properties, namely,

$$|x_{-n}| = |x_n| \tag{1.4a}$$

$$\varphi_{-n} = -\varphi_n \tag{1.4b}$$

Properties of Eq. 1.4 are part of an *if and only if (iff)* relationship. That is, if $x_p(t)$ is real, then Eq. 1.4 holds and if Eq. 1.4 holds, then $x_p(t)$ has to be real. This is because the complex exponentials at $(n f_0)$ and $-(n f_0)$ can be combined into a cosine term. As an example, let the only nonzero coefficients of a periodic signal be $x_{\pm 2}$, $x_{\pm 1}$, x_0. $x_0 = \mathrm{x}^*_0$ implies, x_0 is real and let

$$x_{-2} = 2 e^{j \frac{\pi}{4}} = x_2^* \text{ and } x_{-1} = 3 e^{j \frac{\pi}{3}} = x_1^* \text{ and } x_0 = 1. \text{ Then,}$$

$$x_p(t) = 2 e^{j \frac{\pi}{4}} e^{-j 4 \pi f_0 t} + 3 e^{j \frac{\pi}{3}} e^{-j 2 \pi f_0 t} + 1 + 3 e^{-j \frac{\pi}{3}} e^{j 2 \pi f_0 t} + 2 e^{-j \frac{\pi}{4}} e^{j 4 \pi f_0 t}$$

Combining the appropriate terms results in,

$$x_p(t) = 4 \cos\left(4 \pi f_0 t - \frac{\pi}{4}\right) + 6 \cos\left(2 \pi f_0 t - \frac{\pi}{3}\right) + 1$$

which is a real signal. The above form of representing $x_p(t)$, in terms of cosines is called the *Trigonometric form of the Fourier series.*

We shall illustrate the calculation of the Fourier coefficients using the periodic rectangular pulse train (This example is to be found in almost all the textbooks on communication theory).

Example 1.1

For the unit amplitude rectangular pulse train shown in Fig. 1.2, let us compute the Fourier series coefficients.

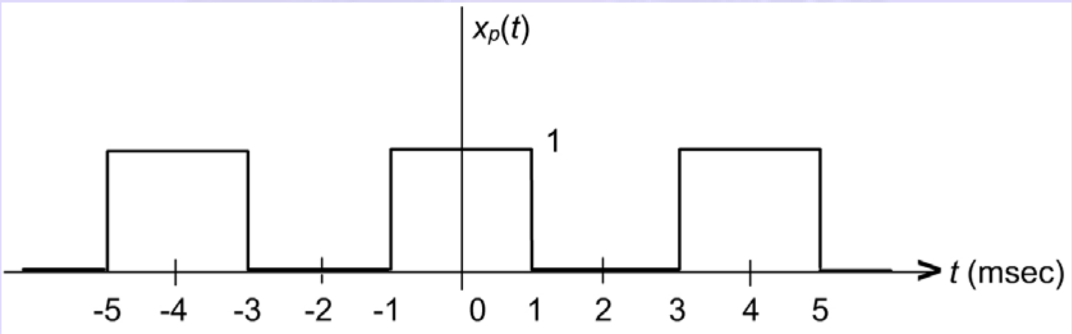


Fig. 1.2: Periodic rectangular pulse train

$x_p(t)$ has a period $T_0 = 4$ milliseconds and is **ON** for half the period and **OFF** during the remaining half. The fundamental frequency f_0 = 250 Hz.

From Eq. 1.2(b), we have

$$x_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j 2 \pi n f_0 t} \, dt$$

$$= \frac{1}{T_0} \int_{-\frac{T_0}{4}}^{\frac{T_0}{4}} e^{-j 2 \pi n f_0 t} dt$$

$$= \frac{1}{T_0} \frac{\sin\left(\frac{n\pi}{2}\right)}{n \pi f_0}$$

$$= \frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi}$$

As can be seen from the equation for x_n, all the Fourier coefficients are real but could be bipolar (+ve or –ve). Hence φ_n is either zero or $\pm\pi$ for all n.

Fig. 1.3 shows the plots of magnitude and phase spectrum.

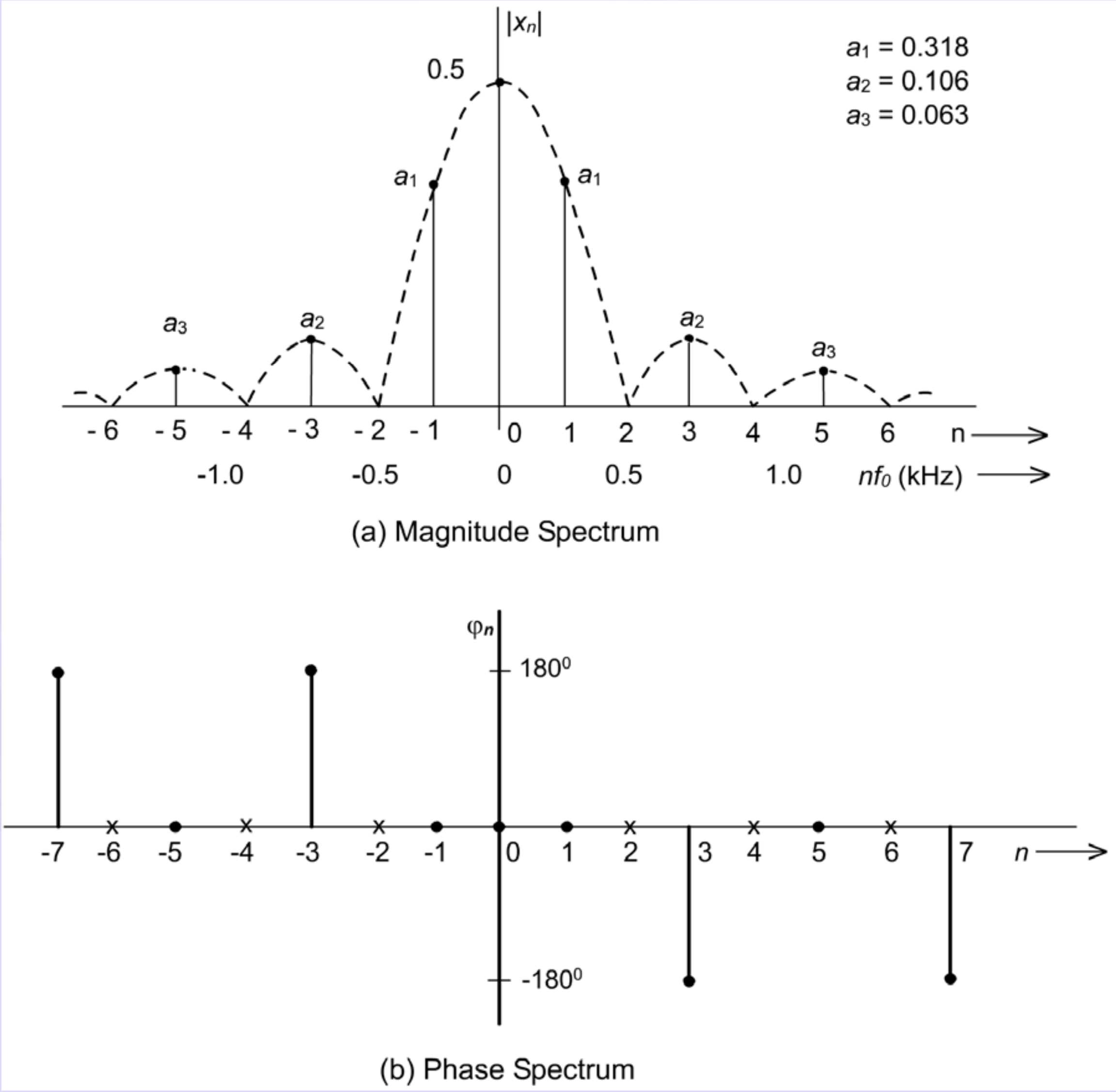


Fig. 1.3: Magnitude and phase spectra for the $x_p(t)$ of example 1.1

From Fig. 1.3, we observe:

i) x_0, the average or the DC value of the pulse train is $\frac{1}{2}$. For any periodic signal, the average value is $\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_p(t)\, dt$.

ii) spectrum exists only at discrete frequencies, namely, $f = n f_0$, with $f_0 = 250\, Hz$. Such a spectrum is called the *discrete spectrum* (or line spectrum).

iii) the curve drawn with broken line in Fig. 1.3(a) is the envelope of the magnitude spectrum. The envelope consists of several lobes and the maximum value of each lobe keeps decreasing with increase in frequency.

iv) the plot of $|x_n|$ vs. frequency is symmetric and the plot of φ_n vs. frequency is anti-symmetric. This is because $x_p(t)$ is real.

v) φ_n at $n = \pm 2, \pm 4$ etc. is undefined as $|x_n| = 0$ for these n. This is indicated with a cross on the phase spectrum plot. ♦

One of the functions that is useful in the study of Fourier analysis is the $\text{sinc}(\;)$ function defined by

$$\text{sinc}\,\lambda = \text{sinc}(\lambda) = \frac{\sin(\pi\lambda)}{\pi\lambda} \tag{1.5}$$

A plot of the $\text{sinc}\,\lambda$ vs. λ along with a table of values are given in appendix A1.1, at the end of the chapter.

In terms of $\text{sinc}(\lambda)$, the Fourier coefficient of example 1.1 can be written as, $x_n = \frac{1}{2}\text{sinc}\left(\frac{n}{2}\right)$.

Exercise 1.1

For the $x_p(t)$ of Fig .1.4, show that $x_n = \left(\frac{\tau}{T_0}\right)\text{sinc}(n f_0 \tau)$

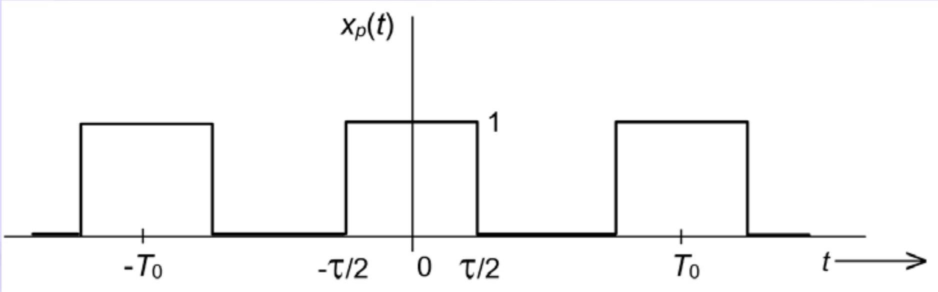
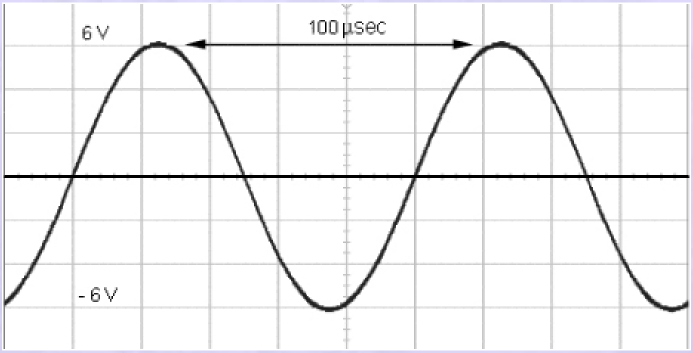


Fig. 1.4: $x_p(t)$ of Exercise 1.1

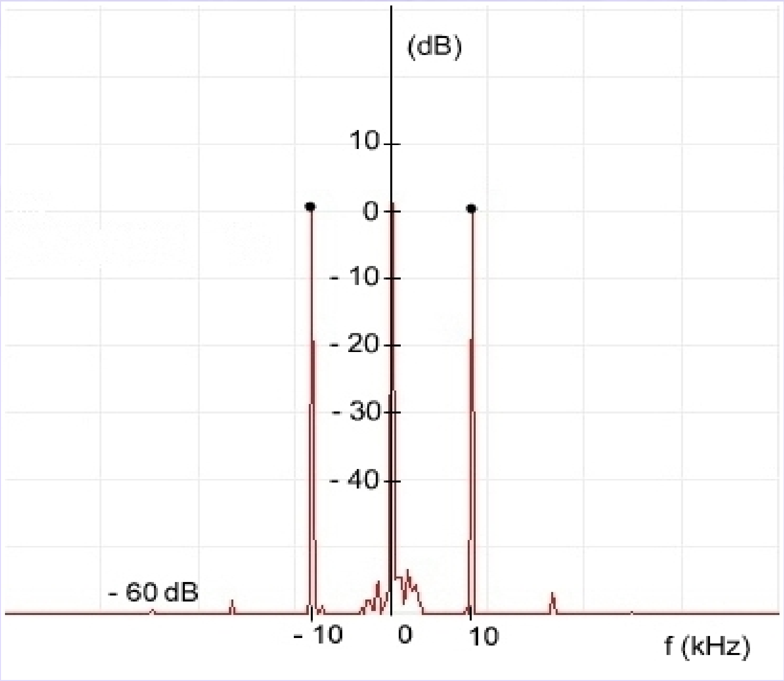
Spectrum analyzer is an important laboratory instrument, which can be used to obtain the magnitude spectrum of periodic signals (frequency resolution, frequency range over which the spectrum can be measured etc. depend on that particular instrument). We have given below a set of four waveforms (output of a function generator) and their line spectra, as indicated by a spectrum analyzer.

The spectral plots 1 to 4 give the values of $20\log_{10}\left[\frac{|x_n|}{|x_1|}\right]$ for the waveforms 1 to 4 respectively. The units for the above quantities are in decibels (dB).

1.



Waveform 1

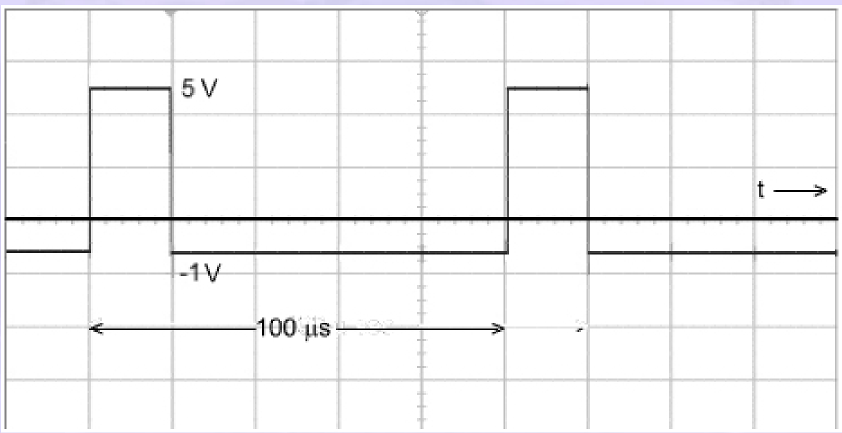


Spectral Plot 1

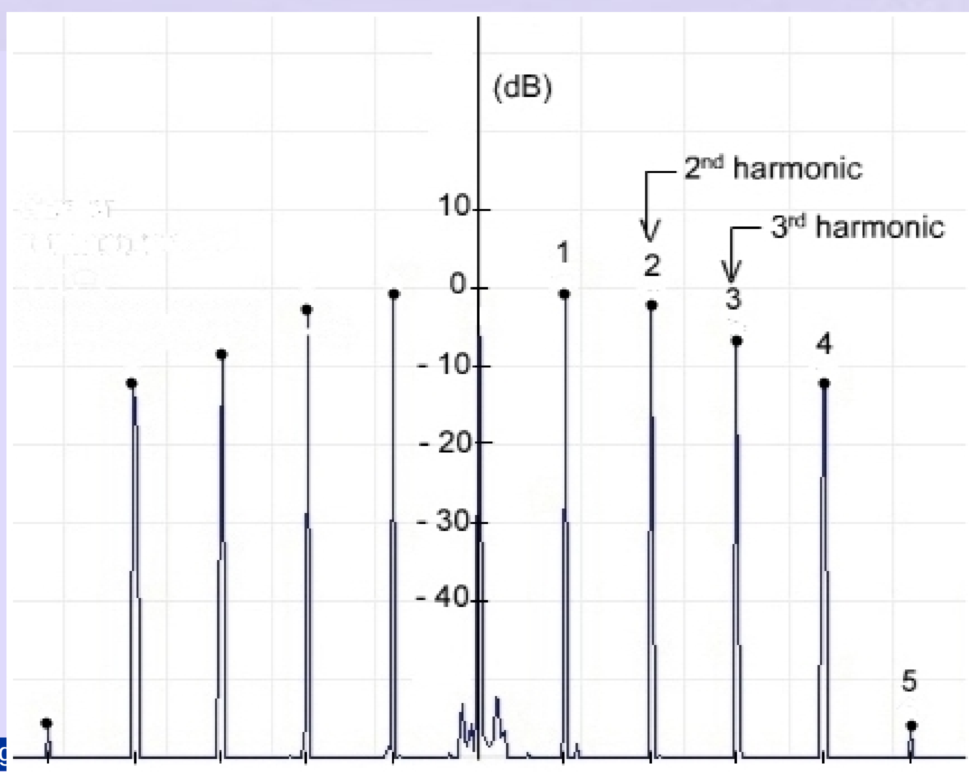
Waveform 1: A cosine signal (frequency 10 kHz).

Comments on spectral plot 1: Waveform 1 has only two Fourier coefficients, namely, x_{-1} and x_1. Also, we have $|x_{-1}| = |x_1|$. Hence the spectral plot has only two lines, namely at ± 10 kHz, and their values are $20 \log_{10} \frac{|x_1|}{|x_1|} = 0$ dB.

2.



Waveform 2



Waveform 2: Periodic square wave with $\frac{\tau}{T_0} = \frac{1}{5}$; $T_0 = 0.1$ msec.

Comments on Spectral Plot 2: Values of various spectral components are:

i) fundamental: 0 dB

ii) second harmonic: $20 \log_{10} \left[\frac{\text{sin}c(0.4)}{\text{sin}c(0.2)}\right]$

$$= 20 \log_{10} 0.809 = -0.924 \, dB$$

iii) third harmonic: $20 \log_{10} \left[\frac{\text{sin}c(0.6)}{\text{sin}c(0.2)}\right]$

$$= 20 \log_{10} \left[\frac{0.504}{0.935}\right]$$

$$= 20 \log_{10} (0.539) \, dB$$

$$= -5.362$$

iv) fourth Harmonic: $20 \log_{10} \left[\frac{\text{sin}c(0.8)}{\text{sin}c(0.2)}\right]$

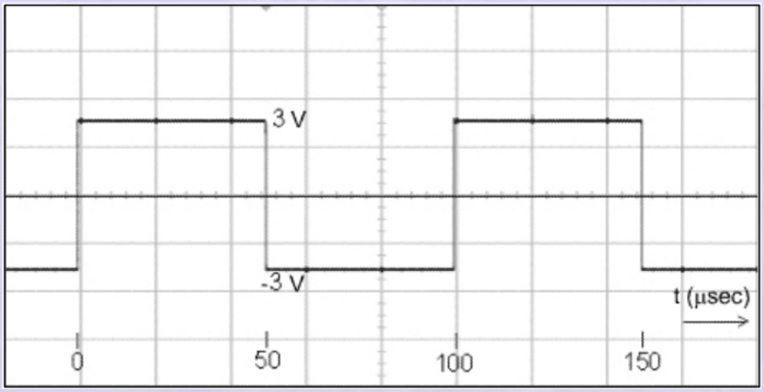
$$= -12.04 \, dB$$

v) fifth harmonic: $20 \log_{10} \left[\frac{\text{sin}c(1)}{\text{sin}c(0.2)}\right]$

$$= 20 \log_{10} (0) = -\infty$$

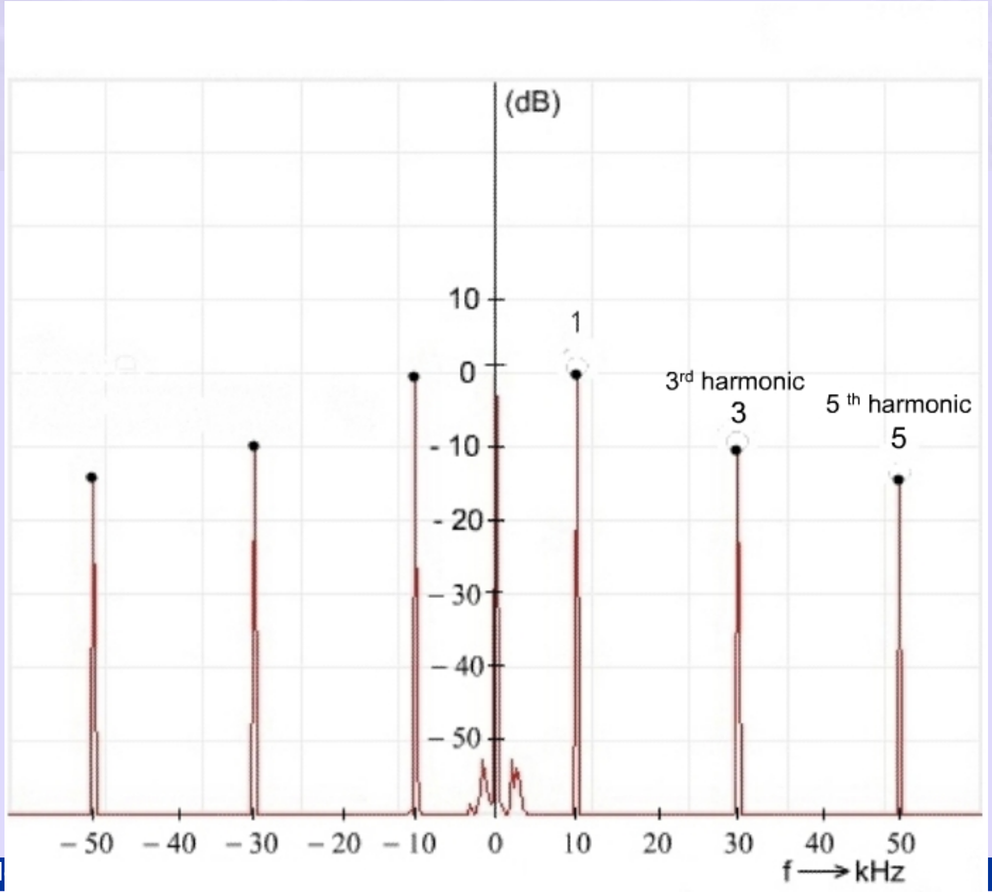
because of the limitations of the instrument, we see a small spike at -60 dB. Similarly, the values of other components can be calculated.

3.

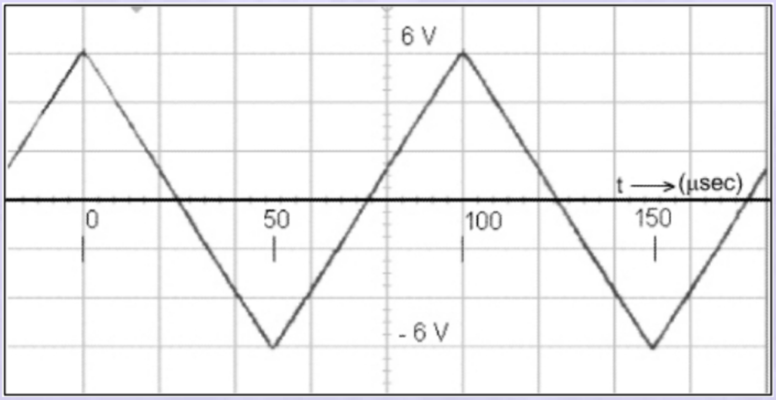


Waveform 3

Values of Spectral Components: Exercise

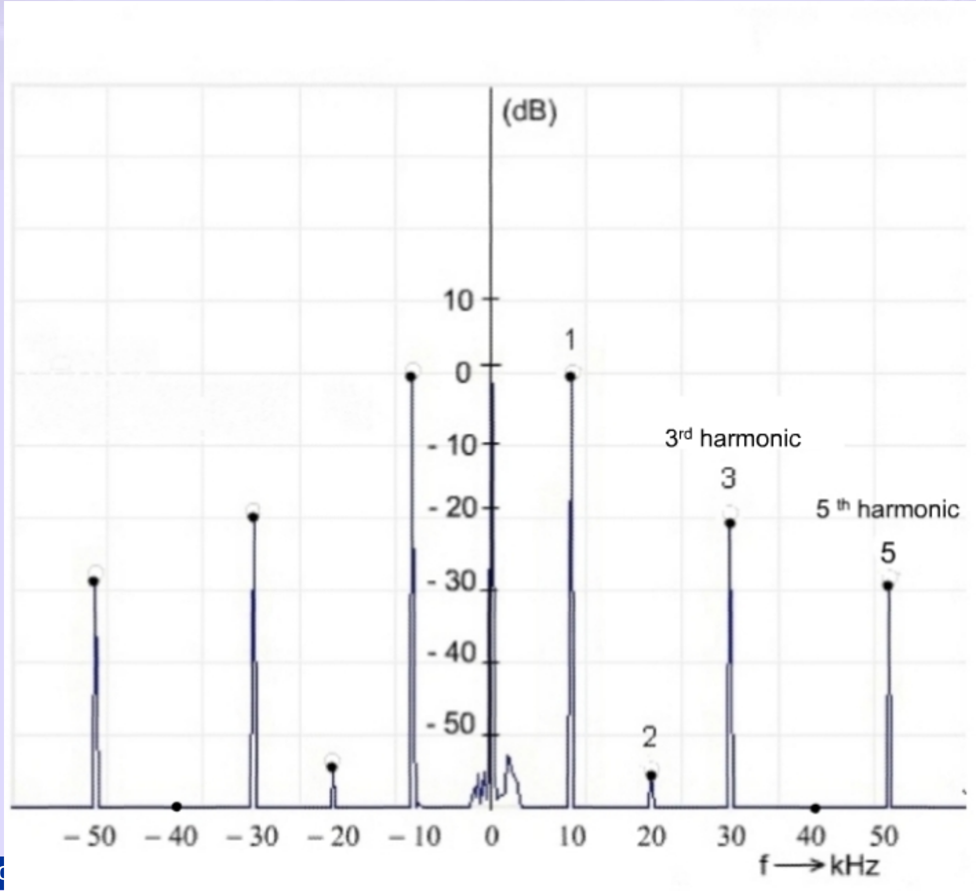


4.



Waveform 4

Values of Spectral Components: Exercise



The Example 1.1 and the periodic waveforms 1 to 4 all have fundamental as part of their spectra. Based on this, we should not surmise that every periodic signal must necessarily have a nonzero value for its fundamental. As a counter to the conjuncture, let $x_p(t) = \cos(20\pi t)\cos(2000\pi t)$.

This is periodic with period 100 msec. However, the only spectral components that have nonzero magnitudes are at 990 Hz and 1010 Hz. That is, the first 99 spectral components (inclusive of DC) are zero!

Let $x_p(t)$ be a periodic voltage waveform across a 1 Ω resistor or a current waveform flowing in a 1 Ω resistor. We now define its (normalized) average power, denoted by P_{x_p}, as

$$P_{x_p} = \frac{1}{T_0} \int_{T_0} \left| x_p(t) \right|^2 dt$$

Parseval's (Power) Theorem relates P_{x_p} to x_n as follows:

$$P_{x_p} = \sum_{n=-\infty}^{\infty} \left| x_n \right|^2$$

(The proof of this relation is left as an exercise.)

If $x_p(t)$ consists of a single complex exponential, ie,

$$x_p(t) = x_n\, e^{j(2\pi n f_0 t + \varphi)}$$

then, $P_{x_p} = \left|x_n^2\right|$

In other words, Parseval's power theorem implies that the total average power in $x_p(t)$ is superposition of the average powers of the complex exponentials present in it.

When the periodic signal exhibits certain symmetries, Fourier coefficients take special forms. Let us first define some of these symmetries (We assume $x_p(t)$ to be real).

Def. 1.2(a): A periodic signal $x_p(t)$ is even, if $x_p(-t) = x_p(t)$ (1.6a)

Def. 1.2(b): A periodic signal $x_p(t)$ is odd, if $x_p(-t) = -x_p(t)$ (1.6b)

Def. 1.2(c): A periodic signal $x_p(t)$ has half-wave symmetry, if

$$x_p\left(t \pm \frac{T_0}{2}\right) = -x_p(t) \qquad (1.6c)$$

♦

With respect to the symmetries defined by Eq. 1.6, we have the following special forms for the coefficients x_n:

$x_p(t)$ even: x_n's are purely real and even with respect to n

$x_p(t)$ odd: x_n's are purely imaginary and odd with respect to n

$x_p(t)$ half-wave symmetric: x_n's are zero for n even.

A proof of these properties is as follows:

i) $x_p(t)$ even:

$$x_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_p(t)\, e^{-j 2 \pi n f_0 t}\, dt$$

$$= \frac{1}{T_0}\left[\int_{-T_0/2}^{0} x_p(t) e^{-j 2\pi n f_0 t}\, dt + \int_{0}^{T_0/2} x_p(t) e^{-j 2\pi n f_0 t}\, dt\right]$$

Changing t to $-t$ in the first integral, and noting $x_p(-t) = x_p(t)$,

$$= \frac{1}{T_0}\left[\int_{0}^{T_0/2} x_p(t) e^{j 2\pi n f_0 t}\, dt + \int_{0}^{T_0/2} x_p(t)\, e^{-j 2\pi n f_0 t}\, dt\right]$$

$$= \frac{2}{T_0}\left[\int_{0}^{T_0/2} x_p(t)\,(\cos 2\pi n f_0 t)\, dt\right]$$

The above integral is real and as $\cos\left[2\pi(-n) f_0 t\right] = \cos(2\pi n f_0 t)$, $x_{-n} = x_n$.

ii) $x_p(t)$ odd:

$$x_n = \frac{1}{T_0}\left[\int_{-T_0/2}^{0} x_p(t) e^{-j 2\pi n f_0 t}\, dt + \int_{0}^{T_0/2} x_p(t) e^{-j 2\pi n f_0 t}\, dt\right]$$

Changing t to $-t$, in the first integral and noting that $x_p(-t) = -x_p(t)$, we have

$$= \frac{1}{T_0}\left[\int_{0}^{T_0/2} - x_p(t)\, e^{j 2\pi n f_0 t}\, dt + \int_{0}^{T_0/2} x_p(t)\, e^{-j 2\pi n f_0 t}\, dt\right]$$

$$= \frac{1}{T_0} \int_{0}^{T_0/2} x_p(t)\left[e^{-j 2\pi n f_0 t} - e^{j 2\pi n f_0 t}\right] dt$$

$$= -\frac{2j}{T_0} \int_{0}^{T_0/2} x_p(t) \sin\left(2\pi n f_0 t\right) dt$$

Hence the result.

iii) $x_p(t)$ has half-wave symmetry:

$$x_n = \frac{1}{T_0}\left[\int_{-T_0/2}^{0} x_p(t) e^{-j 2\pi n f_0 t}\, dt + \int_{0}^{T_0/2} x_p(t) e^{-j 2\pi n f_0 t}\, dt\right]$$

In the first integral, replace t by $t + T_0/2$.

$$x_n = \frac{1}{T_0}\left[\int_{0}^{T_0/2} x_p(t + T_0/2)\, e^{-j(n\omega_0 t + \pi n)} dt + \int_{0}^{T_0/2} x_p(t) e^{-j n \omega_0 t} dt\right]$$

The result follows from the relation

$$e^{-j\pi n} = \begin{cases} -1, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$$

1.2.3 Convergence of Fourier series and Gibbs phenomenon

As seen from Eq. 1.2, the representation of a periodic function in terms of Fourier series involves, in general, an infinite summation. As such, the issue of *convergence of the series* is to be given some consideration.

