An introduction to Information Theory

Adrish Banerjee

Department of Electrical Engineering
Indian Institute of Technology Kanpur
Kanpur, Uttar Pradesh
India

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Lecture #8: Coding of sources with memory

Outline of the lecture

- Discrete stationery source (DSS)

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- Block to variable length coding of DSS

Adrish Banerjee Department of Electrical Engineering Indian Institute of Technology Kanpur Kanpur, Uttar Pradesh India
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- Coding of positive integers

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- Discrete stationery source (DSS)
- Block to variable length coding of DSS
- Coding of positive integers
- Elias-Willems Source Coding

Discrete stationery source

- A K-ary discrete stationary source (DSS) is a device that emits a sequence $U_i, \;\; i = 1, 2, \cdots$ of K-ary random variables such that for every $n \geq 1$ and every $L \geq 1$, the random vectors $[U_1, U_2, \cdots .U_L]$ and $[U_{n+1}, U_{n+2}, \cdots, U_{n+L}]$ have the same probability distribution.

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- A DSS is said to have finite memory μ if, for all $n > \mu$,

  $$P_{U_n|U_1,\cdots,U_{n-1}}(u_n|u_1, \cdots, u_{n-1}) = P_{U_n|U_{n-\mu},\cdots,U_{n-1}}(u_n|u_{n-\mu}, \cdots, u_{n-1})$$

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  holds for all choices of $u_1, \cdots, u_n$ and if μ is the smallest nonnegative integer such that this holds
- The discrete memory source (DMS) is a special case of DSS with memory $\mu = 0$.

Discrete stationery source

- For DSS with memory μ,

$$H(U_n|U_1, \cdots, U_{n-1}) = H(U_n|U_{n-\mu} \cdots, U_{n-1})$$

  for all $n > \mu$.

Discrete stationery source

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$$H_L(U) = \frac{1}{L} H(U_1, U_2, \cdots, U_L)$$

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- Uncertainty per letter of output sequence of L letters is defined as
$$H_L(U) = \frac{1}{L} H(U_1, U_2, \cdots, U_L)$$
- Another way for defining uncertainty per letter of output sequence is $H(U_L|U_1U_2 \cdots U_{L-1})$

Discrete stationery source

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Discrete stationery source

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Proof:

- (ii) $H(U_{L+1}|U_1 \cdots U_L) \leq H(U_{L+1}|U_2 \cdots U_L) = H(U_L|U_1 \cdots U_{L-1})$

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- (i)

$$\begin{aligned} H(U_1U_2\cdots U_L) &= H(U_1) + H(U_2|U_1) + \cdots + H(U_L|U_1\cdots U_{L-1}) \\ &\geq LH(U_L|U_1\cdots U_{L-1}) \\ \implies H_L(U) &\geq H(U_L|U_1\cdots U_{L-1}) \end{aligned}$$

Discrete stationery source

Proof:

- (iii)

$$\begin{aligned} H(U_1U_2\cdots U_{L+1}) &= H(U_1\cdots U_L) + H(U_{L+1}|U_1\cdots U_L) \\ &\leq H(U_1\cdots U_L) + H(U_L|U_1\cdots U_{L-1}) \\ &\leq H(U_1\cdots U_L) + H_L(U) \\ &= LH_L(U) + H_L(U) \\ \implies \frac{H(U_1U_2\cdots U_{L+1})}{L+1} &\leq H_L(U) \\ H_{L+1}(U) &\leq H_L(U) \end{aligned}$$

Discrete stationery source

Proof:

- (iii)

$$\begin{aligned} H(U_1U_2\cdots U_{L+1}) &= H(U_1\cdots U_L)+H(U_{L+1}|U_1\cdots U_L)\\ &\leq H(U_1\cdots U_L)+H(U_L|U_1\cdots U_{L-1})\\ &\leq H(U_1\cdots U_L)+H_L(U)\\ &= LH_L(U)+H_L(U)\\ \implies \frac{H(U_1U_2\cdots U_{L+1})}{L+1} &\leq H_L(U)\\ H_{L+1}(U) &\leq H_L(U) \end{aligned}$$

- (iv) Since $H_L(U)\geq 0$ and $H(U_L|U_1\cdots U_{L-1})\geq 0$ and a non-increasing sequence bounded below always has a limit. Thus (ii) and (iii) ensure that both $\lim_{L\to\infty} H(U_L|U_1\cdots U_{L-1})$ and $\lim_{L\to\infty} H_L(U)$ always exist. (i) implies that

$$\lim_{L\to\infty} H(U_L|U_1\cdots U_{L-1}) \leq \lim_{L\to\infty} H_L(U) = H_\infty(U)$$

Discrete stationery source

Proof (contd.):