There is a set of conditions, known as *Dirichlet conditions* that guarantee convergence. These are stated below.

i) $\int_{T_0} \left|x_p(t)\right| < \infty$

That is, the function is absolutely integrable over any period. It is easy to verify that the above condition results in $\left|x_n\right| < \infty$ for any n.

ii) $x_p(t)$ has only a finite number of maxima and minima over any period T_0.

iii) There are only finite number of finite discontinuities over any period[1].

$$\text{Let } x_M(t) = \sum_{n=-M}^{M} x_n e^{j 2\pi n f_0 t} \qquad (1.7)$$

[1] For examples of periodic signals that do not satisfy one or more of the conditions i) to iii), the reader is referred to [1, 2] listed at the end of this chapter.

and $e_M(t) = x_p(t) - x_M(t)$

then $\lim\limits_{M \to \infty} x_M(t)$ *converges uniformly* to $x_p(t)$ wherever $x_p(t)$ is continuous; that is $\lim\limits_{M \to \infty} e_M(t) = 0$ for all t.

Dirichlet conditions are sufficient but not necessary. Later on, we shall have examples of Fourier series for functions that voilate some of the Dirichet conditions.

If $x_p(t)$ is not absolutely integrable but square integrable, that is, $\int\limits_{T_0} |x_p(t)|^2 dt < \infty$, then the series *converges in the mean.* That is,

$$\lim_{M \to \infty} \int\limits_{T_0} |e_M(t)|^2 dt = 0 \qquad (1.8)$$

Note that Eq. 1.8 does not imply that $\lim\limits_{M \to \infty} e_M(t)$ is zero. There could be nonzero values in $\lim\limits_{M \to \infty} e_M(t)$; but they occur at isolated points, resulting in the integral of Eq. 1.8, being equal to zero.

The limiting behavior of $x_M(t)$ at the points of discontinuity in $x_p(t)$ is somewhat interesting, regardless of $x_p(t)$ being absolutely integrable or square integrable. This is illustrated in Fig. 1.5(a). From the figure, we see that

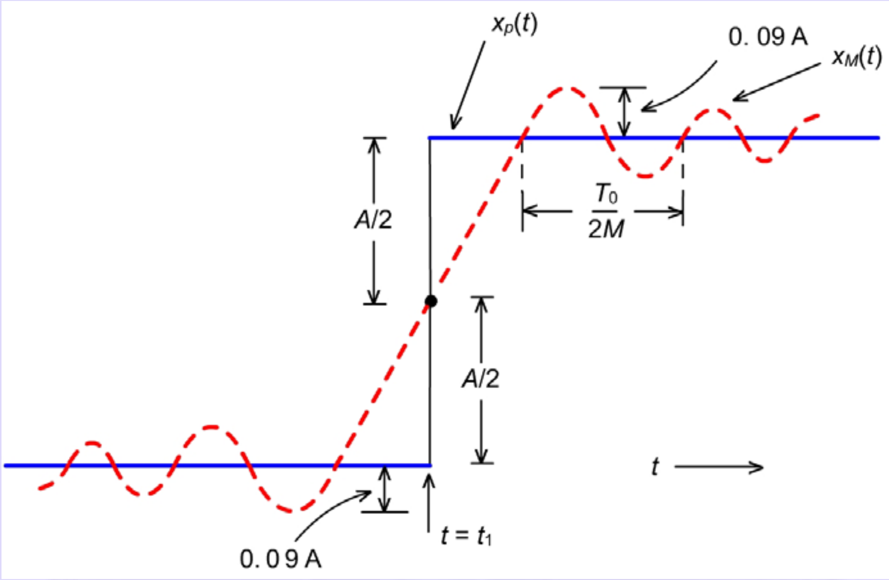


Fig. 1.5(a): Convergence behavior of $x_M(t)$ at a discontinuity in $x_p(t)$

$x_M(t)$ passes through the mid-point of the discontinuity and has a peak overshoot (as well as undershoot) of amplitude 0.09A (We assume M to be sufficiently large). The period of oscillations (whose amplitudes keep decreasing with increasing t, $t > 0$) is $\frac{T_0}{2M}$. These oscillations (with the peak overshoot as well as the undershoot of amplitude 0. 09 A) persist even as $M \to \infty$. In the limiting case, all the oscillations converge in location to the point $t = t_1$ (the point of discontinuity) resulting in what is called as *Gibbs ears* as shown in Fig. 1.5(b).

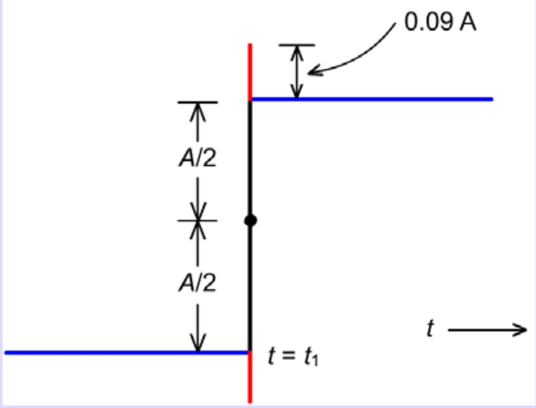


Fig. 1.5(b): Gibbs ears at t = t_1

(In 1898, Albert Michelson, a well-known name in the field of optics, developed an instrument called Harmonic Analyzer (HA), which was capable of computing the first 80 coefficients of the Fourier series. HA could also be used a signal synthesizer. In other words, it has the ability to self-check its calculations by synthesizing the signal using the computed coefficients. When Michelson tried this instrument on signals with discontinuities (with continuous signals, close agreement was found between the original signal and the synthesized signal), he observed a strange behavior: synthesized signal, based on the 80 coefficients, exhibited ringing with an overshoot of about 9% of the discontinuity, in the vicinity of the discontinuity. This behavior persisted even after increasing the number of terms beyond 80. J. W. Gibbs, professor at Yale, investigated and clarified the above behavior by taking the saw-tooth wave as an example; hence the name *Gibbs Phenomenon.)*

The convergence of the Fourier series and the corresponding Gibbs oscillations can be seen from the animation that follows. You have been provided with three options with respect to the number of harmonics M to be summed. These are: $M = 10, 25$ and 75.

1.3 Aperiodic Signals and Fourier Transform

Aperiodic (also called nonperiodic) signals can be of finite or infinite duration. A few of the aperiodic signals occur quote often in theoretical studies. Hence, it behooves us to introduce some notation to describe their behavior.

i) Rectangular pulse, $ga\left(\frac{t}{T}\right)$

$$ga\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| < \frac{T}{2} \\ 0, & |t| > \frac{T}{2} \end{cases} \tag{1.9a}$$

[Rectangular pulse is sometimes referred to as a gate pulse; hence the symbol $ga(\ )$]. In view of the above $A\ ga\left(\frac{t}{T}\right)$ refers to a rectangular pulse of amplitude A and duration T, centered at $t = 0$.

ii) Triangular pulse, $tri\left(\frac{t}{T}\right)$

$$tri\left(\frac{t}{T}\right) = \begin{cases} 1 - \frac{|t|}{T}, & |t| \leq T \\ 0 & , \ outside \end{cases} \tag{1.9b}$$

iii) One-sided (decaying) exponential pulse, $ex1\left(\frac{t}{T}\right)$:

$$ex1\left(\frac{t}{T}\right) = \begin{cases} e^{-\frac{t}{T}}, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases} \tag{1.9c}$$

iv) Two-sided (symmetrical) exponential pulse, $ex2\left(\frac{t}{T}\right)$:

$$ex2\left(\frac{t}{T}\right) = \begin{cases} e^{-\frac{t}{T}}, & t > 0 \\ 1, & t = 0 \\ e^{\frac{t}{T}}, & t < 0 \end{cases} \tag{1.9d}$$

Fig. 1.6 illustrates specific examples of these pulses.

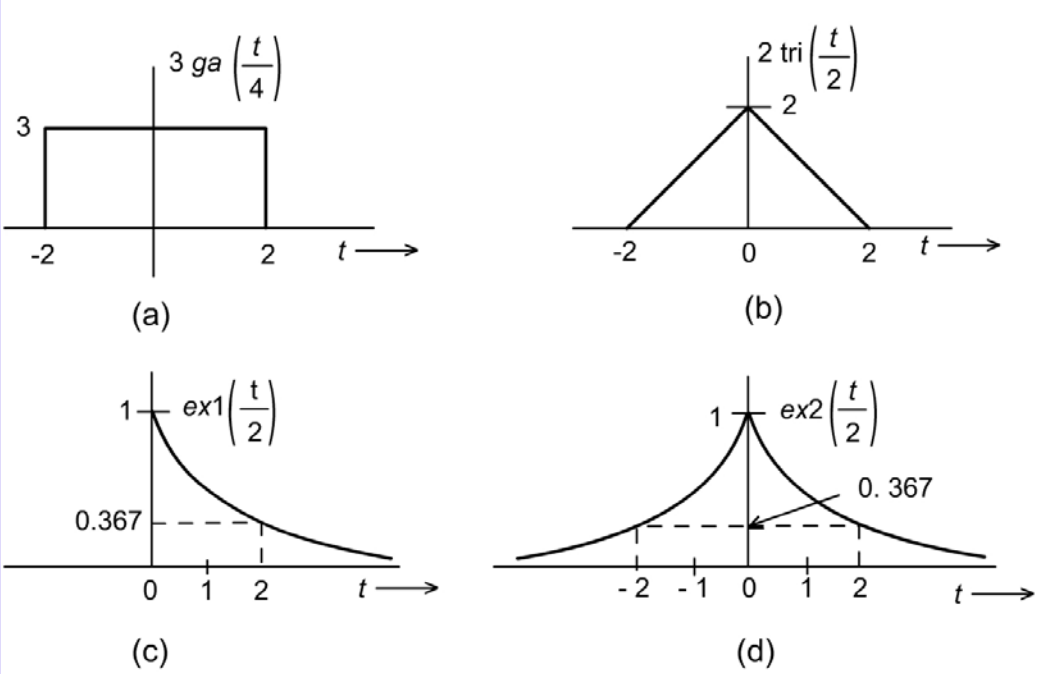


Fig. 1.6: Examples of pulses defined by Eq. 1.9

Let $x(t)$ be any aperiodic signal. We define its normalized energy E_x, as

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{1.10}$$

An aperiodic signal with $0 < E_x < \infty$ is said to be an *energy signal.* (When no specific signal is being referred to, we use the symbol E without any subscript to denote the energy quantity.)

1.3.1 Fourier transform

Like periodic signals, aperiodic signals also can be represented in the frequency domain. However, unlike the discrete spectrum of the periodic case, we have a *continuous spectrum* for the aperiodic case; that is, the frequency components constituting a given signal $x(t)$ lie in a continuous range (or ranges), and quite often this range could be $(-\infty, \infty)$. Eq. 1.2(a) expresses $x_p(t)$ as a sum over a discrete set of frequencies. Its counterpart for the aperiodic case is an integral relationship given by

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j 2 \pi f t} d f \qquad (1.11a)$$

where $X(f)$ is the Fourier transform of $x(t)$.

Eq. 1.11(a) is given the following interpretation. Let the integral be treated as a sum over incremental frequency ranges of width Δf. Let $X(f_i)\Delta f$ be the incremental complex amplitude of $e^{j 2\pi f_i t}$ at the frequency $f = f_i$. If we sum a large number of such complex exponentials, the resulting signal should be a very good approximation to $x(t)$. This argument, carried to its natural conclusion, leads to signal representation with a sum of complex exponentials replaced by an integral, where a continuous range of frequencies, with the appropriate complex amplitude distribution will synthesize the given signal $x(t)$.

Eq. 1.11(a) is called the *synthesis relation* or *Inverse Fourier Transform* (IFT) relation. Quite often, we know $x(t)$ and would want $X(f)$. The companion relation to Eq. 1.11(a) is

$$X(f) = \int_{\infty}^{-\infty} x(t) e^{-j 2\pi f t} dt \qquad (1.11b)$$

Eq. 1.11(b) is referred to as the *Fourier Transform (FT) relation* or, the *analysis equation*, or *forward transform* relation. We use the notation

$$X(f) = F\left[x(t)\right] \qquad (1.12a)$$

$$x(t) = F^{-1}\left[X(f)\right] \qquad (1.12b)$$

Eq. 1.12(a) and Eq. 1.12(b) are combined into the abbreviated notation, namely,

$$x(t) \longleftrightarrow X(f). \qquad (1.12c)$$

$X(f)$ is, in general, a complex quantity. That is,

$$X(f) = X_R(f) + j X_I(f)$$

$$= \left|X(f)\right| e^{j\,\theta(f)}$$

$X_R(f) = \textit{Real part of } X(f)$,

$X_I(f) = \textit{Imaginary part of } X(f)$

$\left|X(f)\right| = \textit{magnitude of } X(f)$

$$= \sqrt{X_R^2(f) + X_I^2(f)}$$

$$\theta(f) = \arg\left[X(f)\right] = arc\tan\left[\frac{X_I(f)}{X_R(f)}\right]$$

Information in $X(f)$ is usually displayed by means of two plots: (a) $\left|X(f)\right|$ *vs.* f, known as *magnitude spectrum* and (b) $\theta(f)$ *vs.* f, known as the *phase spectrum.*

Example 1.2

Let $x(t) = A\, ga\left(\frac{t}{T}\right)$. Let us compute and sketch $X(f)$.

$$X(f) = \int_{-T/2}^{T/2} A\, e^{-j2\pi ft} dt = AT\,\text{sin}c(fT),$$

where

$$\text{sin}c(\lambda) = \frac{\sin(\pi\lambda)}{\pi\lambda}$$

Appendix A1.1 contains the tabulated values of $\text{sin}c(\lambda)$. Its behavior is shown in Fig. A1.1. Note that $\text{sin}c(\lambda) = \begin{cases} 1\,, & \lambda = 0 \\ 0\,, & \lambda = \pm 1, \pm 2 \text{ etc.} \end{cases}$

Fig. 1.7(a) sketches the magnitude spectrum of the rectangular pulse for $AT = 1$.

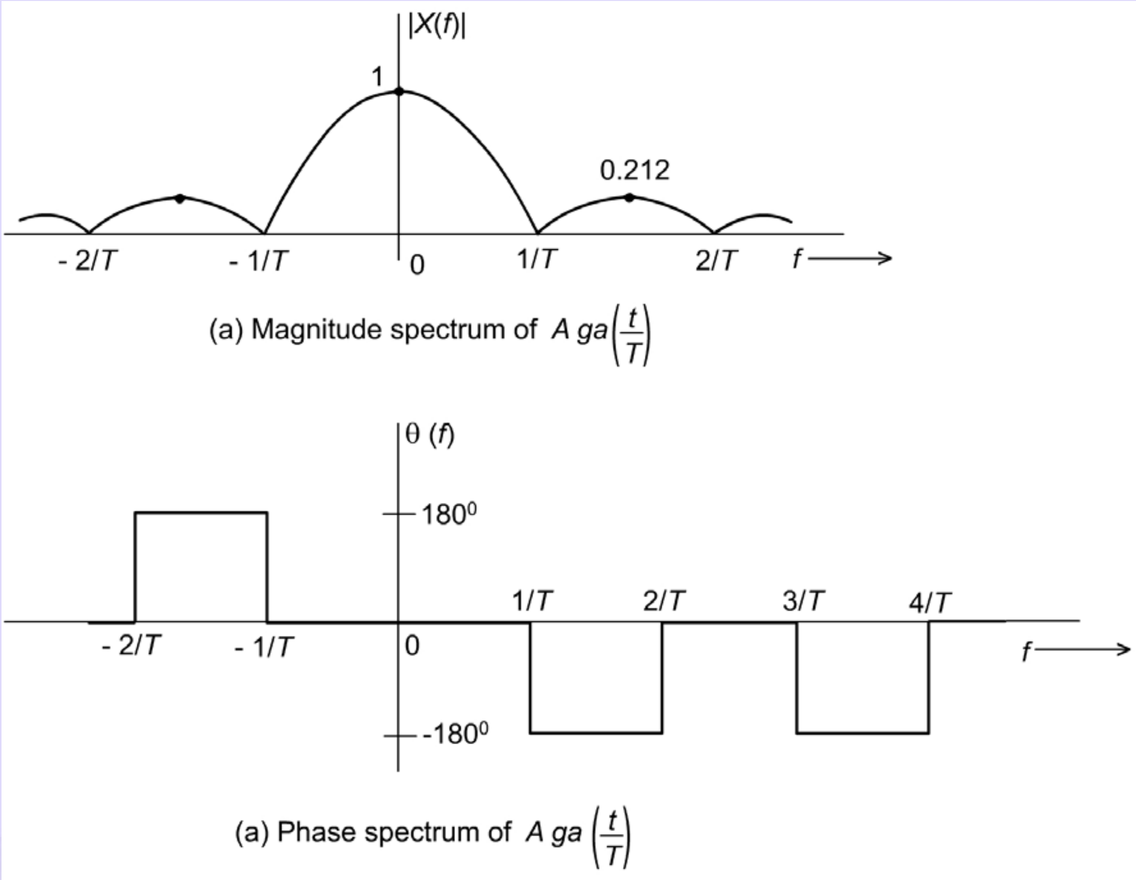


Fig. 1.7: Spectrum of the rectangular pulse

Regarding the phase plot, $\text{sin}c\,(f\,T)$ is real. However it could be bipolar. During the interval, $\frac{m}{T} < |f| < \frac{m+1}{T}$, with m odd, $\text{sin}c\,(f\,T)$ is negative. As the magnitude spectrum is always positive, negative values of $\text{sin}c\,(f\,T)$ are taken care of by making $\theta(f) = \pm 180^0$ for the appropriate ranges, as shown in Fig. 1.7(b).

Remarkable balancing act: A serious look at the magnitude and phase plots reveals a very charming result. From the magnitude spectrum, we find that a rectangular pulse is composed of frequency components in the range $-\infty < f < \infty$, each with its own amplitude and phase. Each of these complex exponentials exist for all t. But when we synthesize a signal using the complex exponentials with the magnitudes and phases as given in Fig. 1.7, they add up to a constant for $|t| < T/2$ and then go to *zero forever*. A very fascinating result indeed!

Fig. 1.7(a) illustrates another interesting result. From the figure, we see that most of the energy, (that is, the range of strong spectral components) of the signal lies in the interval $|f| < \frac{1}{T}$, where T is the duration of the rectangular pulse. Hence, if T is reduced, then its spectral width increases and vice versa (As we shall see later, this is true of other pulse types, other than the rectangular). That is, *more compact is the signal in the time-domain, the more wide-spread it would be in the frequency domain and vice versa.* This is called the phenomenon of **reciprocal spreading.**

♦

Example 1.3

Let $x(t) = ex1(t)$. Let us find $X(f)$ and sketch it.

$$X(f) = \int_0^{\infty} e^{-t} e^{-j2\pi f t} dt = \frac{1}{1 + j2\pi f}$$

Hence, $|X(f)| = \frac{1}{\sqrt{1 + (2\pi f)^2}}$

$$\theta(f) = -arc\tan(2\pi f)$$

A plot of the magnitude and the phase spectrum are given in Fig. 1.8.

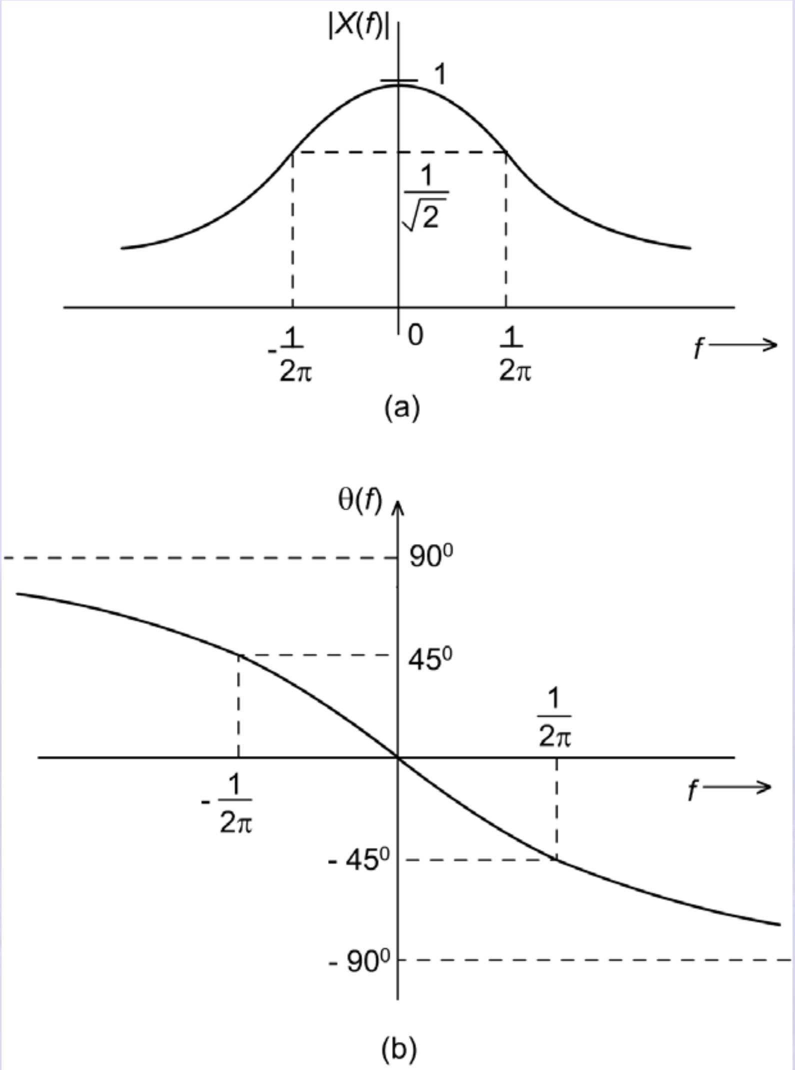


Fig. 1.8: (a) Magnitude spectrum of the decaying exponential

(b) Phase spectrum ⧫

1.3.2 Dirichlet conditions

Given $X(f)$, Eq. 1.11(a) enables us to synthesize the signal $x(t)$. Now the question is: Is the synthesized signal, say $x^{(s)}(t)$, identical to $x(t)$? This leads to the topic of *convergence of the Fourier Integral*. Analogous to the Dirichlet conditions for the Fourier series, we have a set of sufficient conditions, (also called Dirichlet conditions) for the existence of Fourier transform, which are stated below:

(i) $x(t)$ be absolutely integrable; that is,

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$$

This ensures that $|X(f)|$ is finite for all f, because

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, dt$$

$$|X(f)| = \left| \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, dt \right|$$

$$\leq \int_{-\infty}^{\infty} \left| x(t) e^{-j2\pi f t} dt \right| = \int_{-\infty}^{\infty} \left| x(t) \, dt \right| < \infty$$

(ii) $x(t)$ is single valued and has only finite number of maxima and minima with in any finite interval.

(iii) $x(t)$ has a finite number of finite discontinuities with in any finite interval.

If $x^{(s)}(t) = \lim_{W \to \infty} \int_{-W}^{W} X(f) e^{j2\pi f t} \, df$, then $x^{(s)}(t)$ converges to $x(t)$ uniformly wherever $x(t)$ is continuous.

If $x(t)$ is not absolutely integrable but square integrable, that is, $\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$ (Finite energy signal), then we have the *convergence in the mean*, namely

$$\int_{-\infty}^{\infty} \left| x(t) - x^{(s)}(t) \right|^2 dt = 0$$

Regardless of whether $x(t)$ is absolutely integrable or square integrable, $x^{(s)}(t)$ exhibits Gibbs phenomenon at the points of discontinuity in $x(t)$, always passing through the midpoints of the discontinuities.

1.4 Properties of the Fourier Transform

Fourier Transform has a large number of properties, which are developed in the sequel. A thorough understanding of these properties, and the ability to make use of them appropriately, helps a great deal in the analysis of various signals and systems.

P1) Linearity

Let $x_1(t) \longleftrightarrow X_1(f)$ and $x_2(t) \longleftrightarrow X_2(f)$.

Then, for all constants a_1 and a_2, we have

$$a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 X_1(f) + a_2 X_2(f)$$

It is very easy to see the validity of the above transform relationship. This property will be used quite often in the development of this course material.

P2a) Area under $x(t)$

If $x(t) \longleftrightarrow X(f)$, then

$$\int_{-\infty}^{\infty} x(t)\,dt = X(0)$$

The above property follows quite simply by setting $f = 0$ in Eq. 1.11(b). As an example of this property, we have the transform pair

$$ga\left(\frac{t}{T}\right) \longleftrightarrow T\,\mathrm{sinc}(fT)$$

By inspection, area of the time function is T, which is equal to $T\,\mathrm{sinc}(fT)\big|_{f=0}$.

P2b) Area under $X(f)$

If $x(t) \longleftrightarrow X(f)$, then $x(0) = \int_{-\infty}^{\infty} X(f)\, df$

The proof follows by making $t = 0$ in Eq. 1.11(a).

As an illustration of this property, we have

$$x(t) = ex1(t) \longleftrightarrow X(f) = \frac{1}{1 + j2\pi f}$$

Hence $\int_{-\infty}^{\infty} X(f)\, df = \int_{-\infty}^{\infty} \frac{1}{1 + j2\pi f}\, df = \int_{-\infty}^{\infty} \frac{1 - j2\pi f}{1 + (2\pi f)^2}\, df$

Noting that $\int_{-\infty}^{\infty} \frac{2\pi f}{1 + (2\pi f)^2} df = 0$, we have

$$\int_{-\infty}^{\infty} X(f)\, df = \int_{-\infty}^{\infty} \frac{1}{1 + (2\pi f)^2} df = \frac{1}{2}$$, which is the value of $ex1(t)|_{t=0}$

P3) Time Scaling

If $x(t) \longleftrightarrow X(f)$, then $x(\alpha t) \longleftrightarrow \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right)$, where α is a real constant.

Proof: Exercise

The value of α decides the behavior of $x(\alpha t)$. If $\alpha = -1$, $x(\alpha t)$ is a time reversed version of $x(t)$. If $\alpha > 1$, $x(\alpha t)$ is a time compressed version of $x(t)$, where as if $0 < \alpha < 1$, we have a time expanded version of $x(t)$. Let $x(t)$ be as shown in Fig. 1.9(a). Then $x(-2t)$ would be as shown in Fig. 1.9(b).

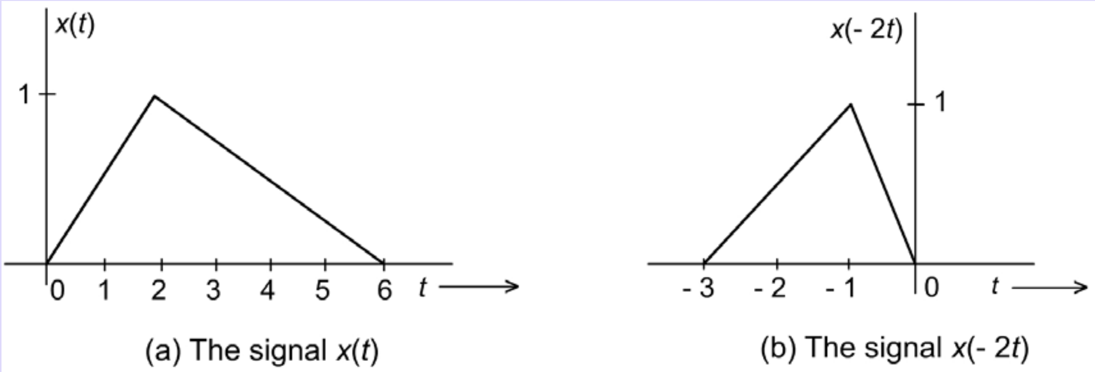


Fig. 1.9: A triangular pulse and its time compressed and reversed version

For the special case $\alpha = -1$, we have the transform pair $x(-t) \longleftrightarrow X(-f)$. That is, both the time function and its Fourier transform undergo reversal. As an example, we know that

$$ex1(t) \longleftrightarrow \frac{1}{1 + j2\pi f}$$

Hence $ex1(-t) \longleftrightarrow \dfrac{1}{1 - j2\pi f}$

$$ex1(t) + ex1(-t) = ex2(t)$$

Using the linearity property of the Fourier transform, we obtain the transform pair

$$ex2(t) = \exp(-|t|) \longleftrightarrow \frac{1}{1 + j2\pi f} + \frac{1}{1 - j2\pi f}$$

$$= \frac{2}{1 + (2\pi f)^2}$$

Consider $x(\alpha t)$ with $\alpha = 2$. Then,

$$x(2t) \longleftrightarrow \frac{1}{2} X\left(\frac{f}{2}\right)$$

Let us compare $X(f)$ with $\frac{1}{2} X\left(\frac{f}{2}\right)$ by taking $x(t) = ex1(t)$, and $y(t) = ex1(2t)$. That is,

$$y(t) = \begin{cases} e^{-2t}, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases}$$

$$Y(f) = \frac{1}{2}\left[\frac{1}{1 + j2\pi(f/2)}\right]$$

$$= \frac{1}{2 + j2\pi f}$$

Let $X(f) = |X(f)| e^{\theta_x(f)}$ and

$Y(f) = |Y(f)| e^{\theta_y(f)}$, where $|Y(f)| = \dfrac{1}{2\sqrt{1 + \pi^2 f^2}}$

$$\theta_y(f) = -arc\tan(\pi f)$$

Fig. 1.10 gives the plots of $X(f)$ and $Y(f)$. In Fig. 1.10(a), we have the plots $|X(f)|\, vs. f$ and $|Y(f)|\, vs. f$. In Fig. 1.10(b), we have the plot of $|Y(f)|$ normalized to have the maximum value of unity. This plot is denoted by $|Y_N(f)|$. Fig. 1.10(c) gives the plots of $\theta_x(f)$ and $\theta_y(f)$. From Fig. 1.10(b), we see that

i) $y(t) = x(2t)$ has a much wider spectral width as compared to the spectrum of the original signal. (In fact, if $X(f)$ is band limited to W Hz, then $X\left(\frac{f}{2}\right)$ will be band limited to $2W$ Hz.)

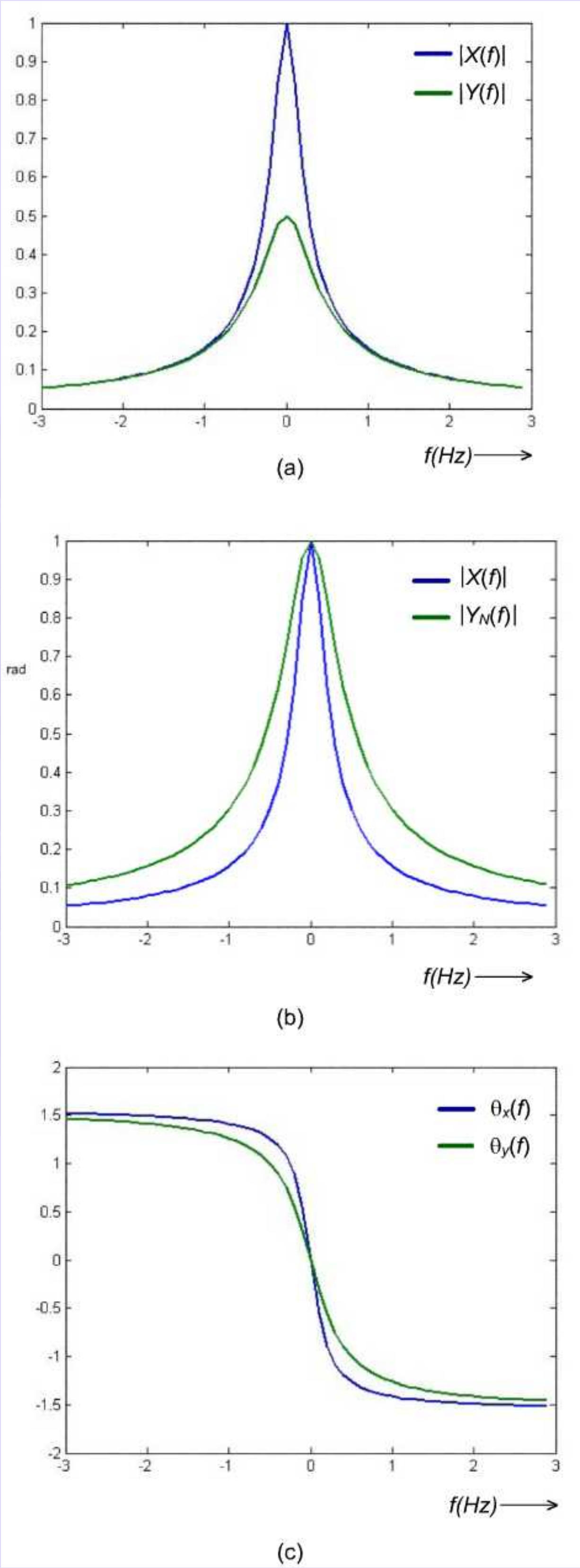


Fig 1.10: Spectral plots of $ex1(t)$ and $ex1(2t)$

(ii) Let $E(f) = |X(f)| - |Y(f)|$. Then the value of $E(f)$ is dependent on f; that is, the original spectral magnitudes are modified by different amounts at different frequencies (Note that $|Y(f)| \neq k|X(f)|$ where k is a constant). In other words, $|Y(f)|$ is a distorted version of $|X(f)|$.

(iii) From Fig. 1.10(c), we observe that $\theta_y(f) \neq \theta_x(f)$ and their difference is a function of frequency; that is $\theta_y(f)$ is a distorted version of $\theta_x(f)$.

In summary, time compression would result either in the introduction of new, higher frequency components (if the original signal is band limited) or making the latter part of the original spectrum much more significant; the remaining spectral components are distorted (both in amplitude and phase). On the other hand, time expansion would result either in the loss or attenuation of higher frequency components, and distortion of the remaining spectrum.

Let $x(t)$ represent an audio signal band limited to 10 kHz. Then $x(2t)$ will have a spectral components upto 20 kHz. These higher frequency components will impart shrillness to the audio, besides distorting the original signal. Similarly, if the audio is compressed, we have loss of “sharpness” in the resulting signal and severe distortion. This property of the FT will now be demonstrated with the help of a recorded audio signal.

P4a) Time shift

If $x(t) \longleftrightarrow X(f)$ then,

$$x(t - t_0) \longleftrightarrow e^{-j2\pi f t_0} X(f)$$

If t_0 is positive, then $x(t - t_0)$ is a delayed version of $x(t)$ and if t_0 is negative, the $x(t - t_0)$ is an advanced version of $x(t)$. In any case, time shifting will simply result in the multiplication of $X(f)$ by a linear phase factor. This implies that $x(t)$ and $x(t - t_0)$ have the same magnitude spectrum.

Proof: Let $\lambda = (t - t_0)$. Then,

$$F\left[x(t - t_0)\right] = \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f(\lambda + t_0)} d\lambda$$

$$= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f \lambda} d\lambda$$

$$= e^{-j2\pi f t_0} X(f) \qquad \blacklozenge$$

P4b) Frequency Shift

If $x(t) \longleftrightarrow X(f)$, then

$$e^{\pm j2\pi f_c t} x(t) \longleftrightarrow X(f \mp f_c)$$

where f_c is a real constant. (This property is also known as *modulation theorem*).

Proof: Exercise

As an application of the above result, let us consider the spectrum of $y(t) = 2\cos(2\pi f_c t)\, x(t)$; that is, we want the Fourier transform of $\left[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right] x(t)$. From the frequency shift theorem, we have $y(t) = 2\cos(2\pi f_c t)\, x(t) \longleftrightarrow Y(f) = X(f - f_c) + X(f + f_c)$. If $X(f)$ is as shown in Fig. 1.11(a), then $Y(f)$ will be as shown in Fig. 1.11(b) for $f_c = W$.

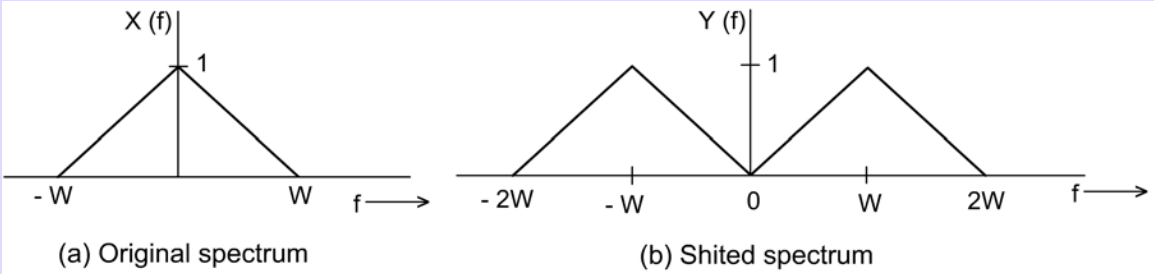


Fig.1.11: Illustration of modulation theorem

P5) Duality

If we look fairly closely at the two equations constituting the Fourier transform pair, we find that there is a great deal of similarity between them. In Eq.1.11a, $'f'$ is the variable of integration where as in Eq. 1.11b, it is the variable $'t'$. The sign of the exponent is positive in Eq. 1.11a where as it is negative in Eq. 1.11b. Both t and f are variables of the continuous type. This results in an interesting property, namely, the *duality property*, which is stated below.

If $x(t) \longleftrightarrow X(f)$, then

$$X(t) \longleftrightarrow x(-f) \text{ and } X(-t) \longleftrightarrow x(f)$$

Note: This is one instance, where the variable t is associated with a function denoted using the upper case letter and the variable f is associated with a function denoted using a lower case letter.

Proof: $x(t) = \int\limits_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$

$$x(-t) = \int\limits_{-\infty}^{\infty} X(f) e^{-j2\pi f t} df$$

The result follows by interchanging the variables t and f. The proof of the second part of the property is similar. ♦

Duality theorem helps us in creating additional transform pairs, from the given set. We shall illustrate the duality property with the help of a few examples.

Example 1.4

If $z(t) = A \operatorname{sinc} 2Wt$, let us use duality to find $Z(f)$.

We look for a transform pair, $x(t) \longleftrightarrow X(f)$ where in $X(f)$, if we replace f by t, we have $z(t) = A \operatorname{sinc} 2Wt$.

We know that if $x(t) = \dfrac{A}{2W} ga\left(\dfrac{t}{2W}\right)$, then,

$$X(f) = A \operatorname{sinc}(2Wf) \text{ and}$$

$$X(t) = A \operatorname{sinc}(2Wt) = z(t) \Rightarrow$$

$$Z(f) = x(-f) = \frac{A}{2W} ga\left(-\frac{f}{2W}\right)$$

As $ga(-f) = ga(f)$, we have

$$Z(f) = \left(\frac{A}{2W}\right) ga\left(\frac{f}{2W}\right).$$

♦

Example 1.5

Find the Fourier transform of $z(t) = \dfrac{2}{1+t^2}$.

We know that if $y(t) = e^{-|t|}$, then $Y(f) = \dfrac{2}{1+(2\pi f)^2}$

Let $x(t) = y(\alpha t)$, with $\alpha = 2\pi$.

Then $X(f) = \dfrac{1}{2\pi} Y\left(\dfrac{f}{2\pi}\right)$

$$= \frac{1}{2\pi} \frac{2}{1+f^2}$$

or $2\pi X(f) = \dfrac{2}{1+f^2}$

As $z(t) = \dfrac{2}{1+f^2}$ with f being replaced by t, we have

$$Z(f) = 2\pi x(-f)$$

$$= 2\pi e^{-|2\pi f|}$$

Hence $\dfrac{2}{1+t^2} \longleftrightarrow 2\pi e^{-|2\pi f|}$ ♦

P6) Conjugate functions

If $x(t) \longleftrightarrow X(f)$, then $x^*(t) \longleftrightarrow X^*(-f)$

Proof: $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$

$$X^*(f) = \int_{-\infty}^{\infty} x^*(t) e^{j2\pi f t} dt$$

$$X^*(-f) = \int_{-\infty}^{\infty} x^*(t) e^{-j2\pi f t} dt$$

Hence the result. ♦

From the time reversal property, we get the additional relation, namely

$x^*(-t) \longleftrightarrow X^*(f)$

Exercise 1.2: Let $x(t) = x_R(t) + j x_I(t)$

where $x_R(t)$ is the real part and $x_I(t)$ is the imaginary part of $x(t)$. Show that

$$x_R(t) \longleftrightarrow \frac{1}{2}\left[X(f) + X^*(-f)\right]$$

$$x_I(t) \longleftrightarrow \frac{1}{2j}\left[X(f) - X^*(-f)\right]$$

Def. 1.3(a): A signal $x(t)$ is called **conjugate symmetric**, if $x(-t) = x^*(t)$.

If $x(t)$ is real, then $x(t)$ is **even** if $x(-t) = x(t)$.

Def. 1.3(b): A signal $x(t)$ is said to be **conjugate anti-symmetric** if

$$x(-t) = -x^*(t).$$

If $x(t)$ is real, then $x(t)$ is **odd** if $x(-t) = -x(t)$. ♦

If $x(t)$ is real, then $x(t) = x^*(t)$.

As a result, $X(f) = X^*(-f)$ or $X^*(f) = X(-f)$.

Hence, the spectrum for the negative frequencies is the complex conjugate of the positive part of the spectrum. This implies, that for real signals,

$$|X(-f)| = |X(f)|$$

$$\theta(-f) = -\theta(f)$$

Going one step ahead, we can show that if $x(t)$ is real and even, then $X(f)$ is also real and even. (Example: $e^{-|t|} \longleftrightarrow \dfrac{2}{1+(2\pi f)^2}$). Similarly, if $x(t)$ is real and odd, its transform is purely imaginary and odd (See Example 1.7).

P7a) Multiplication in the time domain

If $x_1(t) \longleftrightarrow X_1(f)$

$x_2(t) \longleftrightarrow X_2(f)$

then, $x_1(t)\, x_2(t) \longleftrightarrow \displaystyle\int_{-\infty}^{\infty} X_1(\lambda)\, X_2(f-\lambda)\, d\lambda = \int_{-\infty}^{\infty} X_2(\lambda)\, X_1(f-\lambda)\, d\lambda$

The integrals on the R.H.S represent the convolution of $X_1(f)$ and $X_2(f)$. We denote the convolution of $X_1(f)$ and $X_2(f)$ by $X_1(f) * X_2(f)$.

(Note that $*$ in between two functions represents the convolution of the two quantities where as a superscript, it denotes the complex conjugate)

Proof: Exercise

P7b) Multiplication of Fourier transforms

Let $x_1(t) \longleftrightarrow X_1(f)$

$x_2(t) \longleftrightarrow X_2(f)$

then, $F^{-1}\left[X_1(f)\, X_2(f)\right] = \int\limits_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$

$$= \int\limits_{-\infty}^{\infty} x_2(\lambda)\, x_1(t-\lambda)\, d\lambda$$

As any one of the above integrals represent the convolution of $x_1(t)$ and $x_2(t)$, we have

$x_1(t) * x_2(t) \longleftrightarrow X_1(f)\; X_2(f)$

Proof: Let $x_3(t) = x_1(t) * x_2(t)$

That is, $x_3(t) = \int\limits_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$

$$X_3(f) = F\left[x_3(t)\right] = \int\limits_{-\infty}^{\infty} x_3(t)\, e^{-j2\pi f t}\, dt = \int\limits_{-\infty}^{\infty}\left[\int\limits_{-\infty}^{\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda\right] e^{-j2\pi f t}\, dt$$

Rearranging the integrals,

$$X_3(f) = \int\limits_{-\infty}^{\infty} x_1(\lambda)\left[\int\limits_{-\infty}^{\infty} x_2(t-\lambda)\, e^{-j2\pi f t}\, dt\right] d\lambda$$

But the bracketed quantity is the Fourier transform of $x_2(t-\lambda)$. From the property P4(a), we have

$x_2(t-\lambda) \longleftrightarrow e^{-j2\pi f\lambda}\, X_2(f)$

Hence,

$$X_3(f) = \int_{-\infty}^{\infty} x_1(\lambda)\, X_2(f) e^{-j2\pi f \lambda}\, d\lambda$$

$$= X_2(f) \int_{-\infty}^{\infty} x_1(\lambda) e^{-j2\pi f \lambda}\, d\lambda$$

$$= X_1(f)\; X_2(f)$$ ♦

Property P7(b), known as the *Convolution theorem*, is one of the very useful properties of the Fourier transform.

The concept of convolution is very basic in the theory of signals and systems. As will be shown later, the input and output of a linear, time- invariant system are related by the convolution integral. For a fairly detailed treatment of the properties of systems, convolution integral etc. the student is advised to refer to [1 - 3].

Example 1.6

In this example, we will find the Fourier transform of $T\, tri\left(\frac{t}{T}\right)$.

$T\, tri\left(\frac{t}{T}\right)$ can be obtained as the convolution of $ga\left(\frac{t}{T}\right)$ with itself. That is,

$$ga\left(\frac{t}{T}\right) * ga\left(\frac{t}{T}\right) = T\, tri\left(\frac{t}{T}\right)$$

As $ga\left(\frac{t}{T}\right) \longleftrightarrow T \operatorname{sinc}(fT)$, we have

$$T\, tri\left(\frac{t}{T}\right) \longleftrightarrow \left[T \operatorname{sinc}(fT)\right]^2$$ ♦

P8) Differentiation

P8a) Differentiation in the time domain

Let $x(t) \longleftrightarrow X(f)$,

then, $\frac{d}{dt}\left[x(t)\right] \longleftrightarrow j2\pi f\left[X(f)\right]$

Generalizing, $\frac{d^n x(t)}{dt^n} \longleftrightarrow (j2\pi f)^n X(f)$

Proof: We shall prove the first part; generalization follows as a consequence this. We have,

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

$$\frac{d}{dt}\left[x(t)\right] = \frac{d}{dt}\left[\int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df\right]$$

Interchanging the order of differentiation and integration on the RHS,

$$\frac{d}{dt}\left[x(t)\right] = \int_{-\infty}^{\infty} X(f)\left[\frac{d}{dt} e^{j2\pi f t}\right] df$$

$$= \int_{-\infty}^{\infty} \left[j2\pi f \, X(f)\right] e^{j2\pi f t} df$$

From the above, we see that $F^{-1}\left[j2\pi f \, X(f)\right]$ is $\frac{d}{dt} x(t)$. Hence the property. ♦

Example 1.7

Let us find the FT of the doublet pulse $x(t)$ shown in Fig. 1.12 below.

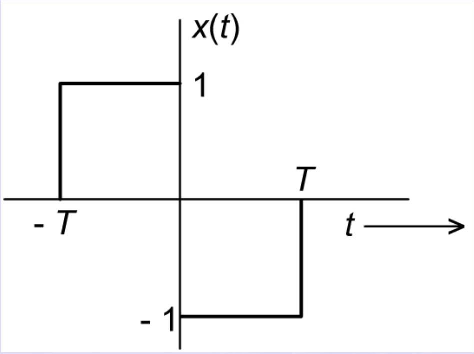


Fig 1.12: $x(t)$ of Example 1.7

$x(t) = \frac{d}{dt}\left[T\, tri\left(\frac{t}{T}\right)\right]$. Hence,

$$X(f) = j2\pi f\, F\left[T\, tri\left(\frac{t}{T}\right)\right]$$

$$= (j2\pi f)T^2 \operatorname{sinc}^2(fT), \text{ (using the result of example 1.6)}$$

$$= (j2\pi f)T^2 \frac{\sin^2(\pi f T)}{(\pi f T)(\pi f T)}$$

$$= j2T \frac{\sin(\pi f T)}{(\pi f T)} \sin(\pi f T)$$

$$= j2T \operatorname{sinc}(\pi f T)\ \sin(\pi f T)$$ ♦

As a consequence of property P8(a), we have the following interesting result.

Let $x_3(t) = x_1(t) * x_2(t)$

Then $X_3(f) = X_1(f)\, X_2(f)$

$$(j2\pi f)\, X_3(f) = \left[(j2\pi f)\, X_1(f)\right] X_2(f)$$

$$= X_1(f)\left[j2\pi f\, X_2(f)\right]$$

That is,

$$\frac{d}{dt}\left[x_1(t) * x_2(t)\right] = \frac{d}{dt}\left[x_1(t)\right] * x_2(t)$$

$$= x_1(t) * \frac{d}{dt}\left[x_2(t)\right]$$

P8b) Differentiation in the frequency domain

Let $x(t) \longleftrightarrow X(f)$.

Then, $\left[(-j 2\pi t) x(t)\right] \longleftrightarrow \dfrac{d\, X(f)}{df}$

Generalizing, $\left[(-j 2\pi t)^n x(t)\right] \longleftrightarrow \dfrac{d^n\, X(f)}{df^n}$

Proof: Exercise

The generalized property can also be written as

$$t^n x(t) \longleftrightarrow \left(\frac{j}{2\pi}\right)^n \frac{d^n X(f)}{df^n}$$

Example 1.8

Find the Fourier transform of $x(t) = t \; ex1\left(\frac{t}{T}\right)$.

We have, $ex1(t) \longleftrightarrow \dfrac{1}{1 + j 2\pi f}$

Hence, $ex1\left(\frac{t}{T}\right) \longleftrightarrow T \; \dfrac{1}{1 + j 2\pi f T}$

$$t\, ex1\left(\frac{t}{T}\right) \longleftrightarrow \frac{j}{2\pi} \frac{d}{df}\left[\frac{T}{1 + j 2\pi f T}\right]$$

$$\longleftrightarrow \frac{j}{2\pi} T \frac{-(j 2\pi T)}{(1 + j 2\pi f T)^2}$$

$$\longleftrightarrow \left[\frac{T^2}{(1+j2\pi f T)^2}\right] \qquad \blacklozenge$$

P9) Integration in time domain

This property will be developed subsequently.

P10) Rayleigh's energy theorem

This theorem states that, E_x, energy of the signal $x(t)$, is

$$E_x = \int_{-\infty}^{\infty} |X(f)|^2 df$$

This result follows from the more general relationship, namely,

$$\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = \int_{-\infty}^{\infty} X_1(f)\, X_2^*(f) df$$

Proof: We have

$$\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = \int_{-\infty}^{\infty} x_1(t) \left[\int_{-\infty}^{\infty} X_2(f)\, e^{j2\pi f t} df\right]^* dt$$

$$= \int_{-\infty}^{\infty} X_2^*(f) \left[\int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi f t} dt\right] df$$

$$= \int_{-\infty}^{\infty} X_2^*(f)\, X_1(f) df$$

If $x_1(t) = x_2(t) = x(t)$, then

$$\int_{-\infty}^{\infty} \left|x(t)^2\right| dt = E_x = \int_{-\infty}^{\infty} \left|X(f)^2\right| df \qquad \blacklozenge$$

Note: If $x_1(t)$ and $x_2(t)$ are real, then,

$$\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt = \int_{-\infty}^{\infty} X_1(f)\, X_2(-f)\, df = \int_{-\infty}^{\infty} X_2(f)\, X_1(-f)\, df$$

Property P10 enables us to compute the energy of a signal from its magnitude spectrum. In a few situations, this may be easier than computing the energy in the time domain. (Some authors refer to this result as Parseval's theorem)

Example 1.9

Let us find the energy of the signal $x(t) = 2AW \operatorname{sinc}(2Wt)$.

$$E_x = \int_{-\infty}^{\infty} \left[2AW \operatorname{sinc}(2Wt)\right]^2 dt$$

In this case, it would be easier to compute E_x based on $X(f)$. From Example 1.4, $X(f) = A\ ga\left(\frac{f}{2W}\right)$. Hence,

$$E_x = \int_{-W}^{W} A^2\, df = 2W A^2 \qquad \blacklozenge$$

More important than the calculation of the energy of the signal, Rayleigh's energy theorem enables to treat $|X(f)|^2$ as the ***energy spectral density*** of $x(t)$. That is, $|X(f_1)|^2 df$ is the energy in the incremental frequency interval df, centered at $f = f_1$. Let $\int_{-W}^{W} |X(f)|^2 df = 0.9 E_x$. Then, 90 percent of the energy of signal is confined to the interval $|f| \le W$. Consider the rectangular pulse $ga\left(\frac{t}{T}\right)$. The first nulls of the magnitude spectrum occur at $f = \pm\frac{1}{T}$. The evaluation of

$\int_{-1/T}^{1/T} |X(f)|^2 df = \int_{-1/T}^{1/T} T \operatorname{sinc}^2(fT)\, df$ will yield the value $0.92\,T$, which is 92 percent of the total energy of $ga\left(\frac{t}{T}\right)$. Hence, the frequency range $\left(-\frac{1}{T}, \frac{1}{T}\right)$ can be taken to be the spectral width of the rectangular pulse. [The interval $\left(-\frac{2}{T}, \frac{2}{T}\right)$ may result in about 95 percent of the total energy].

1. 5 Unified Approach to Fourier Transform

So far, we have represented the periodic functions by Fourier series and the aperiodic functions by Fourier transform. The question arises: is it possible to unify these two approaches and talk only in terms of say, Fourier transform? The answer is yes provided we are willing to introduce **Impulse Functions** both in time and frequency domains. This would also enable us to have Fourier transforms for signals that do not satisfy one or more of the Dirichlet's conditions (for the existence of the Fourier transform).

1.5.1 Unit impulse (Dirac delta function)

Impulse function is not a function in its strict sense [Note that a function $f(\ )$, takes a number y and a produces another number, $f(y)$]. It is a *distribution* or *generalized function*. A distribution is defined in terms of its effect on another function. The symbol $\delta(t)$ is fairly common in the technical literature to denote the unit impulse. We define the unit impulse as any (generalized) function that satisfies the following conditions:

(i) $\delta(t) = 0, \quad t \neq 0$ (1.13a)

(ii) $\delta(t) = \infty, \quad t = 0$ (1.13b)

(iii) Let $p(t)$ be any ordinary function, then

$$\int_{-\infty}^{\infty} p(t)\,\delta(t)\,dt = \int_{-\varepsilon}^{\varepsilon} p(t)\,\delta(t)\,dt = p(0), \quad \varepsilon > 0 \tag{1.13c}$$

(ε could be infinitesimally small)

If $p(t) = 1$, for $|t| \leq \varepsilon$, then we have

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt = \int_{-\infty}^{\infty} \delta(t)\,dt = 1 \tag{1.13d}$$

From Eq. 1.13(c), we see that $\delta(t)$ operates on a function such as $p(t)$ and produces the number, namely, $p(0)$. As such $\delta(t)$ falls between a function and a transform (A transform operates on a function and produces a function).

A number of conventional functions have a limiting behavior that approaches $\delta(t)$. We cite a few such functions below:

Let

(a) $p_1(t) = \dfrac{1}{\varepsilon}\, ga\left(\dfrac{t}{\varepsilon}\right)$ (b) $p_2(t) = \dfrac{1}{\varepsilon}\, tri\left(\dfrac{t}{\varepsilon}\right)$ (c) $p_3(t) = \dfrac{1}{\varepsilon}\, \mathrm{sin}c\left(\dfrac{t}{\varepsilon}\right)$

Then, $\lim\limits_{\varepsilon \to 0} p_i(t) = \delta(t)$, $i = 1, 2, 3$. $p_3(t)$ is shown below in Fig. 1.13.

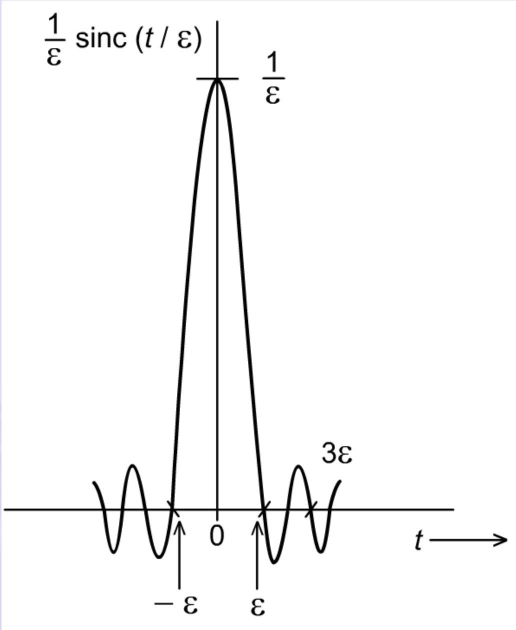


Fig. 1.13: $\text{sin}c\,(\;)$ with the limiting behavior of $\delta(t)$

$\left(\text{Note that } \frac{1}{\varepsilon}\,\text{sin}c\left(\frac{t}{\varepsilon}\right) \longleftrightarrow ga\,(\varepsilon f)\text{. Hence the area under the time function } = 1.\right)$

From the above examples, we see that the shape of the function is not very critical; its area should remain at 1 in order to approach $\delta(t)$ in the limit.

By delaying $\delta(t)$ by t_0 and scaling it by A, we have $A\,\delta(t - t_0)$. This is normally shown as a spear (Fig. 1.14) with the *weight* or *area* of the impulse shown in parentheses very close to it.

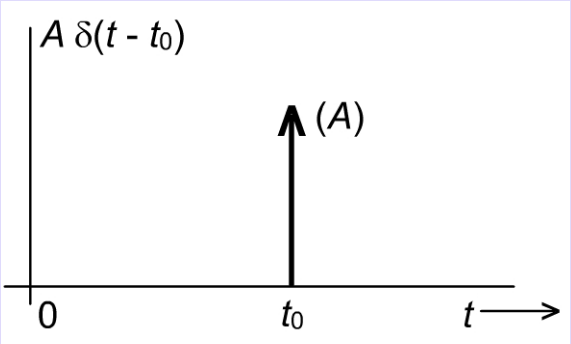


Fig. 1.14: Symbol for $A\,\delta(t - t_0)$

Some properties of unit impulse

P1) Sampling (or sifting) property

Let $p(t)$ be any ordinary function. Then for $a < t_0 < b$,

$$\int_a^b p(t)\,\delta(t - t_0)\,dt = p(t_0)$$

(This is generalization of condition (iii)). Proof follows from making the change of variable $t - t_0 = \tau$ and noting $\delta(\tau)$ is zero for $\tau \neq 0$. Note that for the sampling property, the values of $p(t)$, $t \neq t_0$ are of no consequence.

P2) Replication property

Let $p(t)$ be any ordinary function. Then,

$$p(t) * \delta(t - t_0) = p(t - t_0)$$

The proof of this property follows from the fact, that in the process of convolution, every value of $p(t)$ will be sampled and shifted by t_0 resulting in $p(t - t_0)$.

(Note: Some authors use this property as the operational definition of impulse function.)