- Now we will have to show that equality holds. We know that

$$\begin{aligned} H(U_1\cdots U_L\cdots U_{L+n}) &= H(U_1\cdots U_L)+H(U_{L+1}|U_1\cdots U_L)+\cdots\\ &\quad +H(U_{L+n}|U_1\cdots U_{L+n-1})\\ &\leq H(U_1\cdots U_L)+nH(U_{L+1}|U_1\cdots U_L)\\ \frac{H(U_1\cdots U_L\cdots U_{L+n})}{n+L} &\leq \frac{H(U_1\cdots U_L)}{n+L}+\frac{n}{n+L}H(U_{L+1}|U_1\cdots U_L) \end{aligned}$$

Discrete stationery source

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- Taking the limit as $n \to \infty$ on both sides we get,

$$H_\infty(U) \leq H(U_{L+1}|U_1 \cdots U_L)$$

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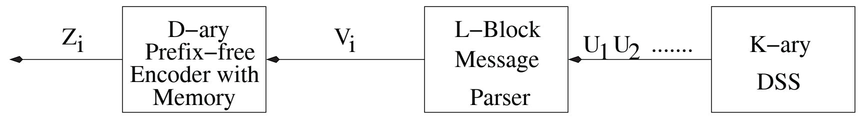
- Taking the limit as $n \to \infty$ on both sides we get,

$$H_\infty(U) \leq H(U_{L+1}|U_1 \cdots U_L)$$

- Now taking the limit as $L \to \infty$ on both sides we get,

$$H_\infty(U) \leq \lim_{L \to \infty} H(U_{L+1}|U_1 \cdots U_L)$$

Block to variable length coding of DSS



- If block length L messages from a DSS are encoded by a D-ary prefix-free code, where the code used for each message may depend on previous messages, then the length W_i of the codeword for the $i-$th message satisfies

$$\frac{E[W_i]}{L} \geq \frac{H_\infty(U)}{\log D}$$

where $H_\infty(U)$ is the uncertainty per letter of the DSS.

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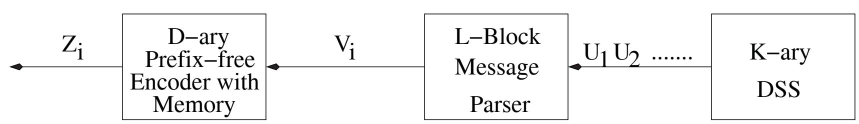

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Proof:

- Since we are using $L-$block message parser, the $i-$th message is encoded as

$$V_i = [U_{iL-L+1}, \cdots, U_{iL-1}, U_{iL}]$$

Block to variable length coding of DSS

- Given that $[V_1, \cdots, V_{i-1}] = [v_1, \cdots, v_{i-1}]$, it follows that the length W_i of the codeword Z_i for the message V_i must satisfy

$$E\left[W_i | [V_1, \cdots, V_{i-1}] = [v_1, \cdots, v_{i-1}]\right] \geq \frac{H(V_i | [V_1, \cdots, V_{i-1}] = [v_1, \cdots, v_{i-1}])}{\log D}$$

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- Multiplying by $P_{V_1 \cdots V_{i-1}}(v_1, \cdots, v_{i-1})$ and summing over all values $v_1, \cdots, v_{i-1}$ gives

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- We can write

$$\begin{aligned} H(V_i | V_1 \cdots V_{i-1}) &= H(U_{iL-L+1} \cdots U_{iL} | U_1 \cdots U_{iL-L}) \\ &= H(U_{iL-L+1} | U_1 \cdots U_{iL-L}) + \cdots + \\ &\quad H(U_{iL} | U_1 \cdots U_{iL-1}) \\ &\geq LH(U_{iL} | U_1 \cdots U_{iL-1}) \\ &\geq LH_\infty(U) \end{aligned}$$

Coding of positive integers

- Binary representation

n	B(n)
1	1
2	1 0
3	1 1
4	1 0 0
5	1 0 1
6	1 1 0
7	1 1 1
8	1 0 0 0

Coding of positive integers

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- Every codeword as 1 as the first digit.
- The length of the codeword $L(n) = \lfloor \log_2 n \rfloor + 1$.

Coding of positive integers

- Elias code 1

n	$B_1(n)$
1	1
2	0 1 0
3	0 1 1
4	0 0 1 0 0
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- This is a prefix-free code.
- No of leading zeros tell how many digits follow the leading 1.

Coding of positive integers

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- This is a prefix-free code.
- No of leading zeros tell how many digits follow the leading 1.
- The length of the codeword $L_1(n) = 2\lfloor \log_2 n \rfloor + 1$.