P3) Scaling Property

$$\delta(\alpha t) = \frac{1}{|\alpha|}\,\delta(t), \quad \alpha \neq 0$$

Proof: (i) Let $\alpha t = \tau$, $\alpha > 0$

$$\int_{-\infty}^{\infty} \delta(\alpha t)\, dt = \int_{-\infty}^{\infty} \frac{1}{\alpha}\,\delta(\tau)\, d\tau = \frac{1}{\alpha}$$

$$= \frac{1}{\alpha}\int_{-\infty}^{\infty} \delta(t)\, dt$$

$$= \frac{1}{|\alpha|}\int_{-\infty}^{\infty} \delta(t)\, dt$$

(ii) Let $\alpha < 0$; that is $\alpha = -|\alpha|$, and let $-|\alpha|\, t = \tau$.

$$\int_{-\infty}^{\infty} \delta(\alpha t)\, dt = \int_{-\infty}^{\infty} \frac{1}{|\alpha|}\,\delta(\tau)\, d\tau = \frac{1}{|\alpha|}$$

$$= \frac{1}{|\alpha|}\int_{-\infty}^{\infty} \delta(t)\, dt \qquad \blacklozenge$$

It is easy to show that $\delta\left[\alpha(t - t_0)\right] = \frac{1}{|\alpha|}\,\delta(t - t_0)$.

Special Case: If $\alpha = -1$, we have the result $\delta(-t) = \delta(t)$.

The above result is not surprising, especially if we look at the examples $p_1(t)$ to $p_3(t)$, which are all even functions of t. Hence some authors call this as the *even sided delta function.* It is also possible to come up with delta functions as a limiting case of functions that are not even; that is, as a limiting case of one-sided functions. In such a situation we have a *left-sided delta function* or *right-sided delta function* etc. Left-sided delta function will prove to be useful in the context of probability density functions of certain random variables, subject matter of chapter 2.

Example 1.10

Find the value of

(a) $\int_{-4}^{4} t^3 \, \delta(t-5) \, dt$

(b) $\int_{4.9}^{5.1} t^3 \, \delta(t-5) \, dt$

(a) $\delta(t-5)$ is nonzero only at $t = 5$. The range of integration does not include the impulse. Hence the integral is zero.

(b) As the range of integration includes the impulse, we have a nonzero value for the product $t^3 \, \delta(t-5)$. As $\delta(t-5)$ occurs at $t = 5$, we can write

$$t^3 \, \delta(t-5) = 5^3 \, \delta(t-5).$$

Hence,

$$\int_{4.9}^{5.1} t^3 \, \delta(t-5) \, dt = 5^3 \int_{4.9}^{5.1} \delta(t-5) \, dt = 125$$ ♦

Example 1.11

Let $p(t) = tri\left(\frac{t}{4}\right)$. Find $\left[p(t) * \delta(2t-1)\right]$.

$$\delta(2t-1) = \delta\left[2\left(t - \frac{1}{2}\right)\right] = \frac{1}{2}\left[\delta\left(t - \frac{1}{2}\right)\right]$$

$$p(t) * \frac{1}{2}\left[\delta\left(t - \frac{1}{2}\right)\right] = \frac{1}{2} \, p\left(t - \frac{1}{2}\right) = \frac{1}{2} \, tri\left(\frac{t - 1/2}{4}\right)$$ ♦

Let us now compute the Fourier transform of $\delta(t)$. From Eq. 1.11(b), we have,

$$F\left[\delta(t)\right] = \int_{-\infty}^{\infty} \delta(t) \, e^{-j2\pi f t} \, dt = 1 \qquad (1.14a)$$

How do we interpret this result? *The spectrum of the unit impulse consists of frequency components in the range* $(-\infty, \infty)$, *all with unity magnitude and zero phase shift,* a fascinating result indeed! Hence exciting any electric network or system with a unit impulse is equivalent to exciting the network simultaneously with complex exponentials of all possible frequencies, all with the same magnitude (unity in this case) and zero phase shift. That is, the unit impulse response of a linear network is the synthesis of responses to the individual complex exponentials and we intuitively feel that the impulse response of a network should be able to characterize the system in the time domain. (We shall see a little later that if the network is linear and time invariant, a simple relation exists between the input to the network, its impulse response and the output).

The dual of the Fourier transform pair of Eq. 1.14(a) gives us

$$1 \longleftrightarrow \delta(-f) = \delta(f) \tag{1.14b}$$

Based on Eq. 1.14(a) and Eq. 1.14(b), we make the following observation: *a constant in one domain will transform into an impulse in the other domain.*

Eq. 1.14(b) is intuitively satisfying; a constant signal has no time variations and hence its spectral content ought to be confined to $f = 0$; $\delta(f)$ is the proper quantity for the transform because it is zero for $f \neq 0$ and its inverse transform yields the required constant in time (note that only an impulse can yield a nonzero value when integrated over zero width).

Because of the transform pair,

$$1 \longleftrightarrow \delta(f),$$

we obtain another transform pair (from modulation theorem)

$$e^{j 2 \pi f_0 t} \longleftrightarrow \delta(f - f_0) \tag{1.15a}$$

$$e^{-j 2 \pi f_0 t} \longleftrightarrow \delta(f + f_0) \tag{1.15b}$$

As $\cos \omega_0 t = \dfrac{e^{j 2\pi f_0 t} + e^{-j 2\pi f_0 t}}{2}$, we have,

$$\cos \omega_0 t \longleftrightarrow \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] \quad (1.16)$$

Similarly,

$$\sin \omega_0 t \longleftrightarrow \frac{1}{2j}\left[\delta(f - f_0) - \delta(f + f_0)\right] \quad (1.17)$$

$F\left[\cos \omega_0 t\right]$ and $F\left[\sin \omega_0 t\right]$ are shown in Fig. 1.15.

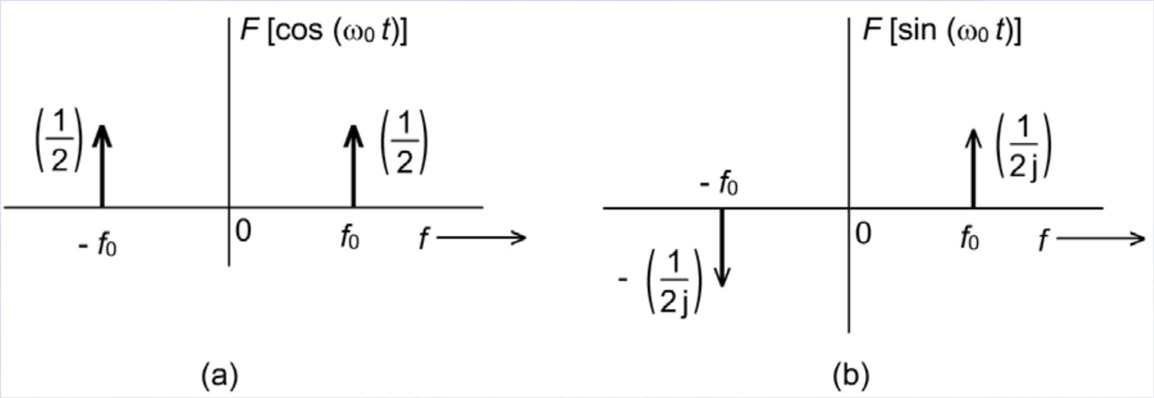


Fig. 1.15: Fourier transforms of (a) $\cos \omega_0 t$ and (b) $\sin \omega_0 t$

Note that the impulses in Fig. 1.15(b) have weights that are complex. It is fairly conventional to show the spectrum of $\sin \omega_0 t$ as depicted in Fig. 1.15(b); or else we can make two separate plots, one for magnitude and the other for phase, where the magnitude plot is identical to that shown in Fig. 1.15(a) and the phase plot has values of $-\frac{\pi}{2}$ at $f = f_0$ and $+\frac{\pi}{2}$ at $f = -f_0$.

In summary, we have found the Fourier transform of $\delta(t)$ (a time function with a discontinuity that is not finite), and using impulses in the frequency domain, we have developed the Fourier transforms of the periodic signals such as $e^{\pm j 2 \pi f_0 t}$, $\cos \omega_0 t$ and $\sin \omega_0 t$, which are neither absolutely integrable nor square integrable.

We are now in a position to present both Fourier series and Fourier transform in a unified framework and talk only of Fourier transform whether the signal is aperiodic or not. This is because, for a periodic signal $x_p(t)$, we have the Fourier series relation,

$$x_p(t) = \sum_{n = -\infty}^{\infty} x_n \, e^{j 2 \pi n f_0 t}$$

Taking the Fourier transform on both the sides,

$$F\left[x_p(t)\right] = X_p(f) = F\left[\sum_{n = -\infty}^{\infty} x_n \, e^{j 2 \pi n f_0 t}\right]$$

$$= \sum_{n = -\infty}^{\infty} x_n \, F\left[e^{j 2 \pi n f_0 t}\right]$$

$$= \sum_{n = -\infty}^{\infty} x_n \, \delta(f - n f_0) \qquad (1.18)$$

FT of $x_p(t)$ is a function of the continuous variable f, whereas, in the FS representation of $x_p(t)$, x_n is a function of the discrete variable n. However, as $X_p(f)$ is purely impulsive, spectral components exist only at $f = n f_0$, with complex weights x_n. As inversion of $X_p(f)$ requires integration, we require impulses in the spectrum. As such, the differences between the line spectrum of sec. 1.1 and spectral representation given by Eq. 1.18 are only minor in nature. They both provide the same information, differing essentially only in notation.

There is an interesting relation between x_n and the Fourier transform of one period of a periodic signal. Let,

$$x(t) = \begin{cases} x_p(t) \ , & -\frac{T_0}{2} < t < \frac{T_0}{2} \\ 0 \ , & outside \end{cases}$$

$$x_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_p(t)\, e^{-j 2\pi n f_0 t}\, dt$$

$$= \frac{1}{T_0} \left[\int_{-T_0/2}^{T_0/2} x(t)\, e^{-j 2\pi n f_0 t}\, dt \right]$$

$$= \frac{1}{T_0} \left[\int_{-\infty}^{\infty} x(t)\, e^{-j 2\pi n f_0 t}\, dt \right] = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t)\, e^{-j 2\pi \left(\frac{n}{T_0}\right) t}\, dt$$

The bracketed quantity is $X(f)\Big|_{f = \frac{n}{T_0}}$

Hence, $x_n = \frac{1}{T_0} X\left(\frac{n}{T_0}\right)$ (1.19)

Example 1.12

Find the Fourier transform of the uniform impulse train $x_p(t) = \sum_{n = -\infty}^{\infty} \delta(t - nT_0)$ shown in Fig 1.16 below.

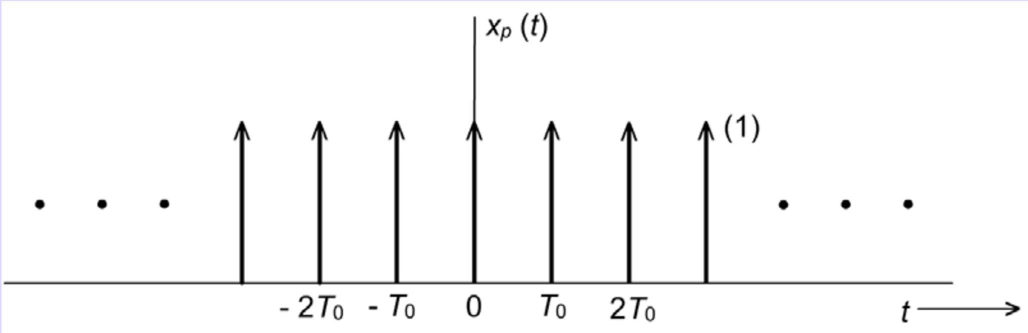


Fig. 1.16: Uniform impulse train

Let $x(t)$ be one period of $x_p(t)$ in the interval, $-\frac{T_0}{2} < t < \frac{T_0}{2}$. Then, $x(t) = \delta(t)$ for this example. But as $\delta(t) \longleftrightarrow 1$, from Eq. (1.19) we have, $x_n = \frac{1}{T_0}$ for all n. Hence,

$$X_p(f) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - n f_0) \qquad \blacklozenge \qquad (1.20)$$

From Eq. 1.20, we have another interesting result:

A uniform periodic impulse train in either domain will transform into another uniform impulse train in the other domain.

From the transform pair, $\delta(t) \longleftrightarrow 1$, we have

$$F^{-1}[1] = \int_{-\infty}^{\infty} e^{j 2\pi f t}\, df = \delta(t)$$

As $\delta(-t) = \delta(t)$, we have, $\int_{-\infty}^{\infty} e^{-j 2\pi f t}\, df = \delta(t)$

That is, $$\int_{-\infty}^{\infty} e^{\pm j 2\pi f t}\, df = \delta(t) \qquad (1.21)$$

Using Eq. 1.21, we show that $x(t)$ and $X(f)$ constitute a transform pair.

Let $$\hat{x}(t) = \int_{-\infty}^{\infty} X(f)\, e^{j 2\pi f t}\, df$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\lambda)\, e^{-j 2\pi f \lambda}\, d\lambda \right] e^{j 2\pi f t}\, df$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\lambda)\, e^{j 2\pi f (t - \lambda)}\, df\, d\lambda$$

$$= \int_{-\infty}^{\infty} x(\lambda) \left[\int_{-\infty}^{\infty} e^{j 2\pi f (t - \lambda)}\, df \right] d\lambda$$

$$= \int_{-\infty}^{\infty} x(\lambda)\, \delta(t - \lambda)\, d\lambda = x(t)$$

As $\hat{x}(t) = x(t)$, we have $X(f)$ uniquely representing $x(t)$.

1.5.2 Impulse response and convolution

Let $x(t)$ be the input to a Linear, Time-Invariant (LTI) system resulting in the output, $y(t)$. We shall now establish a relation between $x(t)$ and $y(t)$.

From the replication property of the impulse, we have,

$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t - \tau)\, d\tau$$

Let $\mathcal{R}\left[\delta(t)\right]$ denote the output (response) of the LTI system, when the input is exited by the unit impulse $\delta(t)$. This is generally denoted by the symbol $h(t)$ and is called the *impulse response* of the system. That is, when $\delta(t)$ is input to an LTI system, its output $y(t) = \mathcal{R}\left[\delta(t)\right] = h(t)$. As the system is time invariant, $\mathcal{R}\left[\delta(t - \tau)\right] = h(t-\tau)$.

As the system is linear, $\mathcal{R}\left[x(\tau)\, \delta(t - \tau)\right] = x(\tau)\, h(t - \tau)$

and $\mathcal{R}\left[x(t)\right] = y(t) = \mathcal{R}\left[\int_{-\infty}^{\infty} x(\tau)\, \delta(t - \tau)\, d\tau\right] = \int_{-\infty}^{\infty} x(\tau)\, h(t - \tau)\, d\tau$

That is, $y(t) = x(t) * h(t)$ (1.22)

The following properties of convolution can be established:

Convolution operation

P1) is commutative

P2) is associative

P3) distributes over addition

P1) implies that $x_1(t) * x_2(t) = x_2(t) * x_1(t)$

That is, $\int\limits_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau = \int\limits_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau$.

P2) implies that $\left[x_1(t) * x_2(t)\right] * x_3(t) = x_1(t) * \left[x_2(t) * x_3(t)\right]$, where the bracketed convolution is performed first. Of course, we assume that every convolution pair gives rise to bounded output.

P3) implies that $x_1(t) * \left[x_2(t) + x_3(t)\right] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$.

Note that the properties P1) to P3) are valid even if the independent variable is other than t.

Taking the Fourier transform on both sides of Eq. 1.22, we have,

$$Y(f) = X(f)\, H(f) \tag{1.23}$$

where $h(t) \longleftrightarrow H(f)$. The quantity $H(f)$ is referred to (quite obviously) as the *frequency response* of the system and describes the frequency domain behavior of the system. (As $H(f) = \dfrac{Y(f)}{X(f)}$, it is also referred to as the transfer function of the LTI system). As $H(f)$ is, in general, complex, it is normally shown as two different plots, namely, the magnitude response: $\left|H(f)\right|$ *vs.* f and the phase response: $\arg\left[H(f)\right]$ *vs.* f.

If $H_1(f) \neq H_2(f)$, we then have $F^{-1}\left[H_1(f)\right] \neq F^{-1}\left[H_2(f)\right]$. That is, $h_1(t) \neq h_2(t)$. In other words, the impulse response of any LTI system can be used to uniquely characterize the system in the time domain.

Example 1.13

RC-lowpass filter (RC-LPF) is one among the quite often used LTI systems in the study of communication theory. This network is shown in Fig. 1.17. Let us find its frequency response as well as the impulse response.

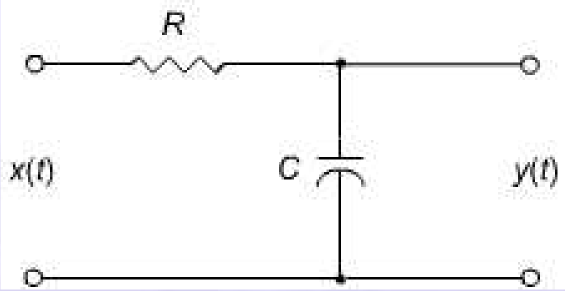


Fig. 1.17: The RC-lowpass filter

One of the important properties of any LTI system is: if the input $x(t) = e^{j 2 \pi f_0 t}$, then the output $y(t)$ is also a complex exponential given by $y(t) = H(f_0) e^{j 2 \pi f_0 t}$.

But, $y(t) = \dfrac{\dfrac{1}{j 2 \pi f_0 C}}{R + \dfrac{1}{j 2 \pi f_0 C}} e^{j 2 \pi f_0 t}$

or $\quad y(t) = \dfrac{1}{1 + j 2 \pi f_0 R C} e^{j 2 \pi f_0 t}$, when $x(t) = e^{j 2 \pi f_0 t}$

Generalizing this result, we obtain,

$$H(f) = \frac{1}{1 + j 2 \pi f R C} \tag{1.24}$$

That is,

$$|H(f)| = \sqrt{\frac{1}{1 + (2 \pi f R C)^2}} \tag{1.25a}$$

$$\theta(f) = \arg\left[H(f)\right] = - \arctan(2 \pi f R C) \tag{1.25b}$$

Let $\frac{1}{2\pi RC} = F$. Then,

$$|H(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{F}\right)^2}} \quad (1.26)$$

A plot of $|H(f)|$ *vs.* f and $\arg[H(f)]$ *vs.* f is shown in Fig. 1.18.

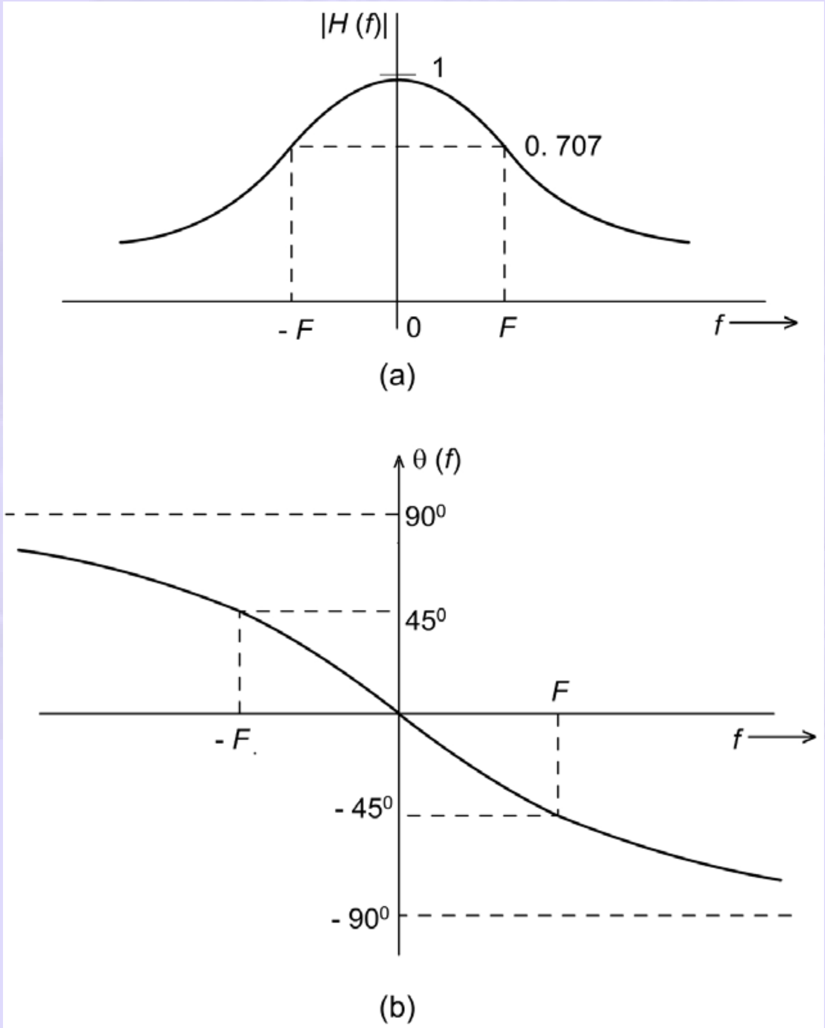


Fig. 1.18 RC-LPF: (a) Magnitude response

(b) Phase response

Let us now compute $h(t) = F^{-1}\left[H(f)\right]$. We have the transform pair

$$ex1(t) \longleftrightarrow \frac{1}{1 + j\,2\pi f}$$

Hence, $\dfrac{1}{RC}\, ex1\left(\dfrac{t}{RC}\right) \longleftrightarrow \dfrac{1}{1 + j\,2\pi f\, RC}$

That is,

$$\left[h(t)\right]_{RC\,-\,LPF} = \frac{1}{RC}\, ex1\left(\frac{t}{RC}\right) \qquad (1.27)$$

This is shown in Fig. 1.19.

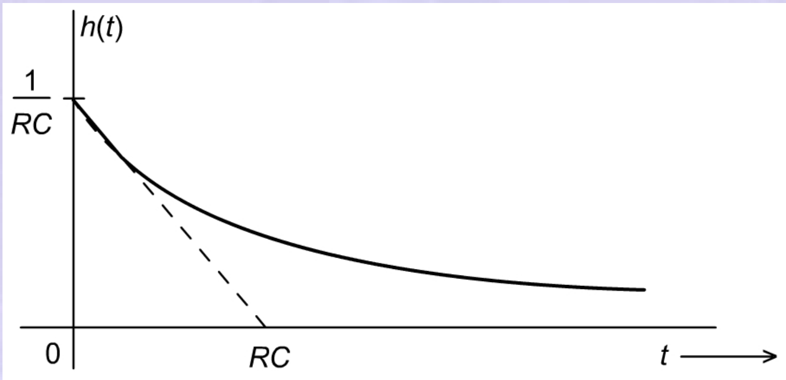


Fig. 1.19: Impulse response of an RC-LPF

♦

Example 1.14

The input $x(t)$ and the impulse response $h(t)$ of an LTI system are as shown in Fig. 1.20. Let us find the output $y(t)$ of the system.

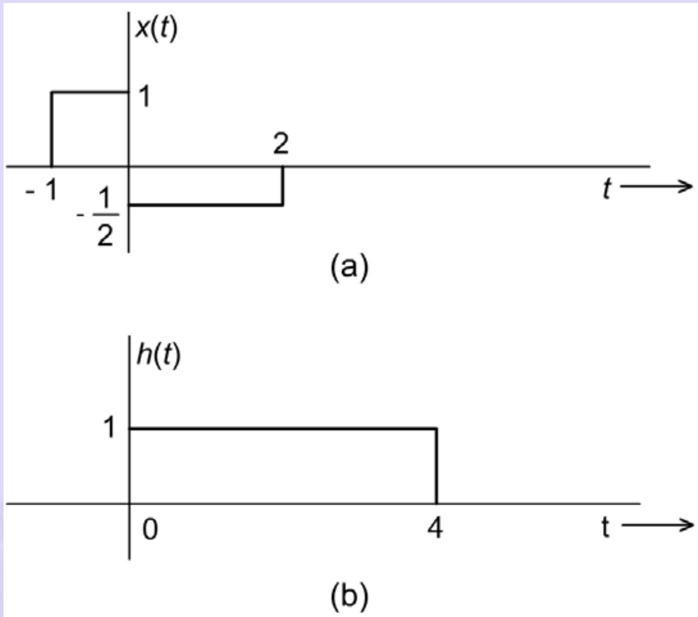


Fig. 1.20: The input $x(t)$ and the impulse response $h(t)$ of an LTI system

Let $y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t - \tau)\, d\tau$

To compute $h(t)$ we have to perform the following three steps:

i) Obtain $h(\tau)$ and $x(t - \tau)$ for a given $t = t_1$.

ii) Take the product of the quantities in (i).

iii) Integrate the result of (ii) to obtain $y(t_1)$.

$h(\tau)$ is the same as $h(t)$ with the change of variable from t to τ. Note that τ is the variable of integration. $x\big[(t - \tau)\big]$ is actually $x\big[-(\tau - t)\big]$; that is, we first reverse $x(\tau)$ to get $x(-\tau)$ and then shift by t, the time instant for which $y(t)$ is desired. This completes the operations in step (i). The operations involved in steps (ii) and (iii) are quite easy to understand.

In quite a few situations, where convolution is to be performed, it would be of great help to have the plots of the quantities in step (i). These have been shown in Fig. 1.21 for three different values of t, namely $t = 0$, $t = -1$ and $t = 2$.

From Fig.1.21(c), we see that if $t < -1$, then $h(\tau)$ and $x(t - \tau)$ do not overlap; that $y(t) = 0$, for $t < -1$. For $-1 < t \leq 0$, overlap of $h(\tau)$ and $x(t - \tau)$ increases as t increases and the integral of the product (which is positive) increases linearly reaching a value of 1 for $t = 0$. For $0 < t \leq 2$, net area of the product $h(\tau)\, x(t - \tau)$, keeps decreasing and at $t = 2$, we have, $\int_{-\infty}^{\infty} h(\tau)\, x(t - \tau)\, d\tau = 0$.

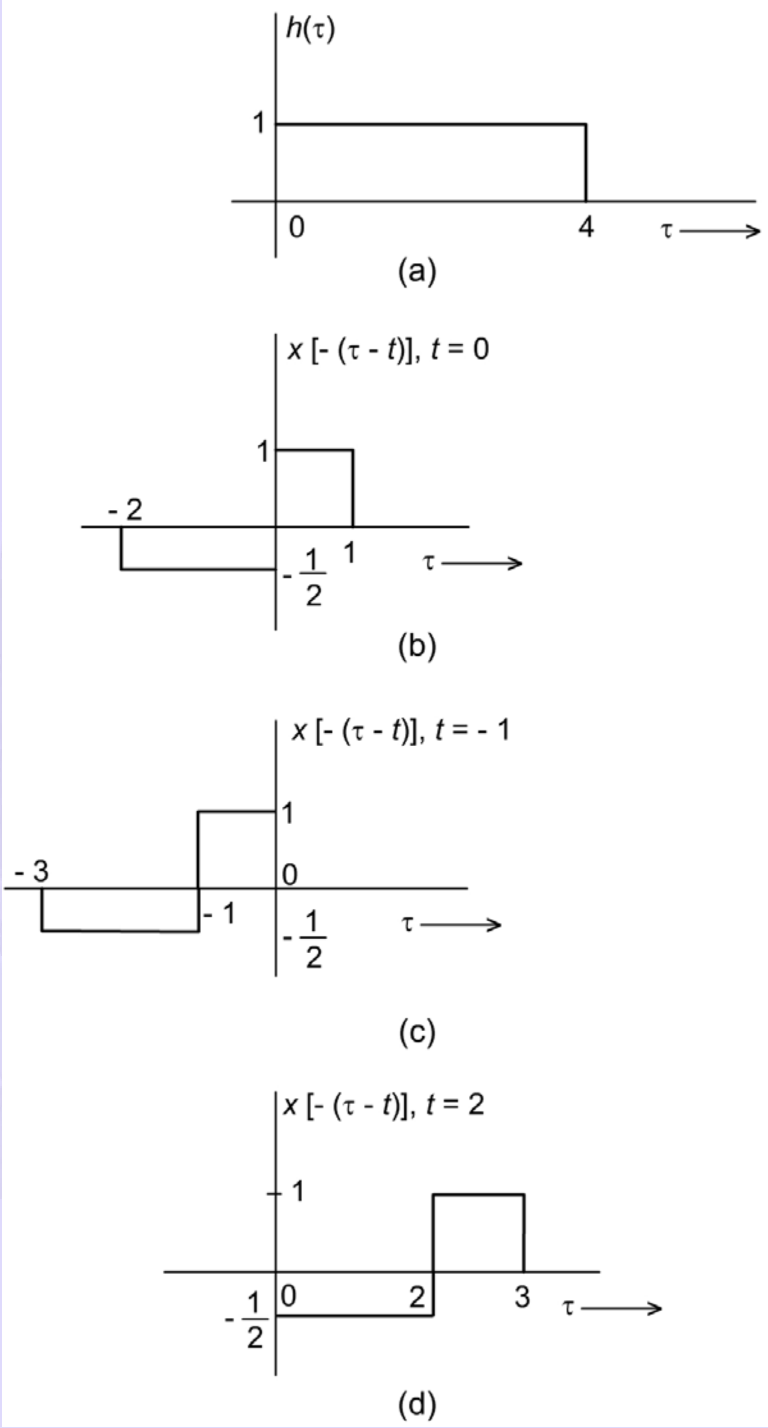


Fig. 1.21: A few plots to implement step (i) of convolution:

(a): $h(\tau)$

(b), (c), (d): $x(t - \tau)$ for $t = 0$, $t = -1$ and $t = 2$ respectively.

Using the similar arguments, $y(t)$ can be computed for $t > 2$. The result of the convolution is indicated in Fig.1.22.

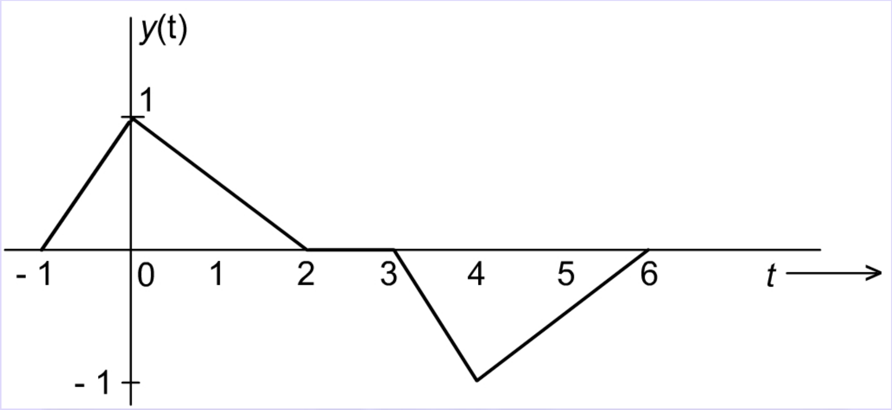


Fig. 1.22: Complete output $y(t)$ of Example 1.14

(Sometimes, computing $y(t) = x(t) * h(t)$, could be very tricky and might even be sticky[1].) ♦

Exercise 1.3

Let $x(t) = \begin{cases} e^{-\alpha t}, & t \geq 0 \\ 0, & otherwise \end{cases}$

$$h(t) = \begin{cases} e^{-\beta t}, & t \geq 0 \\ 0, & otherwise \end{cases}$$

where $\alpha, \beta > 0$. Find $y(t) = x(t) * h(t)$ for

(i) $\alpha = \beta$ and (ii) $\alpha \neq \beta$

[1] Some people claim that convolution has driven many electrical engineering students to contemplate theology either for salvation or as an alternative career (IEEE Spectrum, March 1991, page 60). For an interesting cartoon expressing the student reaction to the convolution operation, see [3].

Exercise 1.4

Find $y(t) = x(t) * h(t)$ where

$$x(t) = \begin{cases} 2\,, & |t| < 2 \\ 0\,, & outside \end{cases}$$

$$h(t) = \begin{cases} 2\,e^{-t}\,, & t \geq 0 \\ 0\,, & outside \end{cases}$$

Exercise 1.5

Let $$X(f) = \begin{cases} \frac{1}{5}\,, & 950 < f < 1050\ Hz \\ \frac{1}{10}\,, & -1050 < f < -950\ Hz \\ 0\,, & elsewhere \end{cases}$$

(a) Compute and sketch $Y(f) = X(f) * X(f)$

(b) Let $f_1 = 50\ Hz$, $f_2 = 2050\ Hz$ and $f_3 = 20\ Hz$

Verify that $Y(f_1) = Y(f_2) = 2$ and $Y(f_3) = 1$.

1.5.3 Signum function and unit step function

Def. 1.4: Signum Function

We denote the signum function by $\operatorname{sgn}(t)$ and define it as,

$$\operatorname{sgn}(t) = \begin{cases} 1\,, & t > 0 \\ 0\,, & t = 0 \\ -1, & t < 0 \end{cases} \qquad \blacklozenge \tag{1.28}$$

Def. 1.5: Unit Step Function

We denote the unit step function by $u(t)$, and define it as,

$$u(t) = \begin{cases} 1 & , t > 0 \\ \frac{1}{2} & , t = 0 \\ 0 & , t < 0 \end{cases} \quad \blacklozenge \qquad (1.29)$$

We shall now develop the Fourier transforms of $\text{sgn}(t)$ and $u(t)$.

***F*[sgn(*t*)]:**

Let $x(t) = e^{-\alpha t}\, u(t) - e^{\alpha t}\, u(-t)$ (1.30)

where α is a positive constant. Then $\text{sgn}(t) = \lim\limits_{\alpha \to 0} x(t)$, as can be seen from Fig. 1.23.

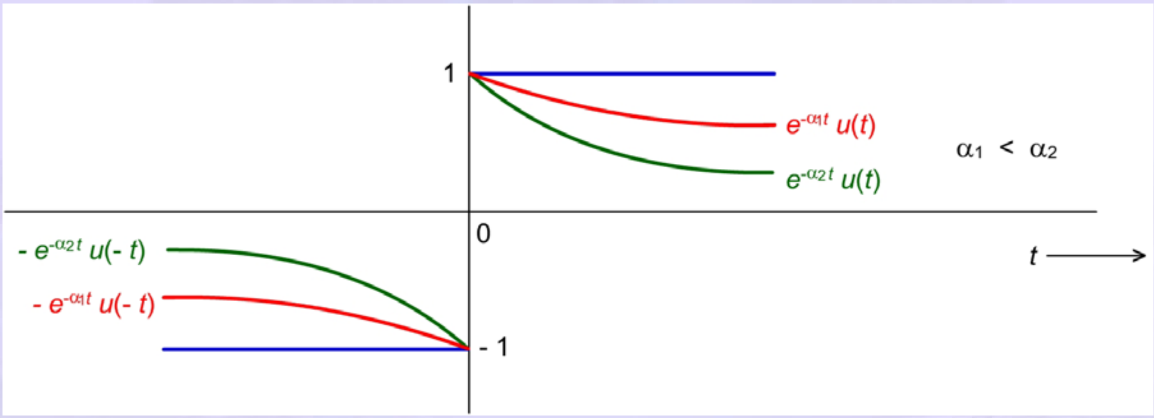


Fig. 1.23: $\text{sgn}(t)$ as a limiting case of $x(t)$ of Eq. 1.30

$$X(f) = \frac{1}{\alpha + j 2\pi f} - \frac{1}{\alpha - j 2\pi f} = \frac{-j 4\pi f}{\alpha^2 + (2\pi f)^2}$$

$$F\left[\text{sgn}(t)\right] = \lim_{\alpha \to 0} X(f) = \frac{1}{j \pi f} \qquad (1.31)$$

***F*[*u*(*t*)]**

As $u(t) = \frac{1}{2}\left[1 + \text{sgn}(t)\right]$, we have

$$U(f) = \frac{1}{2}\delta(f) + \frac{1}{j\,2\pi f} \quad (1.32)$$

We shall now state and prove the FT of the integral of a function.

Properties of FT continued...

P9) Integration in the time domain

Let $x(t) \longleftrightarrow X(f)$

Then, $$\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{1}{j\,2\pi f}\,X(f) + \frac{X(0)}{2}\,\delta(f) \quad (1.33)$$

Proof:

Consider $x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau$. As $u(t-\tau) = 0$ for $\tau > t$,

$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$. But $F\left[x(t) * u(t)\right] = X(f)\,U(f)$.

Hence, $$\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow X(f)\left[\frac{1}{j\,2\pi f} + \frac{\delta(f)}{2}\right]$$

Here there are two possibilities:

(i) $X(0) = 0$; then $\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \dfrac{X(f)}{j\,2\pi f}$

(ii) $X(0) \neq 0$; then $\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \dfrac{X(f)}{j\,2\pi f} + \dfrac{X(0)\,\delta(f)}{2}$

Let $p_1(t) = \frac{1}{\varepsilon}\,ga\left(\frac{t}{\varepsilon}\right)$. Then, we know that $\lim_{\varepsilon \to 0} p_1(t) = \delta(t)$.

But $$\lim_{\varepsilon \to 0} \int_{-\varepsilon/2}^{0} p_1(t)\,dt = \int_{-\infty}^{0} p_1(t)\,dt = \frac{1}{2}$$

and $\lim_{\varepsilon \to 0} \int_{-\varepsilon/2}^{\varepsilon/2} p_1(t)\, dt = \int_{-\infty}^{\infty} p_1(t)\, dt = 1.$

That is,

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t) \tag{1.34a}$$

or $$\frac{d\,u(t)}{d\,t} = \delta(t) \tag{1.34b}$$

We shall now give an alternative proof for the FT relation,

$$u(t) \longleftrightarrow \frac{1}{2}\delta(f) + \frac{1}{j\,2\pi f}.$$

As $\dfrac{d\,u(t)}{d\,t} = \delta(t)$,

$$j\,2\pi f\, U(f) = 1$$

or $$U(f) = \frac{1}{j\,2\pi f}$$

But this is valid only for $f \neq 0$ because of the following argument.

$u(t) + u(-t) = 1$. Therefore,

$$U(f) + U(-f) = \delta(f)$$

As $\delta(f)$ is nonzero only for $f = 0$, we have

$U(0) + U(-0) = 2U(0) = \delta(f)$ or

$U(0) = \dfrac{1}{2}\delta(f)$. Hence,

$$U(f) = \begin{cases} \dfrac{1}{2}\delta(f), & f = 0 \\ \dfrac{1}{j\,2\pi f}, & f \neq 0 \end{cases}$$

Example 1.15

(a) For the scheme shown in Fig. 1.24, find the impulse response (This system is referred to as Zero-Order-Hold, ZOH).

(b) If two such systems are cascaded, what is the overall impulse response (cascade of two ZOHs is called a First-Order-Hold, FOH).

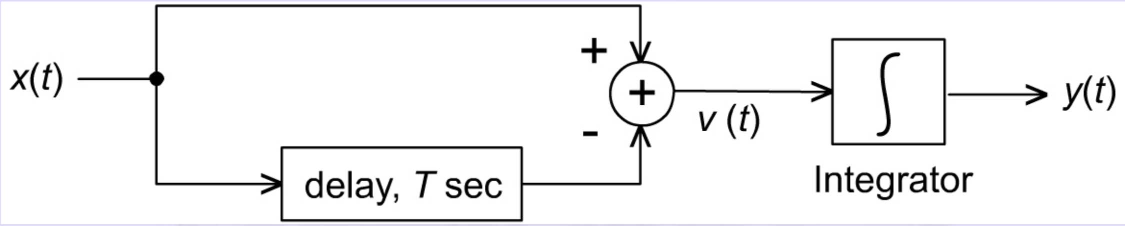


Fig. 1.24: Block schematic of a ZOH

a) $\boldsymbol{h(t)}$ **of ZOH:**

When $x(t) = \delta(t)$, we have

$$v(t) = \delta(t) - \delta(t - T)$$

Hence $y(t) = \left[h(t)\right]_{ZOH}$, the impulse response of the ZOH, is

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau - \int_{-\infty}^{t} \delta(\tau - T)\, d\tau = u(t) - u(t - T) = ga\left(\frac{t - \frac{T}{2}}{T}\right)$$

b) Impulse response of two LTI systems in cascade is the convolution of the impulse responses of the constituents. Hence,

$$\left[h(t)\right]_{FOH} = ga\left[\frac{t - \frac{T}{2}}{T}\right] * ga\left[\frac{t - \frac{T}{2}}{T}\right]$$

$$= T\ tri\left(\frac{t - T}{T}\right)$$

♦

Eq. 1.34(a) can also be established by working in the frequency domain. From Eq. 1.33, with $x(t) = \delta(t)$ and $X(f) = 1$,

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau \longleftrightarrow \frac{1}{j 2 \pi f} + \frac{1}{2}\delta(f) = U(f).$$ That is,

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

Eq. 1.34(b) helps in finding the derivatives of signals with discontinuities. Consider the pulse $p(t)$ shown in Fig. 1.25(a).

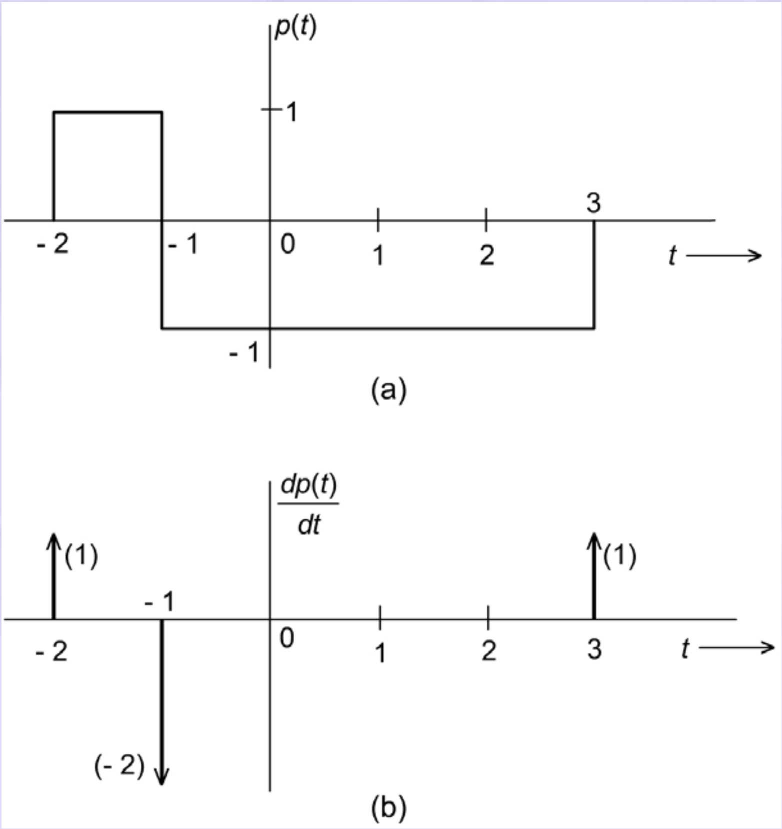


Fig. 1.25: (a) A signal with discontinuities

(b) Derivative of the signal at (a)

$p(t)$ can be written as

$$p(t) = u(t+2) - 2\,u(t+1) + u(t-3)$$

Hence,

$$\frac{d\,p(t)}{d\,t} = \delta(t+2) - 2\,\delta(t+1) + \delta(t-3)$$

which is shown in Fig. 1.25(b). From this result, we note that if there is a step discontinuity of size A at $t = t_1$ in the signal, its derivative will have an impulse of weight A at $t = t_1$.

Example 1.16

Let $x(t)$ be the doublet pulse of Example 1.7 (Fig.1.12). We shall find

$$X(f)$$

from $\frac{d\,x(t)}{dt}$.

$$\frac{d\,x(t)}{dt} = \delta(t+T) - 2\,\delta(t) + \delta(t-T)$$

Taking Fourier transform on both the sides,

$$j\,2\pi f\;X(f) = e^{j\,2\pi f T} - 2 + e^{-j\,2\pi f T}$$

$$= \left(e^{j\pi f T} - e^{-j\pi f T}\right)^2$$

$$X(f) = 2\,j\,T\;\frac{\left(e^{j\pi f T} - e^{-j\pi f T}\right)}{2\,j\,\pi f T}\;\frac{\left(e^{j\pi f T} - e^{-j\pi f T}\right)}{2\,j}$$

$$= 2\,j\,T\;\mathrm{sinc}(f\,T)\;\;\sin(\pi f\,T)$$

♦

Example 1.17

Let $x(t)$, $h(t)$ and $y(t)$ denote the input, impulse response and the output respectively of an LTI system. It is given that,

$$x(t) = t\,e^{-2t}\,u(t) \text{ and } h(t) = e^{-4t}\,u(t).$$

Find a) $Y(f)$

b) $y(t) = F^{-1}\left[Y(f)\right]$

c) $y(t) = x(t) * h(t)$

a) From Eq. 1.22, we obtain

$$Y(f) = X(f)\, H(f)$$

If $z(t) = e^{-2t}\, u(t)$, then $Z(f) = \dfrac{1}{2 + j\,2\pi f}$.

As $x(t) = t\, z(t)$, we have $X(f) = \left(\dfrac{j}{2\pi}\right)\dfrac{d Z(f)}{df} = \dfrac{1}{(2 + j\,2\pi f)^2}$

$$H(f) = \frac{1}{4 + j\,2\pi f}$$

Hence, $Y(f) = \dfrac{1}{(2 + j\,2\pi f)^2}\;\dfrac{1}{4 + j\,2\pi f}$

(b) Using partial fraction expansion,

$$Y(f) = \frac{\left(-\frac{1}{4}\right)}{2 + j\,2\pi f} + \frac{\frac{1}{2}}{(2 + j\,2\pi f)^2} + \frac{\frac{1}{4}}{4 + j\,2\pi f}$$

Hence, $y(t) = \left(-\dfrac{1}{4}\right) e^{-2t}\, u(t) + \dfrac{1}{2}\, t\, e^{-2t}\, u(t) + \dfrac{1}{4}\, e^{-4t}\, u(t)$

(c) $y(t) = \displaystyle\int_{-\infty}^{\infty} x(\tau)\, h(t - \tau)\, d\tau$

$$= \int_{-\infty}^{\infty} \tau\, e^{-2\tau}\, u(\tau)\, e^{-4(t - \tau)}\, u(t - \tau)\, d\tau$$

$y(t) = 0$ for $t < 0$ because for t negative, $u(t - \tau)$ is 1 only for τ negative; but then $u(\tau) = 0$. As $u(t - \tau) = 0$ for $\tau > t$, we have

$$y(t) = \int_0^t \tau\, e^{-2\tau}\, e^{-4(t - \tau)}\, d\tau$$

$$= e^{-4t} \int_0^t \tau \, e^{2\tau} \, d\tau$$

$$= e^{-4t} \left[\tau \frac{e^{2\tau}}{2} - \int \frac{e^{2\tau}}{2} \, d\tau \right]_0^t$$

$$= e^{-4t} \left\{ t \frac{e^{2t}}{2} - \left[\frac{e^{2\tau}}{4} \right]_0^t \right\}$$

$$= t \frac{e^{-2t}}{2} - e^{-4t} \left[\frac{e^{2t} - 1}{4} \right], \quad t \geq 0$$

$$= 0 \text{ for } t < 0$$

♦

Exercise 1.6

Let $x(t) = \begin{cases} 1 + \cos(2\pi f_0 t) & , \quad |t| < \dfrac{T_0}{2} \\ 0 & , \quad |t| > \dfrac{T_0}{2} \end{cases}$

where T_0 is the period of the cosine signal and $f_0 = \dfrac{1}{T_0}$.

(a) Show that

$$\frac{d^3 x(t)}{dt^3} = (2\pi f_0)^2 \left\{ \delta\left(t + \frac{T_0}{2}\right) - \delta\left(t - \frac{T_0}{2}\right) - \frac{d\,x(t)}{dt} \right\}$$

(b) Taking the FT of the equation in (a) above, show that

$$X(f) = \frac{f_0}{f_0^2 - f^2} \operatorname{sin}c(f T_0)$$

1.6 Correlation Functions

Two basic operations that arise in the study of communication theory are: (i) convolution and (ii) correlation. As we have some feel for the convolution operation by now, let us develop the required familiarity with the *correlation* operation.

When we say that there is some correlation between two objects, we imply that there is some similarity between them. We would like to quantify this intuitive notion and come up with a formal definition for correlation so that we have a mathematically consistent and physically meaningful measure for the correlation of the objects of interest to us.

Our interest is in electrical signals. We may like to quantify, say, the similarity between a transmitted signal and the received signal or between two different transmitted or received signals etc. We shall first introduce the cross correlation functions; this will be followed by the special case, namely, *auto-correlation* function.

In the context of correlation functions, we have to distinguish between the energy signals and power signals. Accordingly, we make the following definitions.

1.6.1 Cross-correlation functions (CCF)

Let $x(t)$ and $y(t)$ be signals of the energy type. We now define their cross-correlation functions, $R_{xy}(\tau)$ and $R_{yx}(\tau)$.

Def. 1.6(a): The cross-correlation function $R_{xy}(\tau)$ is given by

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y^*(t-\tau)\, dt \qquad \blacklozenge \qquad (1.35a)$$

Def. 1.6(b): The cross-correlation function $R_{yx}(\tau)$ is given by

$$R_{yx}(\tau) = \int_{-\infty}^{\infty} y(t)\, x^*(t-\tau)\, dt \qquad \blacklozenge \qquad (1.35b)$$

In Eq. 1.35(a), $y^*(t-\tau)$ is a conjugated and shifted version of $y(t)$, τ accounting for the shift in $y^*(t)$. Note that the variable of integration in Eq. 1.35 is t; hence $R_{xy}(\ )$ as well as $R_{yx}(\ )$ is a function of τ, the *shift parameter* (τ is also called the *scanning parameter* or the *search parameter*).

Let $x(t)$ and $y(t)$ be the signals of the power type.

Def. 1.7(a): The cross-correlation function, $R_{xy}(\tau)$ is given by

$$R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t-\tau)\, dt \qquad \blacklozenge \qquad (1.36a)$$

Def. 1.7(b): The cross-correlation function, $R_{yx}(\tau)$ is given by

$$R_{yx}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} y(t)\, x^*(t-\tau)\, dt \qquad \blacklozenge \qquad (1.36b)$$

The power signals that we have to deal with most often are of the periodic variety. For periodic signals, we have the following definitions:

Def. 1.8(a): $$R_{x_p y_p}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_p(t)\, y_p^*(t-\tau)\, dt \qquad \blacklozenge \qquad (1.37a)$$

Def. 1.8(b): $$R_{y_p x_p}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} y_p(t)\, x_p^*(t-\tau)\, dt \qquad \blacklozenge \qquad (1.37b)$$

Let $t - \tau = \lambda$ in Eq. 1.35(a). Then, $t = \tau + \lambda$ and $dt = d\lambda$. Hence,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(\lambda + \tau)\, y^*(\lambda)\, d\lambda$$

$$= R_{yx}^*(-\tau) \qquad (1.38)$$

As $R_{xy}(\tau) \neq R_{yx}(\tau)$, cross-correlation, unlike convolution is not in general, commutative. To understand the significance of the parameter τ, consider the situation shown in Fig. 1.26.

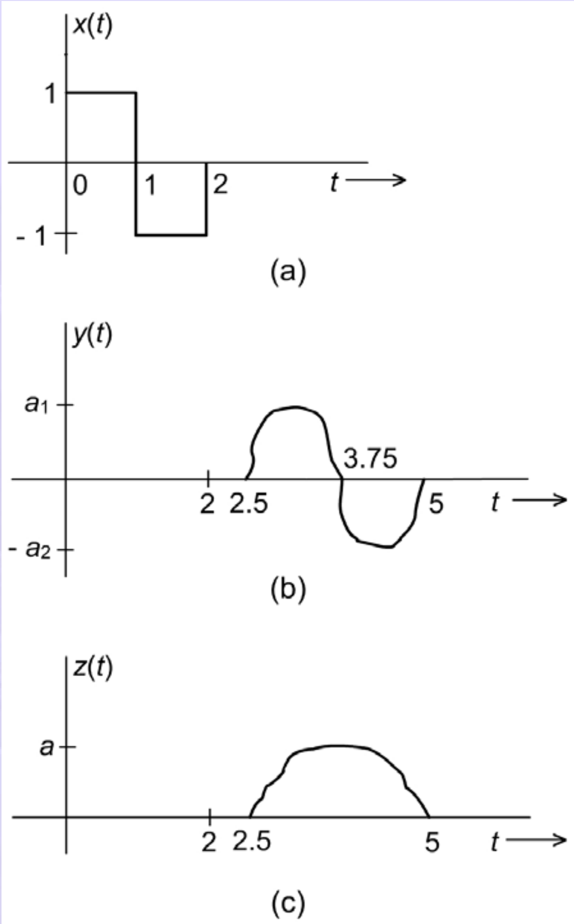


Fig. 1.26: Waveforms used to compute $R_{yx}(\tau)$ and $R_{zx}(\tau)$

If we compute $\int_{-\infty}^{\infty} x(t)\, y(t)\, dt$, we find it to be zero as $x(t)$ and $y(t)$ do not overlap. However, if we delay $x(t)$ by half a unit of time, we find that $x\left(t - \frac{1}{2}\right)$ and $y(t)$ start overlapping and for $\tau > \frac{1}{2}$, we have nonzero value for the integral. For the value of $\tau \simeq 2.75$, we will have a positive value for

$\int_{-\infty}^{\infty} y(t)\, x(t - \tau)\, dt$, which would be about the maximum of $|R_{yx}(\tau)|$ for any τ. For values of $\tau > 2.75$, $|R_{yx}(\tau)|$ keeps decreasing, becoming zero for $\tau > 5$. Similarly, if we compute $R_{zx}(\tau)$, we find that $|R_{zx}(\tau)|_{max}$ would be much smaller than $|R_{yx}(\tau)|_{max}$ (Note that for $\tau \simeq 2.75$, the product quantity, $y(t)\, x(t - \tau)$, is essentially positive for all t). If $y(t)$ is the received signal of a communication system, then we are willing to accept $x(t)$ as the likely transmitted signal (we can treat the received signal as a delayed and distorted version of the transmitted signal) whereas if $z(t)$ is received, it would be difficult for us to accept that $x(t)$ could have been the transmitted signal. Thus the parameter τ helps us to find time-shifted similarities present between the two signals.

From Eq. 1.35(a), we see that computing $R_{xy}(\tau)$ for a given τ, involves the following steps:

(i) Shift $y^*(t)$ by τ

(ii) Take the product of $x(t)$ and $y^*(t - \tau)$

(iii) Integrate the product with respect to t.

The above steps closely resemble the operations involved in convolution. It is not difficult to see that

$R_{xy}(\tau) = x(\tau) * y^*(-\tau)$, because

$$x(\tau) * y^*(-\tau) = \int_{-\infty}^{\infty} x(t)\, y^*[-(\tau - t)]\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, y^*(t - \tau)\, dt = R_{xy}(\tau) \qquad (1.39a)$$

Let $E_{xy}(f) = F[R_{xy}(\tau)]$.

Then, $E_{xy}(f) = F[x(\tau) * y^*(-\tau)] = X(f)\, Y^*(f)$ (1.39b)

Example 1.18

Let $x(t) = \exp1(t)$ and $y(t) = ga\left(\frac{t}{2}\right)$.

Let us find (a) $R_{xy}(\tau)$ and (b) $R_{yx}(\tau)$.

From the results of (a) and (b) above, let us verify Eq. 1.38.

(a) $R_{xy}(\tau)$:

$x(t)$ and $y(t)$ are sketched below (Fig. 1.27).

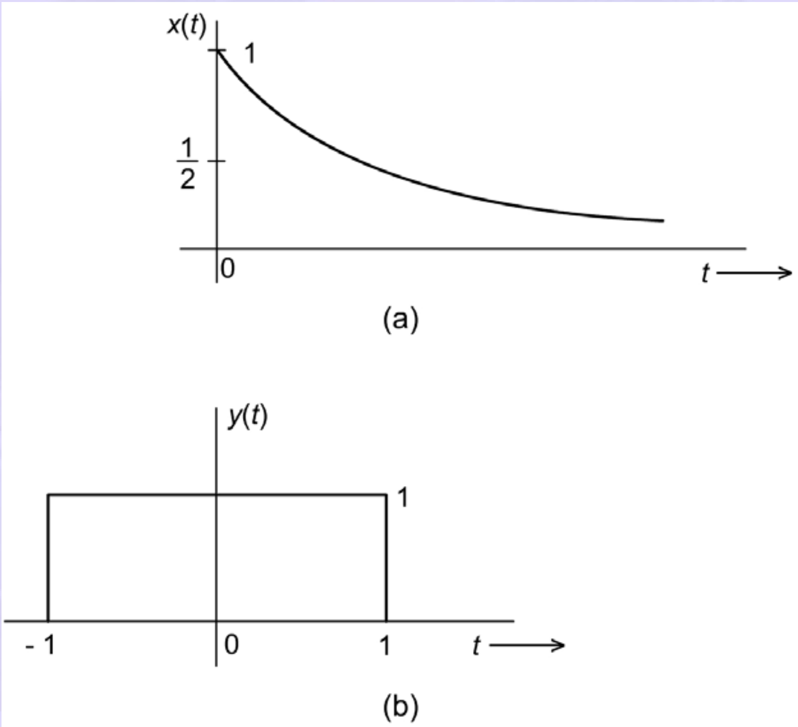


Fig. 1.27: Waveforms of Example 1.18

(i) $\tau < -1$:

$x(t)$ and $y(t - \tau)$ do not overlap and the product is zero. That is, $R_{xy}(\tau) = 0$ for $\tau < -1$.

ii) $-1 \leq \tau < 1$:

$$R_{xy}(\tau) = \int_0^{1+\tau} e^{-t}\, dt = 1 - e^{-(1+\tau)}$$

(iii) $\tau \geq 1$:

$$R_{xy}(\tau) = \int_{\tau - 1}^{\tau + 1} e^{-t}\, dt = e^{-(\tau - 1)} - e^{-(\tau + 1)}$$

$$= e^{-\tau}\left(e - \frac{1}{e}\right)$$

(b) $R_{yx}(\tau)$:

(i) For $\tau > 1$, $y(t)$ and $x(t - \tau)$ do not overlap. Hence, $R_{yx}(\tau) = 0$ for $\tau > 1$.

(ii) For $-1 < \tau \leq 1$,

$$R_{yx}(\tau) = \int_{\tau}^{1} e^{-(t - \tau)}\, dt$$

$$= e^{\tau}\left[e^{-\tau} - e^{-1}\right]$$

$$= 1 - e^{-(1 - \tau)}$$

(iii) For $\tau \leq -1$,

$$R_{yx}(\tau) = \int_{-1}^{1} e^{-(t - \tau)}\, dt$$

$$= e^{\tau}\left(e - \frac{1}{e}\right)$$

$R_{xy}(\tau)$ and $R_{yx}(\tau)$ are plotted in Fig. 1.28.

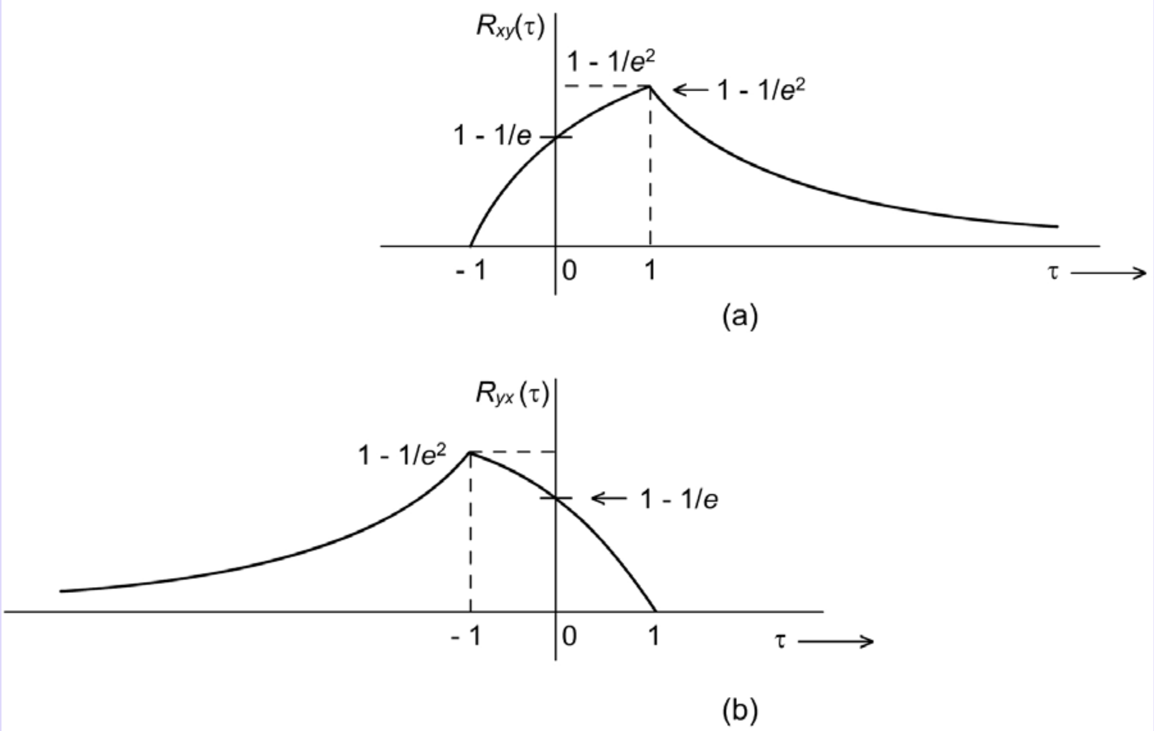


Fig. 1.28: Cross correlation functions of Example 1.18

From the plots of $R_{xy}(\tau)$ and $R_{yx}(\tau)$, it is easy to see that

$$R_{yx}(\tau) = R_{xy}(-\tau)$$ ♦

Def. 1.9: Two signals $x(t)$ and $y(t)$ are said to be **orthogonal** if

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = 0 \quad ♦ \qquad (1.40)$$

Eq. 1.41 implies that for orthogonal signals, say $x(t)$ and $y(t)$

$$R_{xy}(\tau)\Big|_{\tau = 0} = 0 \qquad (1.41a)$$

We have the companion relation to Eq. 1.41(a), namely

$R_{yx}(0) = 0$, if $x(t)$ and $y(t)$ are orthogonal. (1.41b)

Let $x_p(t)$ and $y_p(t)$ be periodic with period T_0. Then, from Eq. 1.37(a), for any integer n,

$$R_{x_p y_p}(\tau + nT_0) = R_{x_p y_p}(\tau)$$

That is, $R_{x_p y_p}(\tau)$ is also periodic with the same period as $x_p(t)$ and $y_p(t)$. Similarly, we find $R_{y_p x_p}(\tau)$. Derivation of the FT of $R_{x_p y_p}(\tau)$ is given in appendix A1.2.

1.6.2 Autocorrelation function (ACF)

ACF can be treated as a special case of CCF. In Eq. 1.35(a), let $x(t) = y(t)$. Then, we have

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t - \tau)\, dt \qquad (1.42a)$$

Instead of $R_{xx}(\tau)$, we use somewhat simplified notation, namely, $R_x(\tau)$ which is called the auto correlation function of $x(t)$.

ACF compares $x(t)$ with a shifted and conjugated version of itself. If $x(t)$ and $x^*(t - \tau)$ are quite similar, we can expect large value for $|R_x(\tau)|$, whereas as a value of $|R_x(\tau)|$ close to zero implies the orthogonality of the two signals. Hence $R_x(\tau)$ can provide some information about the time variations of the signal.

Let $(t - \tau) = \lambda$ in Eq. 1.42(a). We then have,

$$R_x(\tau) = \int_{-\infty}^{\infty} x(\tau + \lambda)\, x^*(\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} x(t + \tau)\, x^*(t)\, dt \qquad (1.42b)$$

Eq. 1.42(b) gives another relation for $R_x(\tau)$. This is quite meaningful because, assuming τ positive, $x^*(t - \tau)$ is a right shifted version of $x^*(t)$. In Eq.

1.42(a), we keep $x(t)$ fixed and move $x^*(t)$ to the right by τ and take the product; in Eq. 1.42(b), we keep $x^*(t)$ fixed and move $x(t)$ to the left by τ. This does not change the integral of Eq. 1.42(a), because if we let $t = t_1$ and $\tau = \tau_1$, the product of Eq. 1.42(a) is $x(t_1)\, x^*(t_1 - \tau_1)$. This product is obtained from Eq. 1.42(b) for $t = t_1 - \tau_1$. For a given $\tau = \tau_1$, as t is varied, all the product quantities are obtained and hence the integral for a given τ_1 remains the same. (Note that shifting a function does not change its area.)

If $x(t)$ is a power signal, then the ACF is special case of Eq. 1.36(a). That is, for the power signals, we have

$$R_x(\tau) = \lim_{T \to \infty} \int_{-T/2}^{T/2} x(t)\, x^*(t - \tau)\, dt \qquad (1.43a)$$

It can easily be shown that,

$$R_x(\tau) = \lim_{T \to \infty} \int_{-T/2}^{T/2} x(t + \tau)\, x^*(t)\, dt \qquad (1.43b)$$

For power signals that are periodic, we have

$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_p(t)\, x_p^*(t - \tau)\, dt \qquad (1.44a)$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_p^*(t)\, x_p(t + \tau)\, dt \qquad (1.44b)$$

Properties of ACF (energy signals)

P1) ACF exhibits conjugate symmetry. That is,

$$R_x(-\tau) = R_x^*(\tau) \qquad (1.45a)$$

Proof: Exercise

Eq. 1.45(a) implies that the real part of $R_x(\tau)$ is an even function of τ where as the imaginary part is an odd function of τ.

P2) $R_x(0) = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = E_x$ (1.45b)

where E_x is the energy of the signal $x(t)$ (Eq. 1.10).

P3) Maximum value of $|R_x(\tau)|$ occurs at the origin. That is, $|R_x(\tau)| \leq R_x(0)$.

Proof: The proof of the above property follows from Schwarz's inequality; which is stated below.

Let $g_1(t)$ and $g_2(t)$ be two energy signals.

Then, $\left| \int_{-\infty}^{\infty} g_1(t) \, g_2(t) \, dt \right|^2 \leq \int_{-\infty}^{\infty} |g_1(t)|^2 \, dt \int_{-\infty}^{\infty} |g_2(t)|^2 \, dt$.

Let $g_1(t) = x(t)$ and $g_2(t) = x^*(t - \tau)$.

From the Schwarz's Inequality,

$$\left| \int_{-\infty}^{\infty} x(t) \, x^*(t - \tau) \, dt \right|^2 \leq \int_{-\infty}^{\infty} |x(t)|^2 \, dt \int_{-\infty}^{\infty} |x^*(t - \tau)|^2 \, dt$$

$$|R_x(\tau)|^2 \leq [R_x(0)]^2 \text{ or}$$

$$|R_x(\tau)| \leq R_x(0) \qquad (1.45c)$$

♦

P4) Let $E_x(f)$ denote the *Energy Spectral Density (ESD)* of the signal $x(t)$.

That is, $\int_{-\infty}^{\infty} E_x(f) \, df = E_x$

Then, $R_x(\tau) \longleftrightarrow E_x(f)$

Proof

From Eq. 1.39(b),

$$E_{xy}(f) = X(f) \, Y^*(f)$$

Let $x(t) = y(t)$; then $E_{xx}(f) = |X(f)|^2$. That is,

$$R_x(\tau) \longleftrightarrow |X(f)|^2$$

But $|X(f)|^2$ is the ESD of $x(t)$. That is,

$$E_x(f) = E_{xx}(f) = |X(f)|^2$$

Hence,

$$R_x(\tau) \longleftrightarrow E_x(f) \quad \blacklozenge \tag{1.45d}$$

It is to be noted that $E_x(f)$ depends only on the magnitude spectrum, $|X(f)|$. Let $x(t)$ and $y(t)$ be two signals such that $|X(f)| = |Y(f)|$. Then, $R_x(\tau) = R_y(\tau)$. Note that if $\theta_x(f) \neq \theta_y(f)$, $x(t)$ may not have any resemblance to $y(t)$; but their ACFs will be the same. In other words, ACF does not provide a unique description of the signal. Given $x(t)$, its ACF is unique; but given some ACF, we can find many signals that have the given ACF.

P5) Let $x(t) = y(t) + v(t)$. Then,

$$R_x(\tau) = R_y(\tau) + R_v(\tau) + R_{yv}(\tau) + R_{vy}(\tau) \tag{1.45e}$$

Proof: Exercise

If $R_{yv}(\tau) = R_{vy}(\tau) \equiv 0$ (that is, $y(t)$ and $v^*(t-\tau)$ are orthogonal for all τ), then, $R_x(\tau) = R_y(\tau) + R_v(\tau)$ (1.45f)

In such a situation, $R_x(\tau)$ is the superposition of the ACFs of the components of $x(t)$. This also leads to the superposition of the ESDs; that is

$$E_x(f) = E_y(f) + E_v(f) \tag{1.45g}$$

Properties of ACF (periodic signals)

We list below the properties of the ACF of periodic signals. Proofs of these properties are left as an exercise.

P1) ACF exhibits conjugate symmetry

$$R_{x_p}(-\tau) = R^*_{x_p}(\tau) \tag{1.46a}$$

P2) $$R_{x_p}(\tau)\Big|_{\tau = 0} = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \left| x_p(t) \right|^2 dt = P_{x_p} \tag{1.46b}$$

where P_{x_p} denotes the average power of $x_p(t)$ (Sec. 1.2.2, Pg. 1.16).

P3) $$\left| R_{x_p}(\tau) \right| \le R_x(0) \tag{1.46c}$$

That is, the maximum value of $\left| R_{x_p}(\tau) \right|$ occurs at the origin.

P4) $$R_{x_p}(\tau \pm n T_0) = R_{x_p}(\tau),\ n = 1, 2, 3, \ldots \tag{1.46d}$$

where T_0 is the period of $x_p(t)$. That is, the ACF of a periodic signal is also periodic with the same period as that of the signal.

P5) Let $P_{x_p}(f)$ denote the *Power Spectral Density (PSD)* of $x_p(t)$. That is,

$\int_{-\infty}^{\infty} P_{x_p}(f)\, df = P_{x_p}$. Then,

$$R_{x_p}(\tau) \longleftrightarrow P_{x_p}(f) \tag{1.46e}$$

As $R_{x_p}(\tau)$ is periodic, we expect the PSD to be purely impulsive.

Exercise 1.7

Let $x(t) = \begin{cases} x_p(t), & -\dfrac{T_0}{2} < t < \dfrac{T_0}{2} \\ 0, & \textit{outside} \end{cases}$ and $x(t) \longleftrightarrow X(f)$.

Show that $F\left[R_{x_p}(\tau)\right] = P_{x_p}(f) = \dfrac{1}{T_0^2} \sum_{n = -\infty}^{\infty} \left| X\left(\dfrac{n}{T_0}\right) \right|^2 \delta\left(f - \dfrac{n}{T_0}\right)$

Example 1.19

a) Let $x(t)$ be the signal shown in Fig. 1.29. Compute and sketch $R_x(\tau)$.

b) $x(t)$ of part (a) is given as the input to an LTI system with the impulse response $h(t)$. If the output $y(t)$ of the system is $R_x(t-3)$, find $h(t)$.

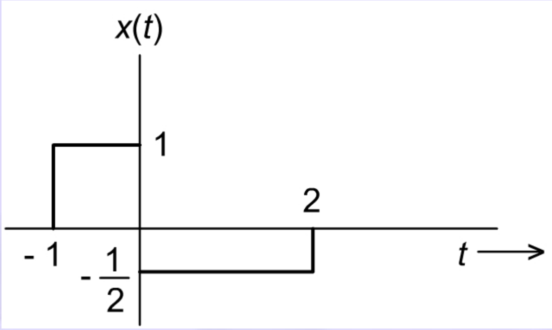


Fig. 1.29: $x(t)$ of Example 1.19

a) Computation of $R_x(\tau)$:

We know that the maximum value of the ACF occurs at the origin; that is, at $\tau = 0$ and $R_x(0) = E_x$.

$$R_x(0) = 1.1 + \frac{1}{4}.2 = 1.5$$

Consider the product $x(t)\,x(t-\tau)$ for $0 < \tau \le 1$. As τ increases in this range, the overlap between the positive parts of the pulses $x(t)$ and $x(t-\tau)$ (and also between the negative parts and these pulses) decreases, which implies a decrease in the positive value for the integral of the product. In addition, a part of $x(t-\tau)$ that is positive overlaps with the negative part of $x(t)$, there by further reducing the positive value of the $\int x(t)\,x(t-\tau)$. (The student is advised to make a sketch of $x(t)$ and $x(t-\tau)$.) This decrease is linear (with a constant slope) until $\tau = 1$. The value of $R_x(1)$ is $\left(-\frac{1}{4}\right)$. For $1 < \tau < 2$, the positive part of $x(t-\tau)$ fully overlaps with the negative part of $x(t)$; this makes $R_x(\tau)$ further negative and the ACF reaches its minimum value at $\tau = 2$. As can easily be checked, $R_x(2) = \left(-\frac{1}{2}\right)$. As τ increases beyond 2, the $R_x(\tau)$ becomes less and less negative and becomes zero at $\tau = 3$. As $R_x(-\tau) = R_x(\tau)$, we have all the information necessary to sketch $R_x(\tau)$, which is shown in Fig. 1.30.

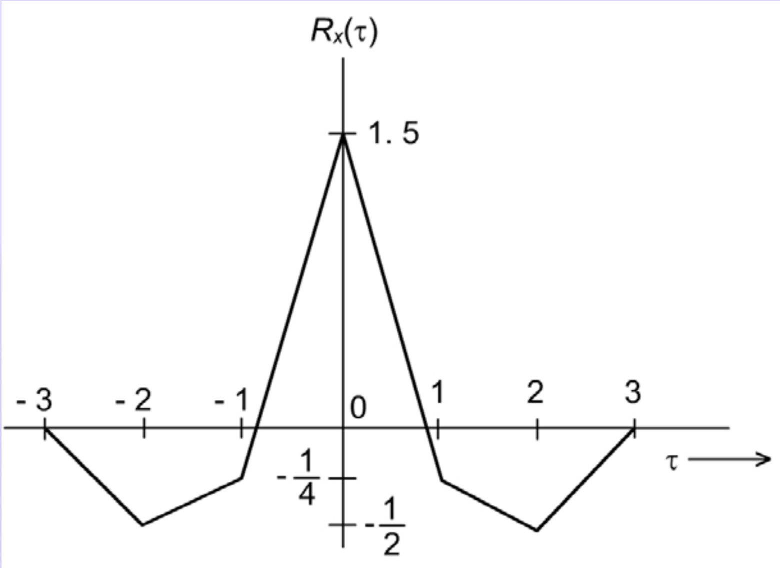


Fig. 1.30: ACF of the signal of Example 1.19

b) Calculating $h(t)$:

We have $y(t) = x(t) * h(t)$

$$= R_x(t-3)$$

$$= R_x(t) * \delta(t-3)$$

But from Eq. 1.39(a), $R_x(t) = x(t) * x(-t)$.

Hence, $y(t) = x(t) * \left[x(-t) * \delta(t-3)\right]$

$$= x(t) * x\left[-(t-3)\right]$$

That is, $h(t) = x\left[-(t-3)\right]$

which is sketched in Fig. 1.31.

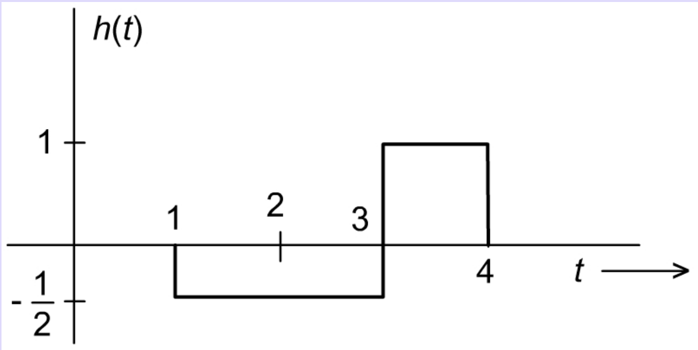


Fig. 1.31 Impulse response of the LTI system of Example 1.19 ♦

Example 1.20

Let $x(t) = A\cos(\omega_0 t + \theta)$. We will find $R_x(\tau)$.

Method 1:

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A^2 \cos(\omega_0 t + \theta)\cos\left[\omega_0 (t - \tau) + \theta\right] dt$$

$$= \frac{A^2}{T_0}\int_{-T_0/2}^{T_0/2} \frac{1}{2}\left\{\cos(2\omega_0 t - \omega_0\tau + 2\theta) + \cos\omega_0\tau\right\} dt$$

$$= \frac{A^2}{2}\cos\omega_0\tau$$

We find that:

(i) $R_x(\tau)$ is periodic with the same period as $x(t)$.

(ii) Its maximum value occurs at $\tau = 0$

(iii) The maximum value is $\frac{A^2}{2}$ which is the average power of the signal.

(iv) $R_x(\tau)$ is independent of θ.

Method 2:

$$A\cos(\omega_0 t + \theta) = \left[\frac{A}{2} e^{j(\omega_0 t + \theta)} + \frac{A}{2} e^{-j(\omega_0 t + \theta)}\right]$$

$$= v(t) + w(t)$$

As $x(t) = v(t) + w(t)$, we have

$$R_x(\tau) = R_v(\tau) + R_w(\tau) + R_{vw}(\tau) + R_{wv}(\tau).$$

It is not difficult to see that $R_{vw}(\tau) = R_{wv}(\tau) = 0$ for all τ. Hence,

$$R_x(\tau) = R_v(\tau) + R_w(\tau)$$

But $R_w(\tau) = R_v^*(\tau)$. Hence,

$$R_x(\tau) = 2\,\mathrm{Re}\left[R_v(\tau)\right]$$

$$R_v(\tau) = \frac{A^2}{4}\left[\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{j(\omega_0 t + \theta)}\, e^{-j\left[\omega_0 (t-\tau) + \theta\right]}\, dt\right] = \frac{A^2}{4T_0}\int_{-T_0/2}^{T_0/2} e^{j\omega_0 \tau}\, dt$$

$$= \frac{A^2}{4} e^{j\omega_0 \tau}$$

Hence $R_x(\tau) = \dfrac{A^2}{2}\cos(\omega_0 \tau)$ ♦

Example 1.21

Let $x(t) = \sin(2\pi t),\ \ 0 \le t \le 2$. Let us find its ACF and sketch it.

In Fig. 1.32, we show $x(t)$ and $x(t-\tau)$ for $0 < \tau < 2$. Note that if $\tau > 2$, $x(t)\,x(t-\tau) = 0$ which implies $R_x(\tau) = 0$ for $|\tau| > 2$.

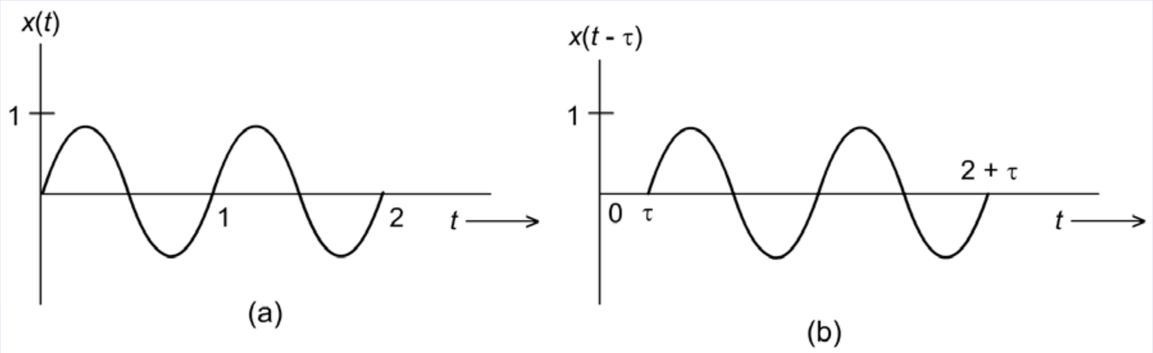


Fig. 1.32: (a) Sinusoidal pulse of Example 1.21 and (b) its shifted version

$$R_x(\tau) = \int_{\tau}^{2} \sin(2\pi t)\sin\left[2\pi(t-\tau)\right] dt$$

$$= \int_{\tau}^{2} \frac{\cos(2\pi\tau) - \cos(4\pi t - 2\pi\tau)}{2} dt$$

$$= \left[\frac{t\cos 2\pi\tau}{2}\right]_{\tau}^{2} - \left[\frac{\sin(4\pi t - 2\pi\tau)}{4\pi}\right]_{\tau}^{2}$$

$$= \cos(2\pi\tau) - \frac{\tau\cos(2\pi\tau)}{2} - \frac{\sin(8\pi - 2\pi\tau) - \sin(2\pi\tau)}{4\pi}$$

$$= \cos(2\pi\tau) - \frac{\tau\cos 2\pi\tau}{2} + \frac{\sin 2\pi\tau}{2\pi}, \quad 0 \le \tau \le 2$$

As $R_x(-\tau) = R_x(\tau)$, we have

$$R_x(\tau) = \begin{cases} \cos(2\pi\tau) - \dfrac{|\tau|\cos(2\pi\tau)}{2} + \dfrac{\sin(2\pi|\tau|)}{2\pi}, & |\tau| \le 2 \\ 0, & otherwise \end{cases}$$

$R_x(\tau)$ is plotted in Fig. 1.33.

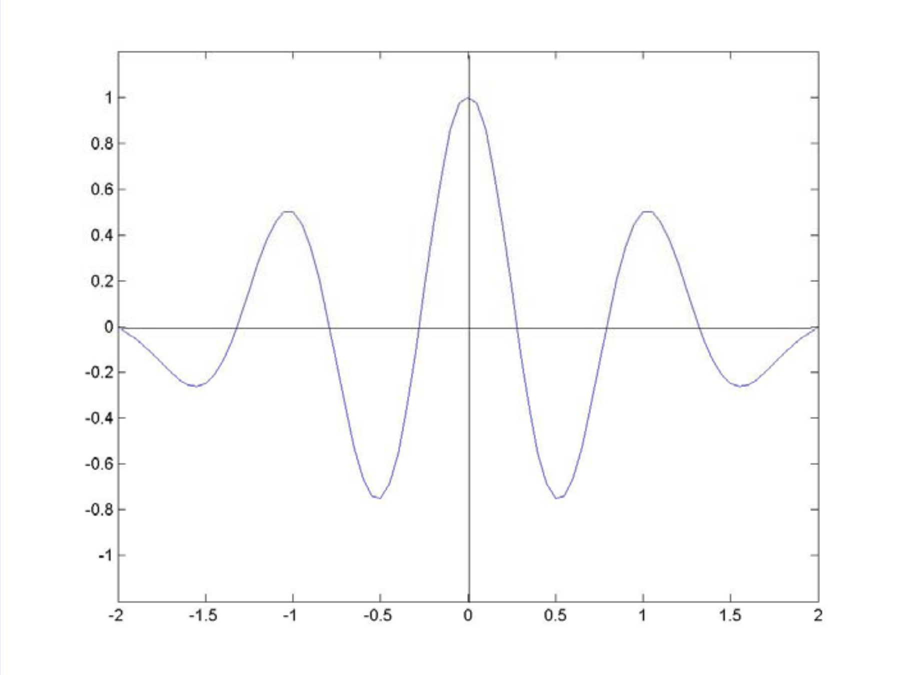


Fig. 1.33: $R_x(\tau)$ of Example 1.21

♦

1.7 Hilbert Transform

Let $x(t)$ be the input of an LTI system with the impulse response $h(t)$. Then, the output $y(t)$ is

$$y(t) = x(t) \ast h(t)$$

or $$Y(f) = X(f) \ast H(f)$$

That is, $$|Y(f)| = |X(f)|\,|H(f)| \quad (1.47a)$$

$$\theta_y(f) = \theta_x(f) + \theta_h(f) \quad (1.47b)$$

From Eq. 1.47 we see that an LTI system alters, in general, both the magnitude spectrum and the phase spectrum of the input signal. However, there are certain networks, called *all pass networks*, which would alter only the (input) phase spectrum. That is, if $H(f)$ is the frequency response of an all pass network, then

$$|Y(f)| = |X(f)|$$

$$\theta_y(f) = \theta_x(f) + \theta_h(f)$$

An interesting case of all-pass network is the ideal delay, with the impulse response $h(t) = \delta(t - t_d)$. Though $\theta_y(f) \neq \theta_x(f)$, phase shift imparted to each input spectral component is proportional to the frequency, the proportionality constant being $2\pi t_d$. Another interesting network is the Hilbert transformer. Its output is characterized by:

(i) $|Y(f)| = |X(f)|$, $f \neq 0$ and

(ii) $\theta_y(f) = \begin{cases} -\dfrac{\pi}{2} + \theta_x(f), & f > 0 \\ \dfrac{\pi}{2} + \theta_x(f), & f < 0 \end{cases}$

That is, a Hilbert transform is essentially a $\left(\pm \dfrac{\pi}{2}\right)$ phase shifter.

Hence, we define the Hilbert transformer in the frequency domain, with the frequency response function

$$H(f) = -j \operatorname{sgn}(f) \tag{1.48a}$$

where $\operatorname{sgn}(f) = \begin{cases} 1 & , f > 0 \\ 0 & , f = 0 \\ -1 & , f < 0 \end{cases}$

As $\left[-j \operatorname{sgn}(f)\right] \longleftrightarrow \dfrac{1}{\pi t}$,

$$h(t) = \frac{1}{\pi t} \tag{1.48b}$$

When $x(t)$ is the input to a Hilbert transformer, we denote its output as $\hat{x}(t)$ where

$$\hat{x}(t) = x(t) * \frac{1}{\pi t}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{(t - \tau)} d\tau \quad (1.49a)$$

[1]

and $\hat{X}(f) = -j \operatorname{sgn}(f) X(f)$ (1.49b)

$\hat{x}(t)$ is called the Hilbert transform of $x(t)$.

Note: Unlike other transforms, both $x(t)$ and $\hat{x}(t)$ are functions of the same variable (t in our case).

Hilbert Transform (HT) will prove quite useful later on in the study of bandpass signals and single sideband signals. For the present, let us look at some examples of HT.

Example 1.22

Hilbert transform $\delta(t)$.

Let $x(t) = \delta(t)$. Let us find $\hat{x}(t)$.

As $\delta(t) \longleftrightarrow 1$, we have

$$F\left[\hat{\delta}(t)\right] = -j \operatorname{sgn}(f) \Rightarrow$$

$$\hat{\delta}(t) = \frac{1}{\pi t}$$

This also establishes the relation, $\left[\delta(t) * \frac{1}{\pi t}\right] = \frac{1}{\pi t}$!! ♦

Example 1.23

HT of a cosine signal.

Let $x(t) = \cos(2\pi f_0 t)$. Let us find $\hat{x}(t)$.

[1] This integral is actually Cauchy's principal value.

$$X(f) = \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

$\hat{X}(f) = -j\,\mathrm{sgn}(f)\,X(f)$. That is,

$$\hat{X}(f) = \frac{1}{2j}\left[\delta(f - f_0) - \delta(f + f_0)\right]$$

That is, $\hat{x}(t) = \sin(2\pi f_0 t)$.

Alternatively,

if $x_1(t) = e^{j2\pi f_0 t}$, then $\hat{x}_1(t) = e^{j\left(2\pi f_0 t - \pi/2\right)}$

and if $x_2(t) = e^{-j2\pi f_0 t}$, then $\hat{x}_2(t) = e^{-j\left(2\pi f_0 t - \pi/2\right)}$

Hence,

$$x(t) = \cos(2\pi f_0 t) = \frac{1}{2}\left[x_1(t) + x_2(t)\right] \text{ has } \hat{x}(t) = \cos\left(\omega_0 t - \frac{\pi}{2}\right)$$

$$= \sin(\omega_0 t)$$

Similarly, we can show that if $x(t) = \sin(2\pi f_0 t)$, then $\hat{x}(t) = -\cos\omega_0 t$. ♦

Example 1.24

Let $x(t) = \dfrac{1}{1+t^2}$. Let us find $\hat{x}(t)$.

$$\hat{x}(t) = x(t) * \frac{1}{\pi t}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{\left(1+\tau^2\right)(t-\tau)}\,d\tau$$

$$= \frac{1}{\pi}\,\frac{1}{1+t^2}\left[\int_{-\infty}^{\infty} \frac{t+\tau}{1+\tau^2}\,d\tau + \int_{-\infty}^{\infty} \frac{d\tau}{(t-\tau)}\right]$$

As $\displaystyle\int_{-\infty}^{\infty} \frac{\tau}{1+\tau^2}\,d\tau = \int_{-\infty}^{\infty} \frac{1}{t-\tau}\,d\tau = 0$, we have

$$\hat{x}(t) = \frac{1}{\pi} \frac{t}{1+t^2} \left[\int_{-\infty}^{\infty} \frac{1}{1+\tau^2} d\tau \right]$$

As the bracketed integral is equal to π, we have

$$\hat{x}(t) = \frac{t}{1+t^2}$$ ♦

Exercise 1.8

(a) Let $x_1(t) = \frac{\sin t}{t}$. Show that $\hat{x}_1(t) = \frac{1 - \cos t}{t}$

(b) Let $x_2(t) = ga(t)$. Show that $\hat{x}_2(t) = -\frac{1}{\pi} \ln \left| \frac{t - \frac{1}{2}}{t + \frac{1}{2}} \right|$

Example 1.25

Let $x(t) = m(t) \cos 2\pi f_c t$ where $m(t)$ is a lowpass signal with $M(f) = 0$ for $|f| > W$ and $f_c > W$. We will show that

$$\hat{x}(t) = m(t) \sin(2\pi f_c t) = m(t) \widehat{\cos}(2\pi f_c t).$$

$$X(f) = \frac{1}{2}\left[M(f - f_c) + M(f + f_c)\right]$$

$$\hat{X}(f) = \frac{1}{2}\left[M(f - f_c) + M(f + f_c)\right]\left[-j\,\text{sgn}(f)\right]$$

But $\left[M(f - f_c)\right]$ is nonzero only for $f > 0$

and $\left[M(f + f_c)\right]$ is nonzero only for $f < 0$.

Hence,

$$\hat{X}(f) = \begin{cases} \frac{1}{2} M(f - f_c)\, e^{-j\pi/2}, & f > 0 \\ \frac{1}{2} M(f + f_c)\, e^{j\pi/2}, & f < 0 \end{cases}$$

Consider $m(t)\sin(2\pi f_c t) \longleftrightarrow M(f) * \frac{1}{2j}\left[\delta(f - f_c) - \delta(f + f_c)\right]$

$$\longleftrightarrow M(f) * \frac{1}{2}\left[\delta(f - f_c)\, e^{-j\frac{\pi}{2}} + e^{j\pi}\, e^{-j\frac{\pi}{2}}\, \delta(f + f_c)\right]$$

$$\longleftrightarrow M(f) * \frac{1}{2}\left[\delta(f - f_c)\, e^{-j\frac{\pi}{2}} + e^{j\frac{\pi}{2}}\, \delta(f + f_c)\right]$$

That is,

$$F\left[m(t)\,\sin(2\pi f_c t)\right] = \begin{cases} \frac{1}{2} M(f - f_c)\, e^{-j\frac{\pi}{2}}, & f > 0 \\ \frac{1}{2} M(f + f_c)\, e^{j\frac{\pi}{2}}, & f < 0 \end{cases}$$

As $F\left[m(t)\,\sin 2\pi f_c t\right] = \hat{X}(f)$, we have

$$\hat{x}(t) = m(t)\,\sin(2\pi f_c t) = m(t)\,\widehat{\cos}(2\pi f_c t) \qquad \blacklozenge$$

Note: It is possible to establish even a stronger result, which is stated below.

Let $x(t) = m(t)\,v(t)$ where $m(t)$ is a lowpass signal with $M(f) = 0$ for $|f| > W$ and $v(t)$ is a high-pass signal with $V(f) = 0$ for $|f| < W$. Then $\hat{x}(t) = m(t)\,\hat{v}(t)$. [We assume that there are no impulses in either $M(f)$ or $V(f)$.

1.7.1 Properties of Hilbert transform

Our area of application of HT is real signals. Hence, we develop the properties of HT as applied to real signals. We assume that the signals under consideration have no impulses in their spectra at $f = 0$.

P1) A signal $x(t)$ and its HT, $\hat{x}(t)$, have the same energy.

Proof: $E_x = \int_{-\infty}^{\infty} |X(f)|^2\, df$

As $\hat{X}(f) = -j \operatorname{sgn}(f) X(f)$

$\left|\hat{X}(f)\right| = \left|X(f)\right|$. Hence $E_x = E_{\hat{x}}$. ♦

Note: Though $\hat{X}(f)\Big|_{f=0}$ is zero, it will not change the value of the integral; and hence the energy.

P2) $\left\{HT\left[\hat{x}(t)\right]\right\} = -x(t)$

That is, applying the HT twice on a given signal $x(t)$ changes the sign of the signal. Intuitively, this is satisfying. Each time we perform HT, we change the phase of a spectral component in $X(f)$ by 90^0. Hence, performing the transformation twice results in a phase shift of 180^0.

Proof: $\hat{x}(t) \longleftrightarrow -j \operatorname{sgn}(f) X(f)$

$$HT\left[\hat{x}(t)\right] \longleftrightarrow -j \operatorname{sgn}(f)\left[-j \operatorname{sgn}(f) X(f)\right]$$

$$= \left(-j \operatorname{sgn}(f)\right)^2 X(f) = -X(f)$$ ♦

Example 1.26

Let $x(t) = \frac{1}{t}$. We shall find $\hat{x}(t)$.

From Example 1.22, we have $\hat{\delta}(t) = \frac{1}{\pi t}$.

Hence,

$$HT\left[\hat{\delta}(t)\right] = -\delta(t) = HT\left[\frac{1}{\pi t}\right].$$ That is,

$$HT\left[\frac{1}{t}\right] = -\pi\delta(t)$$ ♦

P3) A signal $x(t)$ and its HT, $\hat{x}(t)$, are orthogonal.

Proof: We know that

$$\int_{-\infty}^{\infty} x(t)\, \hat{x}(t)\, dt = \int_{-\infty}^{\infty} X(f)\, \hat{X}(-f)\, df$$

(See the note, property P10, Sec. 1.4). Hence,

$$\int_{-\infty}^{\infty} x(t)\, \hat{x}(t)\, dt = \int_{-\infty}^{\infty} X(f) \left[- j \operatorname{sgn}(-f)\, X(-f)\right] df$$

As $x(t)$ is real, $X(-f) = X^{*}(f)$. Therefore,

$$\int_{-\infty}^{\infty} x(t)\, \hat{x}(t)\, dt = \int_{-\infty}^{\infty} - j \operatorname{sgn}(-f) \left|X(f)\right|^2 df$$

$$= 0 \qquad \text{(Note that the integrand on the RHS is odd)}$$

♦

P4) $HT\left[x(\alpha t)\right]$, where α is a nonzero constant is $\left[\operatorname{sgn}(\alpha)\, \hat{x}(\alpha t)\right]$.

Proof: We will first establish that if $y(t) = x(t) * h(t)$ and $z(t) = x(\alpha t) * h(\alpha t)$, then $z(t) = \frac{1}{|\alpha|} y(\alpha t)$.

If $z(t) = x(\alpha t) * h(\alpha t)$, then

$$Z(f) = \frac{1}{|\alpha|^2} X\left(\frac{f}{a}\right) H\left(\frac{f}{a}\right)$$

Also, $y(\alpha t) \longleftrightarrow \frac{1}{|\alpha|} Y\left(\frac{f}{\alpha}\right)$, where

$$Y\left(\frac{f}{\alpha}\right) = X\left(\frac{f}{\alpha}\right) H\left(\frac{f}{\alpha}\right)$$

$$= |\alpha|^2 Z(f)$$

Hence, $y(\alpha t) \longleftrightarrow \frac{1}{|\alpha|} |\alpha|^2 Z(f) = |\alpha|\, Z(f)$

That is, $z(t) = \dfrac{y(\alpha t)}{|\alpha|}$

$$HT\left[x(\alpha t)\right] = x(\alpha t) * \frac{1}{\pi t}$$

$$= x(\alpha t) * \alpha\left(\frac{1}{\alpha \pi t}\right)$$

$$= \alpha\left[x(\alpha t) * \frac{1}{\alpha \pi t}\right]$$

As $\left[x(t) * \frac{1}{\pi t}\right] = \hat{x}(t)$, we have $\left[x(\alpha t) * \frac{1}{\pi \alpha t}\right] = \frac{\hat{x}(\alpha t)}{|\alpha|}$.

Hence $HT\left[x(\alpha t)\right] = \frac{\alpha}{|\alpha|}\hat{x}(\alpha t) = \text{sgn}(\alpha)\, \hat{x}(\alpha t)$. ◆

As a simple illustration of the property, let $x(t) = \frac{1}{1+t^2}$ and $\alpha = 2$. Then

$x(\alpha t) = \frac{1}{1+4t^2}$. Let us obtain $HT\left[\frac{1}{1+4t^2}\right]$ using P4.

As $\hat{x}(t) = \frac{t}{1+t^2}$, $HT\left[\frac{1}{1+4t^2}\right] = \text{sgn}(2)\left[\frac{2t}{1+(2t)^2}\right] = \frac{2t}{1+4t^2}$

If $\alpha = -2$, $x(\alpha t) = \frac{1}{1+4t^2}$ which is the same as with $\alpha = 2$.

With $\alpha = -2$, $\text{sgn}(\alpha)\, \hat{x}(\alpha t) = -\left(\frac{-2t}{1+4t^2}\right)$

$$= \frac{2t}{1+4t^2}, \text{ as required}$$

Exercise 1.9

Using the result $HT\left(\frac{\sin t}{t}\right) = \frac{1-\cos t}{t}$,

find the $HT\left[\text{sin}c(t)\right]$.

P5) The cross correlation function of $x(t)$ and $\hat{x}(t)$, $R_{x\hat{x}}(\tau)$ is the negative of the HT of $\hat{R}_x(\tau)$. That is,

$$R_{x\hat{x}}(\tau) = -\hat{R}_x(\tau)$$

Proof: $F\left[R_{x\hat{x}}(\tau)\right] = F\left[x(\tau) * \hat{x}(-\tau)\right]$

$$= X(f)\left[-j\,\text{sgn}(-f)\,X(-f)\right]$$

$$= |X(f)|^2\left[-j\,\text{sgn}(-f)\right] = |X(f)|^2\left[j\,\text{sgn}(f)\right]$$

That is,

$$R_{x\hat{x}}(\tau) = -\hat{R}_x(\tau)$$

♦

Exercise 1.10

Show that $R_{\hat{x}x}(\tau) = \hat{R}_x(\tau)$.

1.8 Bandpass Signals

Consider a communication system that transmits the signal $s(t) = m(t)\cos 2\pi f_c t$, where $m(t)$ is a (lowpass) *message* signal and $\cos(2\pi f_c t)$ is the (high-frequency) *carrier* term. Then the spectrum $S(f)$ of the transmitted signal is $S(f) = \frac{1}{2}\left[M(f - f_c) + M(f + f_c)\right]$. If $M(f)$ is as shown in Fig 1.34(a), then for a carrier frequency $f_c = 100\,kHz$, $S(f)$ will be as shown in Fig 1.34(b).

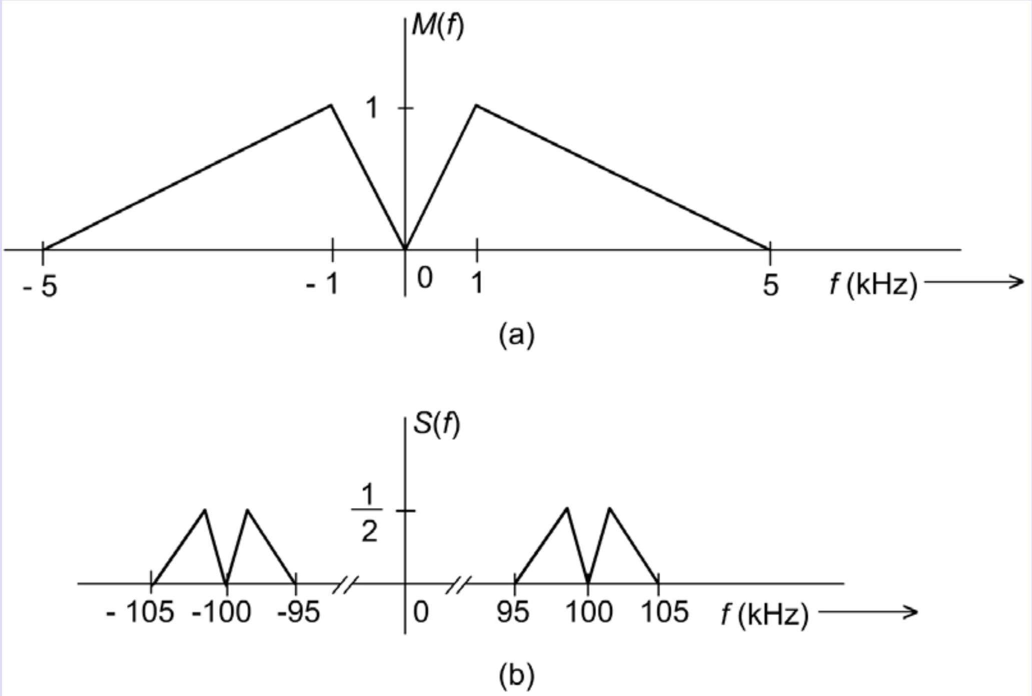


Fig. 1.34: A typical narrowband, bandpass spectrum

We see that the spectrum of $s(t)$ is confined to the frequency interval $95 \leq |f| \leq 105\ kHz$. Where as $m(t)$ is a lowpass signal, $s(t)$ is a bandpass signal. Moreover $s(t)$ is a narrowband, bandpass signal because the spectral width of $S(f)$, $(105 - 95) = 10\ kHz$, is quite small in comparison with the carrier frequency f_c of $100\ kHz$. Hence, we call $s(t)$ as a *Narrowband, Bandpass (NBBP)* signal. Fig 1.35 shows some more spectra that represent NBBP signals.

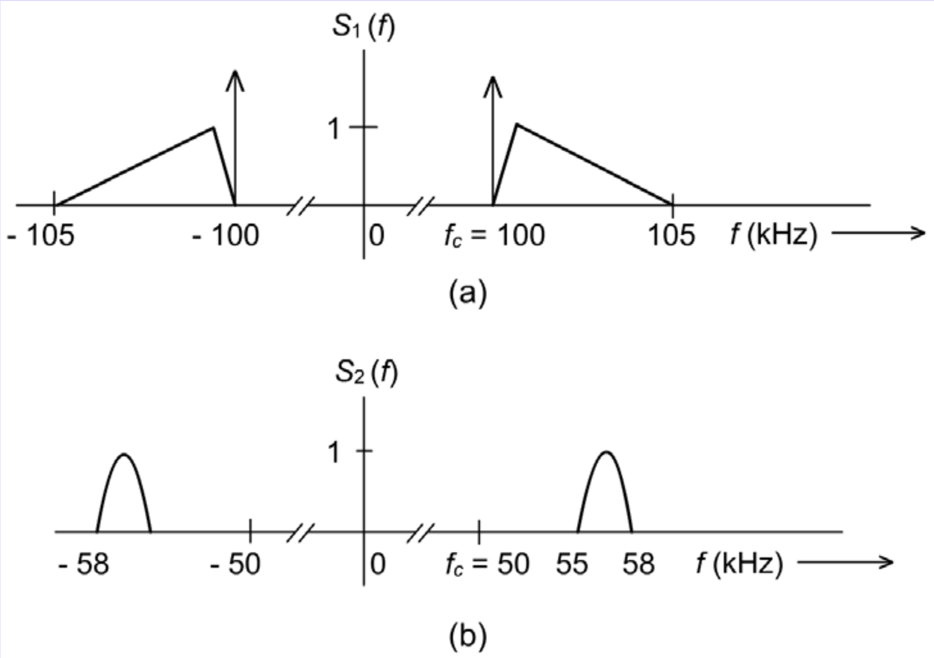


Fig. 1.35: A few more examples of NBBP spectra

NBBP signals play an important role in the communication process. Let us assume that the rest of the communication system (channel, a part of the receiver etc.) is also of the bandpass variety. The study of such transmission-reception schemes becomes a little complicated because of the presence of the carrier term in some form or the other (The term $\cos \omega_c t$ in $m(t) \cos \omega_c t$ is meant only to "carry" the information $m(t)$ and is not part of the information) If we develop tools to study bandpass signals and bandpass systems, independent of the carrier, the analysis of the communications schemes would become somewhat simplified. (That is, bandpass signals and bandpass systems are studied in terms of their *lowpass equivalents*.) The mathematical concepts of pre-envelope and complex envelope have been developed for this purpose. We shall make use of these concepts in our studies on linear modulation and angle modulation.

1.8.1 Pre-envelope

Def. 1.10: Let $x(t)$ be any real signal with the FT, $X(f)$. We define its *pre-envelope* as,

$$x_{pe}(t) = x(t) + j\,\hat{x}(t) \qquad (1.50a)$$

♦

Taking the Fourier transform of Eq. 1.54(a), we have

$$X_{pe}(f) = X(f) + j\left[- j\,\mathrm{sgn}(f)\,X(f)\right]$$

$$= X(f) + \mathrm{sgn}(f)\,X(f)$$

That is,

$$X_{pe}(f) = \begin{cases} 2\,X(f) & , f > 0 \\ X(0) & , f = 0 \\ 0 & , f < 0 \end{cases} \qquad (1.50b)$$

(We assume that $X(f)$ has no impulse at $f = 0$). That is, $x_{pe}(t)$ has spectrum only for $f \geq 0$, even though $X(f)$ is two sided (As $x_{pe}(t)$ has spectral components only for $f \geq 0$, some authors use the symbol $x_{+}(t)$ to denote the pre-envelope of $x(t)$). Of course, $x^{*}_{pe}(t)$ will have spectrum only for $f \leq 0$). Consider the signals $x_1(t)$ and $x_2(t)$ whose spectra are shown in Fig. 1.36.

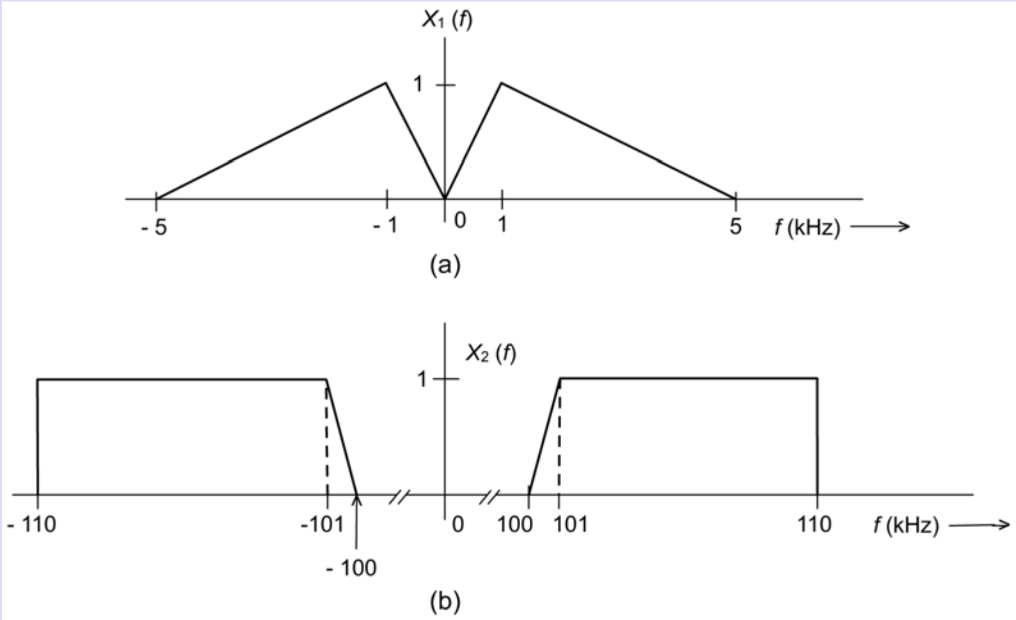


Fig. 1.36: Typical two-sided spectra

The corresponding $X_{1,\,pe}(f)$ and $X_{2,\,pe}(f)$ are as shown in Fig. 1.37(a) and (b) respectively.

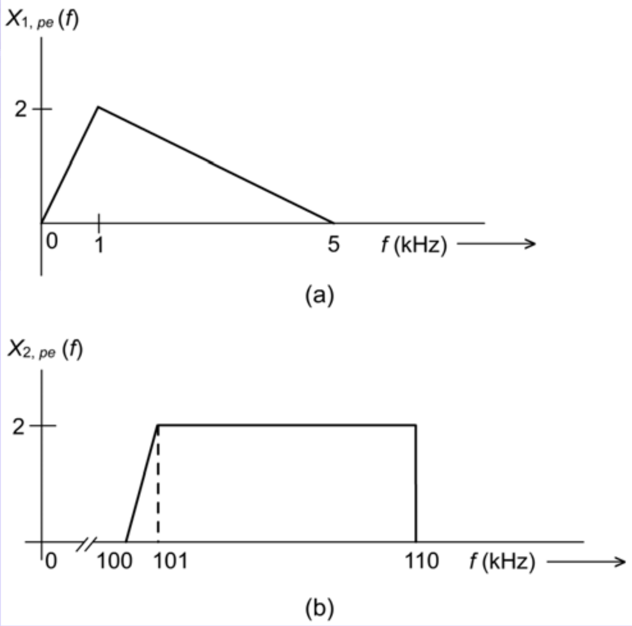


Fig. 1.37: Fourier transform of (a) $x_{1,\,pe}(t)$ and (b) $x_{2,\,pe}(t)$ of the signals in

Because of Eq. 1.50(b), we can write

$$x_{pe}(t) = 2\int_{0}^{\infty} X(f)\, e^{j 2\pi f t}\, df \tag{1.51}$$

Example 1.27

Let $x(t) = \dfrac{1}{1+t^2}$. Let us find $X_{pe}(f)$ and $x_{pe}(t)$.

First, let us compute $X(f)$. From Example 1.5, we know that $\dfrac{1}{1+t^2} \longleftrightarrow \pi\, e^{-|2\pi f|}$.

Hence $X_{pe}(f) = 2\pi\, e^{-2\pi f}\, u(f)$, where $u(f) = \begin{cases} 1\,, & f > 0 \\ \dfrac{1}{2}\,, & f = 0 \\ 0\,, & f < 0 \end{cases}$.

We require $F^{-1}\left[X_{pe}(f)\right]$.

As $ex1(t) \longleftrightarrow \dfrac{1}{1+j\,2\pi f}$

$ex1(2\pi t) \longleftrightarrow \dfrac{1}{2\pi}\,\dfrac{1}{1+jf}$.

From duality, $2\pi\, ex1(2\pi f) = 2\pi e^{-2\pi f}\, u(f) \longleftrightarrow \dfrac{1}{1-jt}$.

That is, $x_{pe}(t) = \dfrac{1}{1-jt} = \dfrac{1+jt}{1+t^2}$

$$= \frac{1}{1+t^2} + j\,\frac{t}{1+t^2}.$$

Then, $\hat{x}(t) = \dfrac{t}{1+t^2}$, which is a known result. ♦

As can be seen from the above discussion, the spectrum of $x_{pe}(t)$ is still bandpass (though one-sided) if $x(t)$ is a bandpass signal. Let $x(t)$ be a bandpass signal with $X_{pe}(f)$ "centered" with respect to $f_c = 102\ kHz$ as shown in Fig. 1.38(a).

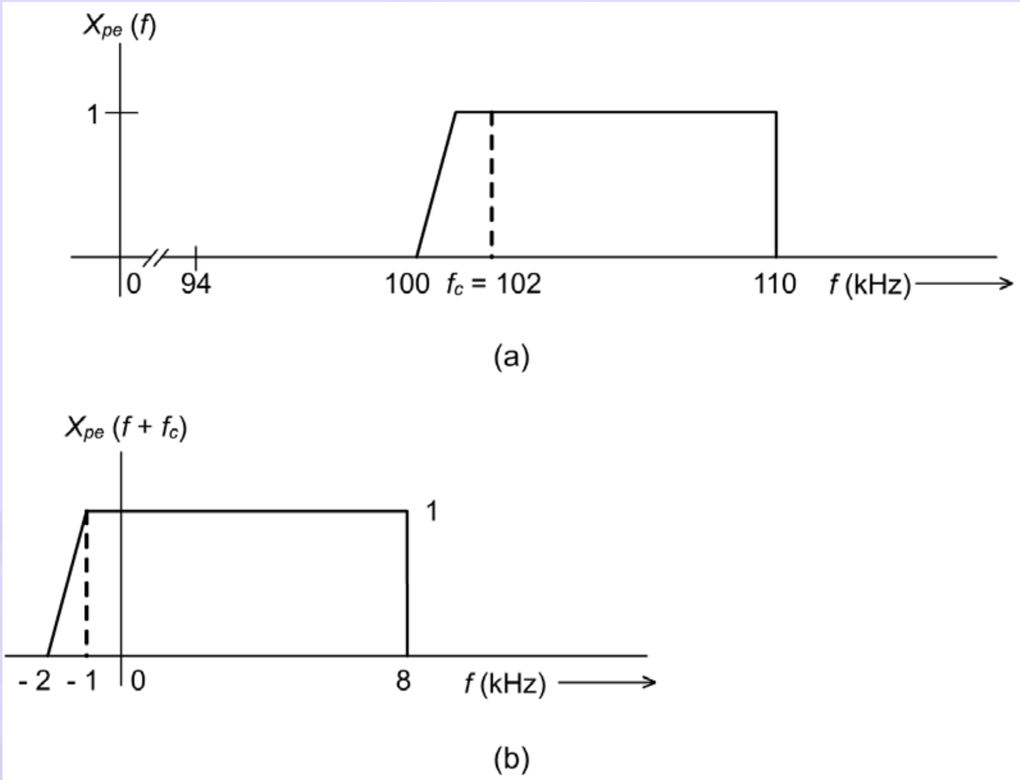


Fig 1.38: A typical $X_{pe}(f)$ and shifted version

(We can treat f_c to be the center frequency in Fig. 1.38(a), by taking the bandpass spectrum from 94 to 110 kHz, though the spectrum is zero for the frequency range 94 to 100 kHz.) From Fig 1.38(b), we see that $X_{pe}(f + f_c)$ is a lowpass spectrum, nonzero in the frequency range (-2) to 8 kHz.

1.8.2 Complex envelope

Def. 1.11: We now define the *complex envelope* of $x(t)$, denoted $x_{ce}(t)$ as

$$x_{ce}(t) = x_{pe}(t)\, e^{-j 2 \pi f_c t} \quad (1.52a)$$

♦

Eq. 1.52(a) implies

$$X_{ce}(f) = X_{pe}(f + f_c) \quad (1.52b)$$

(We assume that we know the center frequency f_c and it is such that $X_{pe}(f + f_c)$ is lowpass in nature).

Equation 1.52(a) also implies

$$x_{pe}(t) = x_{ce}(t)\, e^{j 2 \pi f_c t} \quad (1.53)$$

$x_{ce}(t)$ is also referred to as the *equivalent lowpass signal* of the bandpass signal $x(t)$. In general $x_{ce}(t)$ is complex. Let $x_c(t)$ be the real part and $x_s(t)$ its imaginary part. Then,

$$x_{ce}(t) = x_c(t) + j\, x_s(t) \quad (1.54)$$

We will show a little later that, both $x_c(t)$ and $x_s(t)$ are lowpass in nature. Using Eq. 1.54 in Eq. 1.53, we obtain,

$$x_{pe}(t) = \left[x_c(t) + j\, x_s(t)\right] e^{j 2 \pi f_c t}$$

As the real bandpass signal $x(t)$ is the real part of $x_{pe}(t)$, we have

$$x(t) = x_c(t)\cos(2\pi f_c t) - x_s(t)\sin(2\pi f_c t) \quad (1.55)$$

Eq. 1.55 is referred to as the *canonical representation* of the bandpass signal. $x_c(t)$, which is the coefficient of the cosine term, is usually referred to as the *in-phase* component and $x_s(t)$, the coefficient of the sine term, as the *quadrature* component. Note that $\sin(2\pi f_c t)$ is in phase quadrature to $\cos(2\pi f_c t)$. (It is also common in the literature to use the symbol $x_I(t)$ for the in-phase component and $x_Q(t)$ for the quadrature component.)

To express Eq. 1.54 in polar form, let

$$A(t) = \sqrt{x_c^2(t) + x_s^2(t)} \tag{1.56a}$$

$$\varphi(t) = \tan^{-1}\left[\frac{x_s(t)}{x_c(t)}\right] \tag{1.56b}$$

Then,

$$x_{ce}(t) = A(t)\, e^{j\varphi(t)} \tag{1.56c}$$

$$x(t) = \text{Re}\left[x_{pe}(t)\right] = \text{Re}\left[x_{ce}(t)\, e^{j 2\pi f_c t}\right] \tag{1.57}$$

$$= \text{Re}\left[A(t)\, e^{j\varphi(t)}\, e^{j 2\pi f_c t}\right] \tag{1.58}$$

Eq. 1.58 resembles phasor representation of a sinusoid. We know that,

$$A \cos(2\pi f_c t + \varphi) = \text{Re}\left[A\, e^{j\varphi}\, e^{j 2\pi f_c t}\right] \tag{1.59}$$

$\left[A\, e^{j\varphi}\right]$ is generally referred to the as the *phasor* associated with the sinusoidal signal $A \cos(2\pi f_c t + \varphi)$. (The phasor is a complex number providing information about the amplitude and phase (at $t = 0$) of the sinusoid.) The quantity $\left[A\, e^{j\varphi}\, e^{j 2\pi f_c t}\right]$ can be treated as a rotating vector as shown in Fig. 1.39(a). Comparing Eq. 1.58 with Eq. 1.59, we find that they have a close resemblance. Phasor of the monochromatic (single frequency) signal has constant amplitude A and a fixed phase φ. In the case of the complex envelope (of a narrowband signal) both $A(t)$ and $\varphi(t)$ are, (slowly) time-varying. This is shown in Fig 1.39(b). (Note that a single frequency sinusoid is the extreme case of a narrowband signal with *zero* spectral width!)

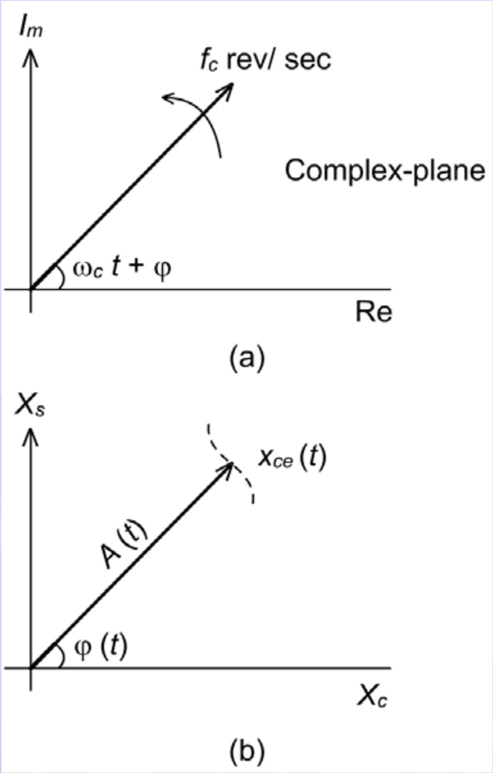


Fig. 1.39: (a) Phasor representation of $A\, e^{j\left(2\pi f_c t \,+\, \varphi\right)}$

(b) Complex envelope as a (slowly) varying amplitude and phase

In other words, complex envelope can be treated as a generalization of the phasor representation used for single frequency sinusoids; the generalization permits the amplitude and phase to change as a function of time. Note, however, that for a given time $t = t_1$, $A(t_1)\, e^{j\varphi(t_1)}$ is a complex number, having the necessary information about the narrowband signal. As we shall see later, different modulation schemes are basically different methods of controlling either $A(t)$ or $\varphi(t)$ (or both) as a function of the message signal $m(t)$.

If $x(t)$ is a NBBP signal with spectrum confined to the frequency range $|f \pm f_c| \leq W$, then $x_c(t)$ and $x_s(t)$ are lowpass signals with spectrum confined to $|f| \leq W$. This is because $X_{ce}(f) = X_{pe}(f + f_c)$ is nonzero only for $|f| \leq W$.

As $x_c(t)$ is the real part of $x_{ce}(t)$, we have

$$x_c(t) = \frac{\left[x_{ce}(t) + x_{ce}^*(t)\right]}{2}$$

or $$F\left[x_c(t)\right] = X_c(f) = \frac{X_{ce}(f) + X_{ce}^*(-f)}{2}$$

As $X_{ce}(f)$ and $X_{ce}^*(-f)$ are zero for $|f| > W$, $X_c(f)$ is also zero for $|f| > W$. Similarly, $x_s(t)$ is also a lowpass signal with $X_s(f) = 0$ for $|f| > W$.

The scheme shown below (Fig 1.40) enables us to obtain $x_c(t)$ and $x_s(t)$ from $x(t)$.

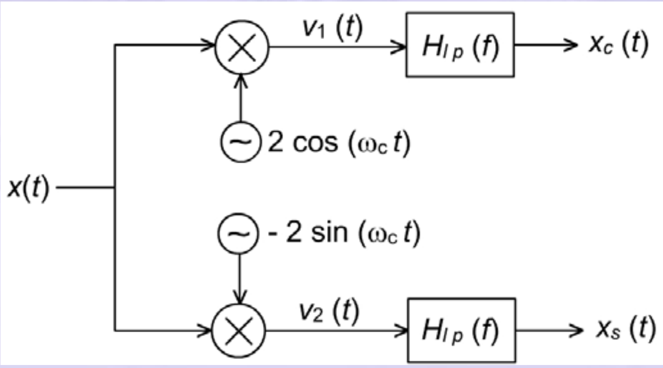


Fig. 1.40: Scheme for the recovery of $x_c(t)$ and $x_s(t)$ from $x(t)$

In Fig 1.40, $x(t)$ a NBBP signal, with the spectrum confined to the interval $|f \pm f_c| \leq W$, where $W << f_c$.

$$v_1(t) = 2x(t)\cos\omega_c t$$

$$= 2\left(x_c(t)\cos\omega_c t - x_s(t)\sin\omega_c t\right)\cos\omega_c t$$

$$= 2\, x_c(t)\cos^2\omega_c t - x_s(t)\sin 2\omega_c t$$

$$= 2\, x_c(t)\left[\frac{1 + \cos 2\omega_c t}{2}\right] - x_s(t) \sin 2\omega_c t$$

$$= x_c(t) + x_c(t)\cos(2\omega_c t) - x_s(t)\sin 2\omega_c t \qquad (1.60)$$

In Fig. 1.40, $H_{lp}(f)$ is an Ideal Lowpass Filter (ILPF) with the frequency response.

$$H_{lp}(f) = \begin{cases} 1\,, & |f| \le W \\ 0\,, & outside \end{cases}$$

$x_c(t)$ and $x_s(t)$ are lowpass signals, band limited to W Hz. $x_c(t)\cos(2\omega_c t)$ and $x_s(t)\sin(2\omega_c t)$ have bandpass spectra centered at $\pm f_c$. These quantities will be filtered out by the ILPF and at the output of the top channel, we obtain $x_c(t)$. Similar analysis will show that the output of the bottom channel is $x_s(t)$.

From Eq. 1.58, we have,

$$x(t) = A(t)\cos\left[\omega_c t + \varphi(t)\right] \qquad (1.61)$$

Eq. 1.61 is referred to as the *envelope and phase* representation of $x(t)$. $A(t)$ is called the *natural envelope* (or simply the *envelope*) of $x(t)$ and $\varphi(t)$, its phase. As we assume that f_c is known, the information about $x(t)$ is contained in either of the quantities $\left(x_c(t), x_s(t)\right)$ or $\left[A(t), \varphi(t)\right]$ and these are lowpass in nature. Note that

$$A(t) = \left|x_{ce}(t)\right| = \left|x_{pe}(t)\right| \qquad (1.62)$$

and is always non-negative.

We shall now illustrate the concepts of $x_{pe}(t)$, $x_{ce}(t)$, $A(t)$ and $\varphi(t)$ with the help of a few examples.

Example 1.28

Let $x(t) = \cos\left[2\pi f_c t\right]$. Let us find $x_{pe}(t)$, $x_{ce}(t)$, $A(t)$ and $\varphi(t)$.

$$X(f) = \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$

From Eq. 1.50(b), we have

Hence $X_{pe}(f) = \delta(f - f_c) \Rightarrow x_{pe}(t) = e^{j 2\pi f_c t}$

As $X_{ce}(f) = \delta(f)$, we obtain $x_{ce}(t) = 1$

As $x_{ce}(t)$ is real and positive, $\varphi(t) = 0$ and $A(t) = \left|x_{ce}(t)\right| = 1$.

♦

Example 1.29

Let $x(t) = ga\left(\frac{t}{T}\right)\cos\omega_c t$. Assume that $f_c T \gg 1$ so that $x(t)$ can be taken as a NBBP signal. We shall find $x_{pe}(t)$, $x_{ce}(t)$ and $A(t)$.

Method 1 (Frequency domain):

Because of the assumption $f_c T \gg 1$, we can take $X(f)$ approximately as

$$X(f) \simeq \begin{cases} \frac{T}{2}\,\text{sinc}\left[(f - f_c)T\right] & , f > 0 \\ \frac{T}{2}\,\text{sinc}\left[(f + f_c)T\right] & , f < 0 \end{cases}$$

From Eq. 1.50(b),

$$X_{pe}(f) = \begin{cases} T\,\text{sinc}\left[(f - f_c)T\right] & , f > 0 \\ 0 & , \textit{otherwise} \end{cases}$$

Hence,

$$x_{pe}(t) = ga\left(\frac{t}{T}\right)e^{j 2\pi f_c t}$$

$$x_{ce}(t) = x_{pe}(t)\, e^{-j 2 \pi f_c t}$$

$$= ga\left(\frac{t}{T}\right)$$

As $x_{ce}(t)$ is real, we have $x_s(t) = 0 \Rightarrow \varphi(t) = 0$.

$$A(t) = |x_{ce}(t)| = g\, a\left(\frac{t}{T}\right)$$

Note that for Examples 1.28 and 1.29, the complex envelope is real valued and is equal to the envelope, $A(t)$.

Method 2 (Time domain):

Comparing the given $x(t)$ with Eq. 1.55, we find that it is already in the canonic form with $x_c(t) = ga\left(\frac{t}{T}\right)$, $x_s(t) = 0$

Hence, $x_{ce}(t) = ga\left(\frac{t}{T}\right)$

And $x_{pe}(t) = x_{ce}(t)\, e^{j 2 \pi f_c t}$

$$= ga\left(\frac{t}{T}\right) e^{j 2 \pi f_c t}$$

$$A(t) = ga\left(\frac{t}{T}\right) \text{ and } \varphi(t) = 0$$

♦

Example 1.30

$x(t)$ is a NBBP signal with $X(f)$ as shown in Fig 1.41. Let us find $x_c(t)$ and $x_s(t)$.

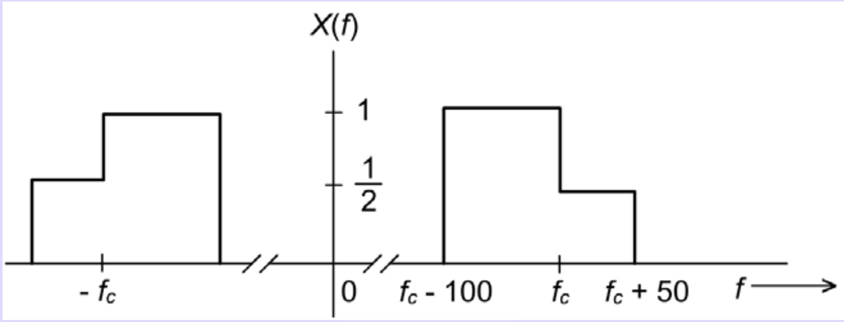


Fig. 1.41: Spectrum of the NBBP signal of Example 1.30

From the given spectrum, we can obtain $X_{ce}(f)$, which is shown below in Fig. 1.42.

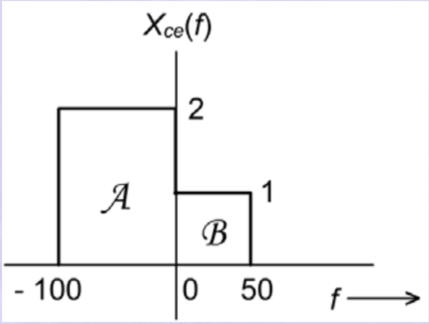


Fig. 1.42: Spectrum of the complex envelope of the signal of Example 1.30

As shown in the Fig 1.42, we can take $X_{ce}(f)$ as the sum of $\mathcal{A}$ and $\mathcal{B}$.

Inverse Fourier transform of $\mathcal{A}$: $200 \operatorname{sinc}(100t)\, e^{-j100\pi t}$

Inverse Fourier transform of $\mathcal{B}$: $50 \operatorname{sinc}(50t)\, e^{j50\pi t}$

Hence $x_{ce}(t) = 200 \operatorname{sinc}(100t)\, e^{-j100\pi t} + 50 \operatorname{sinc}(50t)\, e^{j50\pi t}$

$$x_c(t) = 200 \operatorname{sinc}(100t)\cos(100\pi t) + 50 \operatorname{sinc}(50t)\cos(50\pi t)$$

$$x_s(t) = 50 \operatorname{sinc}(50t)\sin(50\pi t) - 200 \operatorname{sinc}(100t)\sin(100\pi t)$$

♦

Exercise 1.11

Find the pre-envelope of $x(t) = \text{sin}c(t)$.

Hint: use the result of Exercise 1.9.

Exercise 1.12

Let $x(t) = ex1(t)\sin\left[(\omega_c + \Delta\omega)\,t\right]$ where $\omega_c >> \Delta\omega$.

Find $x_{ce}(t)$.

Exercise 1.13

Find the expression for the envelope of

$x(t) = \left[1 + k\cos(\omega_m t)\right]\cos(\omega_c t)$ where k is a real constant and $f_c >> f_m$. Sketch it for $k = 0.5$ and 1.5.

1.9 Bandpass (BP) Systems

Let a signal $x(t)$ be input to an LTI system with impulse response $h(t)$. If $y(t)$ is the output, then $Y(f) = X(f)\,H(f)$ and

$$|Y(f)|^2 = |X(f)|^2\,|H(f)|^2$$

If $E_x(f)$ is the energy spectral density of $x(t)$, then

$$E_y(f) = E_x(f)\,|H(f)|^2 \qquad (1.63a)$$

Let $|H(f)|^2 \longleftrightarrow R_h(\tau)$ be the Fourier transform pair where $R_h(\tau)$ is the ACF of the impulse response $h(t)$. Then,

$$R_y(\tau) = R_x(\tau) * R_h(\tau)$$

$$= R_x(\tau) * \left[h(\tau) * h^*(-\tau)\right] \qquad (1.63b)$$

Let $x(t)$ be a bandpass signal with $X(f)$ zero for $|f \pm f_c| > W$. A bandpass signal is usually processed a *bandpass system*; that is, a system with passband in the interval $|f \pm f_c| < B$ where $B \leq W$. We would like to study the effect of a BP system on a BP input. This study is again greatly facilitated if we were to use complex envelopes of the signals involved.

In appendix A1.3 (Eq A1.3.8), it is shown that

$$y_{ce}(t) = \frac{1}{2}\left[x_{ce}(t) * h_{ce}(t)\right] \tag{1.64}$$

We shall now give an example to illustrate the use of Eq. 1.64.

Example 1.31

Let $x(t)$ be a sinusoidal pulse given by

$$x(t) = \begin{cases} 2\cos\left[2\pi\left(10^6\right)t\right], & 0 \leq t \leq 1\, m\text{sec} \\ 0, & \textit{outside} \end{cases}$$

$x(t)$ is the input to an LTI system with impulse response $h(t) = x(T - t)$, where $T = 1\, m\text{sec}$. Find $y_{ce}(t)$ and $y(t)$.

$x(t)$ can be taken as a NBBP signal. Its complex envelope,

$$x_{ce}(t) = 2,\ 0 \leq t \leq 1\, m\text{sec}$$

$$h(t) = 2\cos\left[2\pi\left(10^6\right)(T - t)\right]$$

$$= 2\left\{\cos\left(2\pi\, 10^6\, T\right)\cos 2\pi\left(10^6\right)t + \sin\left(2\pi\, 10^6\, T\right)\sin 2\pi\left(10^6\right)t\right\}$$

$$= 2\left\{\cos\left(2\pi \times 10^3\right)\cos\left[2\pi\left(10^6\right)t\right] + \sin\left(2\pi \times 10^3\right)\sin 2\pi \times 10^6\, t\right\}$$

$$= 2\cos\left[2\pi\left(10^6\right)t\right],\ 0 \leq t \leq 1\, m\text{sec}$$

Again, we can treat $h(t)$ as a NBBP signal, with

$$h_{ce}(t) = \begin{cases} 2, & 0 \le t \le 1\, m\sec \\ 0, & outside \end{cases}$$

That is, $x_{ce}(t) = h_{ce}(t) = 2\, ga\left(\dfrac{t - T/2}{T}\right)$

From Eq.1.64,

$$y_{ce}(t) = \frac{1}{2}\left[x_{ce}(t) * h_{ce}(t)\right] = \left(2 \times 10^{-3}\right) tri\left(\frac{t - T}{T}\right)$$

$$y(t) = y_{ce}(t)\cos(2\pi f_c t)$$

Obtaining $y(t)$ directly as $\left[x(t) * h(t)\right]$ would be quite cumbersome. ♦

Exercise 1.14

Let $x(t) = \text{sin}c(200\, t)\cos\left[2\pi\left(10^6\right)t\right]$ be the input to an NBBP system with $H(f) = \dfrac{1}{2j}\left[H_1\left(f - 10^6\right) - H_1\left(f + 10^6\right)\right]$ (1.65)

where $H_1(f)$ is as shown in Fig. 1.43.

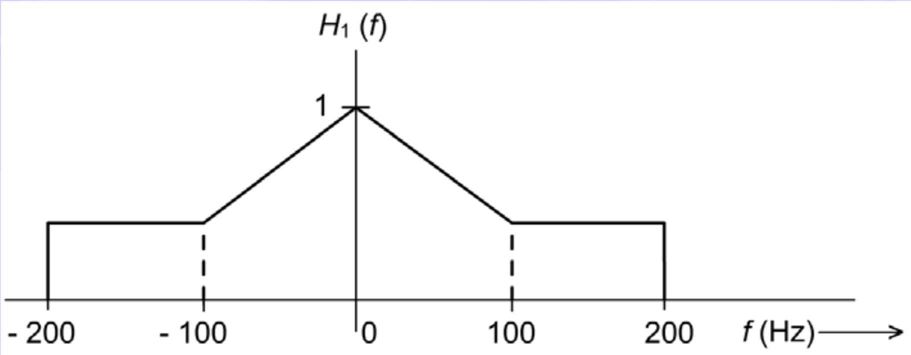


Fig. 1.43: $H_1(f)$ of Eq. 1.65

Find $y_{ce}(t)$, the complex envelope of the output.

Appendix A1.1

Tabulation of sinc (λ)

λ	sinc (λ)	λ	sinc (λ)	λ	sinc (λ)
0.00	1.000000	1.70	-0.151481	3.40	-0.089038
0.05	0.995893	1.75	-0.128616	3.45	-0.091128
0.10	0.983631	1.80	-0.103943	3.50	-0.090946
0.15	0.963397	1.85	-0.078113	3.55	-0.088561
0.20	0.935489	1.90	-0.051770	3.60	-0.084092
0.25	0.900316	1.95	-0.025536	3.65	-0.077703
0.30	0.858393	2.00	0.000000	3.70	-0.069600
0.35	0.810331	2.05	0.024290	3.75	-0.060021
0.40	0.756826	2.10	0.046840	3.80	-0.049237
0.45	0.698645	2.15	0.067214	3.85	-0.037535
0.50	0.636619	2.20	0.085045	3.90	-0.025222
0.55	0.571619	2.25	0.100035	3.95	-0.012607
0.60	0.504550	2.30	0.111964	4.00	-0.000000
0.65	0.436331	2.35	0.120688	4.05	0.012295
0.70	0.367882	2.40	0.126138	4.10	0.023991
0.75	0.300104	2.45	0.128323	4.15	0.034821
0.80	0.233871	2.50	0.127324	4.20	0.044547
0.85	0.170010	2.55	0.123291	4.25	0.052960
0.90	0.109291	2.60	0.116435	4.30	0.059888
0.95	0.052414	2.65	0.107025	4.35	0.065199
1.00	-0.000001	2.70	0.095377	4.40	0.068802
1.05	-0.047424	2.75	0.081847	4.45	0.070650
1.10	-0.089422	2.80	0.066821	4.50	0.070736
1.15	-0.125661	2.85	0.050705	4.55	0.069097
1.20	-0.155915	2.90	0.033919	4.60	0.065811
1.25	-0.180064	2.95	0.016880	4.65	0.060993
1.30	-0.198091	3.00	0.000000	4.70	0.054791
1.35	-0.210086	3.05	-0.016326	4.75	0.047385
1.40	-0.216236	3.10	-0.031730	4.80	0.038979
1.45	-0.216821	3.15	-0.045876	4.85	0.029796
1.50	-0.212203	3.20	-0.058468	4.90	0.020074
1.55	-0.202833	3.25	-0.069255	4.95	0.010059
1.60	-0.189207	3.30	-0.078036	5.00	0.000000
1.65	-0.171888	3.35	-0.084661		

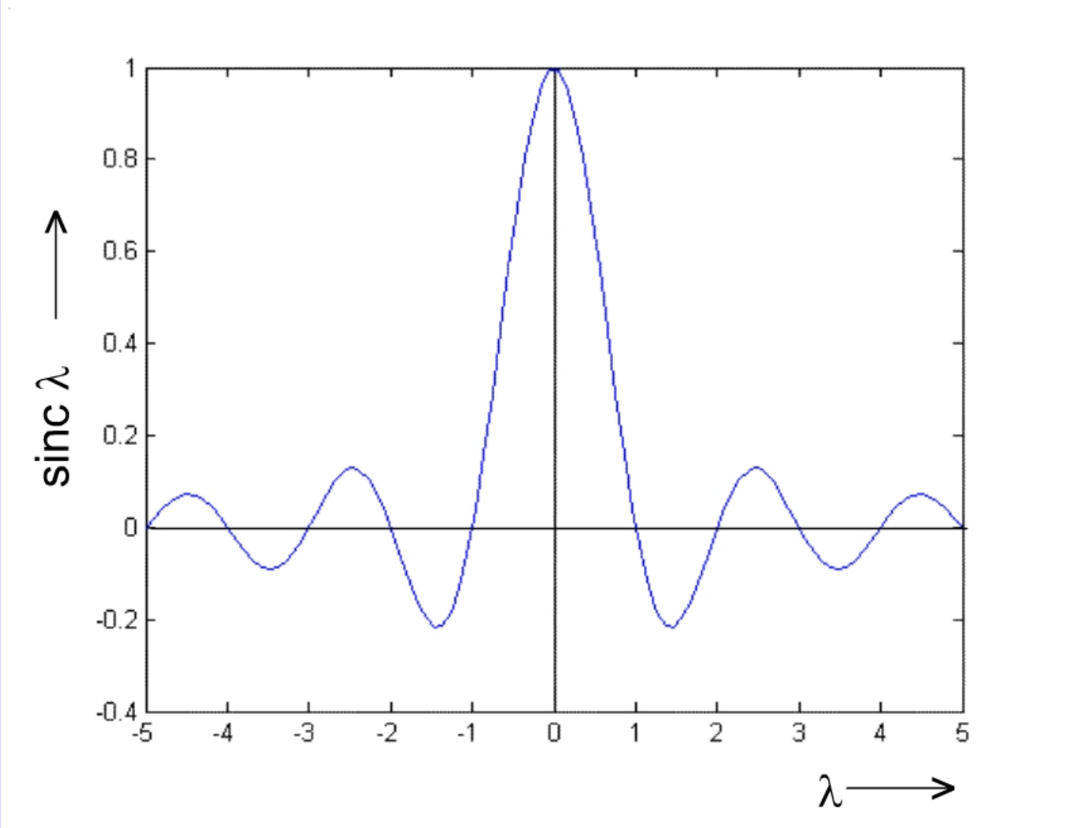


Fig. A1.1: Plot of $\text{sin}c\,(\lambda)$ for $|\lambda| \leq 5$

Appendix A1.2

Fourier transform of $R_{x_p y_p}(\tau)$

As $R_{x_p y_p}(\tau)$ is periodic, we expect the spectrum to be purely impulsive. We have only to decide the weights of these impulses.

$$R_{x_p y_p}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_p(t)\, y_p^*(t-\tau)\, dt$$

Let $x(t) = \begin{cases} x_p(t) , & -\dfrac{T_0}{2} < t < \dfrac{T_0}{2} \\ 0 , & outside \end{cases}$

then, $x_p(t) = \sum_{n=-\infty}^{\infty} x(t - nT_0)$

Similarly, $y_p^*(t-\tau) = \sum_{n=-\infty}^{\infty} y^*(t-\tau-nT_0)$

where $y^*(t) = \begin{cases} y_p^*(t) , & -\dfrac{T_0}{2} < t < \dfrac{T_0}{2} \\ 0 , & outside \end{cases}$

$$R_{x_p y_p}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \left[\sum_{n=-\infty}^{\infty} y^*(t-\tau-nT_0) \right] dt$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\; y^*(t-\tau-nT_0)\, dt$$

As $x(t) = 0$ for $|t| > \dfrac{T_0}{2}$,

$$R_{x_p y_p}(\tau) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t)\, y^*(t - \tau - nT_0)\, dt$$

$$= \frac{1}{T_0} \sum_{n=-\infty}^{\infty} R_{xy}(\tau + nT_0)$$

Taking the Fourier transform on both sides,

$$\mathrm{F}\left[R_{x_p y_p}(\tau)\right] = \frac{1}{T_0} F\left[\sum_{n=-\infty}^{\infty} R_{xy}(\tau + nT_0)\right]$$

As $R_{xy}(\tau) \longleftrightarrow X(f)\, Y^*(f)$,

$$\mathrm{F}\left[R_{x_p y_p}(\tau)\right] = \frac{1}{T_0}\left[\sum_{n=-\infty}^{\infty} X(f)\, Y^*(f)\, e^{j 2\pi n f T_0}\right]$$

$$= \frac{1}{T_0}\left[X(f)\, Y^*(f) \sum_{n=-\infty}^{\infty} e^{j 2\pi n f T_0}\right] \qquad \text{(A1.2.1)}$$

But from Example 1.12, we have

$$\sum_{m=-\infty}^{\infty} \delta(t - mT_0) = F^{-1}\left[\frac{1}{T_0} \sum_{m=-\infty}^{\infty} \delta(f - n f_0)\right] \qquad \text{(A1.2.2)}$$

$$= \frac{1}{T_0}\left[\sum_{m=-\infty}^{\infty} e^{j 2\pi m f_0 t}\right]$$

Replacing t by f, T_0 by f_0, we get the dual relation

$$\sum_{m=-\infty}^{\infty} \delta(f - m f_0) = \frac{1}{f_0}\left[\sum_{m=-\infty}^{\infty} e^{j 2\pi m f T_0}\right]$$

As $f_0 = \frac{1}{T_0}$, we have

$$\frac{1}{T_0} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T_0}\right) = \sum_{m=-\infty}^{\infty} e^{j 2\pi m f T_0} \qquad \text{(A1.2.3)}$$

Using Eq. A1.2.3 in Eq. A1.2.1, we obtain,

$$F\left[R_{x_p y_p}(\tau)\right] = \frac{1}{T_0^2}\left[\sum_{m = -\infty}^{\infty} X\left(\frac{m}{T_0}\right) Y^*\left(\frac{m}{T_0}\right) \delta\left(f - \frac{m}{T_0}\right)\right] \qquad \text{(A1.2.4)}$$

where $X\left(\frac{m}{T_0}\right) = X(f)\Big|_{f = \frac{m}{T_0}}$

and $Y^*\left(\frac{m}{T_0}\right) = Y^*(f)\Big|_{f = \frac{m}{T_0}}$.

Appendix A1.3

Complex envelope of the output of a BP system

Let $x(t)$, a BP signal, be applied as input to a BP system with impulse response $h(t)$. Let the resulting output be denoted $y(t)$, which is also a BP signal. We shall derive a relation for $y_{ce}(t)$ in terms of $x_{ce}(t)$ and $h_{ce}(t)$.

We know that,

$$x(t) = x_c(t)\cos\omega_c t - x_s(t)\sin\omega_c t\,,$$

$$x_{ce}(t) = x_c(t) + j\,x_s(t)\,,$$

$$x(t) = \mathrm{Re}\left[x_{ce}(t)\,e^{j 2\pi f_c t}\right],$$

$$X(f) = \frac{1}{2}\left\{X_{ce}(f - f_c) + X_{ce}^{*}\left[-(f + f_c)\right]\right\}. \qquad \text{(A1.3.1)}$$

Similarly, let

$$h_{ce}(t) = h_c(t) + j\,h_s(t)\,, \qquad \text{(A1.3.2)}$$

$$h(t) = h_c(t)\cos\omega_c t - h_s(t)\sin\omega_c t\,, \qquad \text{(A1.3.3)}$$

$$h(t) = \mathrm{Re}\left[h_{ce}(t)\,e^{j 2\pi f_c t}\right],$$

$$2\,h(t) = h_{ce}\,e^{j 2\pi f_c t} + h_{ce}^{*}\,e^{-j 2\pi f_c t}. \qquad \text{(A1.3.4)}$$

Taking the FT of Eq. A1.3.4, we have

$$2\,H(f) = H_{ce}(f - f_c) + H_{ce}^{*}\left[-(f + f_c)\right] \qquad \text{(A1.3.5)}$$

But $\quad y(t) = \mathrm{Re}\left[y_{ce}(t)\,e^{j 2\pi f_c t}\right].$

Therefore,

$$Y(f) = X(f)\,H(f) = \frac{Y_{ce}(f - f_c) + Y_{ce}^{*}\left[-(f + f_c)\right]}{2} \qquad \text{(A1.3.6)}$$

Because of Eq. A1.3.1 and A1.3.5, Eq. A1.3.6 becomes

$$X(f)\,H(f) = \frac{1}{4}\left\{\left[H_{ce}(f - f_c) + H_{ce}^{*}\left[-(f + f_c)\right]\right]\left[X_{ce}(f - f_c) + X_{ce}^{*}\left[-(f + f_c)\right]\right]\right\}$$

Consider the product term

$$H_{ce}(f - f_c)\ X_{ce}^*\left[-(f + f_c)\right]$$

$H_{ce}(f - f_c)$ has spectrum confined to the range $(f_c - B,\ f_c + B)$. $X_{ce}^*\left[-(f + f_c)\right]$ has non- zero spectral components only in the range $\left\{-(f_c + W),\ -(f_c - W)\right\}$. That is, the spectra $H_{ce}(f - f_c)$ and $X_{ce}^*\left[-(f + f_c)\right]$ do not overlap; hence the product is zero. Similarly, $H_{ce}^*\left[-(f + f_c)\right]\ X_{ce}(f - f_c) = 0$. Hence,

$$\begin{aligned}\frac{Y_{ce}(f - f_c) + Y_{ce}^*\left[-(f + f_c)\right]}{2} &= \frac{1}{4} H_{ce}(f - f_c)\, X_{ce}(f - f_c) \\ &\quad + \frac{1}{4} H_{ce}^*\left[-(f + f_c)\right] X_{ce}^*\left[-(f + f_c)\right]\end{aligned}$$

$Y_{ce}(f - f_c)$ has nonzero spectral components only in the range $(f_c - B,\ f_c + B)$. That is,

$$\frac{1}{2} Y_{ce}(f - f_c) = \frac{1}{4}\left[H_{ce}(f - f_c)\, X_{ce}(f - f_c)\right],$$

and $\quad \dfrac{1}{2} Y_{ce}^*\left[-(f + f_c)\right] = \dfrac{1}{4}\left[H_{ce}^*\left(-(f + f_c)\right)\ X_{ce}^*\left(-(f + f_c)\right)\right]$.

In other words,

$$Y_{ce}(f) = \frac{1}{2} X_{ce}(f)\, H_{ce}(f) \qquad \text{(A1.3.7)}$$

Therefore,

$$y_{ce}(t) = \frac{1}{2}\left[x_{ce}(t) * h_{ce}(t)\right] \qquad \text{(A1.3.8)}$$

From Eq. A1.3.8, we obtain the equations for $y_c(t)$ and $y_s(t)$.

$$y_{ce}(t) = \frac{1}{2}\left\{\left[x_c(t) + j\, x_s(t)\right] * \left[h_c(t) + j\, h_s(t)\right]\right\}$$

Therefore,

$$y_c(t) = \frac{1}{2}\left\{x_c(t) * h_c(t) - x_s(t) * h_s(t)\right\} \qquad \text{(A1.3.9)}$$

$$y_s(t) = \frac{1}{2}\left\{x_c(t) * h_s(t) + x_s(t) * h_c(t)\right\} \qquad \text{(A1.3.10)}$$

and, $\quad y_{ce}(t) = y_c(t) + j\, y_s(t)$.

Exercise A1.3.1

Given the pairs $\left(x_c(t),\ x_s(t)\right)$ and $\left[h_c(t),\ h_s(t)\right]$ suggest a scheme to recover $y_c(t)$ and $y_s(t)$.

References

1. Oppenheim, A. V, Willisky, A. S, with Hamid Nawab, S., Signals and Systems (2nd Edition), PHI, 1997
2. Ashok Ambardar, Analog and Digital Signal Processing (2nd Edition), Brooks/Cole Publishing Company, Thomson Asia Pvt. Ltd., Singapore, 1999
3. Lathi, B. P., Signal Processing and Linear systems, Berkeley-Cambridge Press, 1998

Note: The above three books have a large collection of problems. The student is advised to try to solve them.

Other Suggested Books

1. Couch II, L. W., Digital and Analog Communication Systems (6th Edition) Pearson Asia, 2001
2. Carlson, A. B., Communication Systems (4th Edition), Mc Graw-Hill, 2003
3. Lathi, B. P., Modern Digital and Analog Communication Systems (3rd Edition), Oxford University Press, 1998