Coding of positive integers

- Elias code 2

n	$B_2(n)$
1	1
2	0 1 0 0
3	0 1 0 1
4	0 1 1 0 0
5	0 1 1 0 1
6	0 1 1 1 0
7	0 1 1 1 1
8	0 0 1 0 0 0 0 0

Coding of positive integers

- Elias code 2

n	$B_2(n)$
1	1
2	0 1 0 0
3	0 1 0 1
4	0 1 1 0 0
5	0 1 1 0 1
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- Elias code 2 begins with $B_1(L(n))$ (Elias code 1 to signal the number of following digits in the codeword) followed by B(n) with leading 1 removed.

Coding of positive integers

- Elias code 2

Coding of positive integers

- Elias code 2
- The length of the codeword $L_2(n)$ is given by

$$\begin{aligned} L_2(n) &= L_1(L(n)) + L(n) - 1 \\ &= L_1(\lfloor \log_2 n \rfloor + 1) + \lfloor \log_2 n \rfloor \\ &= \lfloor \log_2 n \rfloor + 2\lfloor \log_2(\lfloor \log_2 n \rfloor + 1) \rfloor + 1 \end{aligned}$$

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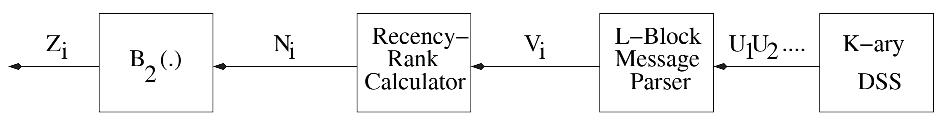
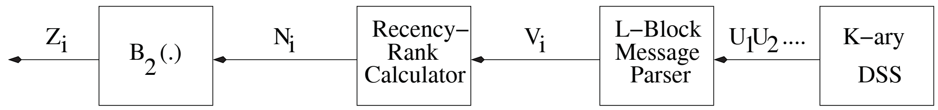
Elias-Willems Source Coding



- The message seen most recently has recency rank of 1, next different message value most recently seen has recency rank 2 and so on.

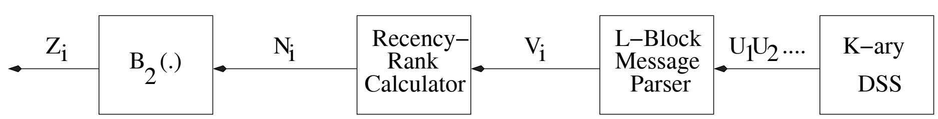
Elias-Willems Source Coding



- The message seen most recently has recency rank of 1, next different message value most recently seen has recency rank 2 and so on.
- Consider $L = 2$, and $K = 2$, i.e. four possible message values are $00, 01, 10, 11$. Consider the following sequence of past messages

$$\cdots|V_{i-2}|V_{i-1} = \cdots|11|01|10|01|01|00|01$$

Elias-Willems Source Coding



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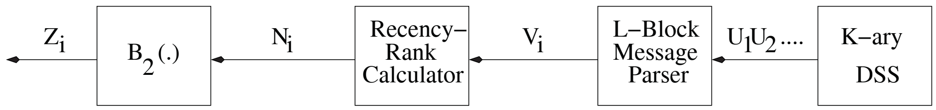
Elias-Willems Source Coding



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- In this example, message 01 has a recency rank of 1, message 00 has a recency rank of 2, message 10 has a recency rank of 3, and message 11 has a recency rank of 4.
- In Elias Willems source coding scheme, recency rank calculator assigns a recency rank, N_i to the message V_i.

Elias-Willems Source Coding

- For the Elias-Willems binary coding scheme for block length L messages from a K-ary ergodic discrete stationary source, the codeword length W_i for all i sufficiently large satisfies

$$\frac{E[W_i]}{L} \leq H_L(U) + \frac{2}{L}\log_2(LH_L(U)+1) + \frac{1}{L}$$

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Proof:

- We define time Δ_i to the recent occurence of message V_i as $\Delta_i = \delta$, where if $V_i = v$, $V_{i-\delta}$ is the most recent past message with the same value of v. Also $N_i \leq \Delta_i$.

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- We can write

$$E[\Delta_i | V_i = v] = \frac{1}{P_V(v)}$$

Elias-Willems Source Coding

- Codeword length W_i can be upper bounded by

$$\begin{aligned} W_i = L_2(N_i) &\leq L_2(\Delta_i) \\ &\leq \log_2(\Delta_i) + 2\log_2(\log_2(\Delta_i)+1)+1 \end{aligned}$$

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- Taking the conditional expectation, given that $V_i = v$ and applying Jensen's inequality, we get

$$E[W_i|V_i = v] \leq \log_2(E[\Delta_i|V_i = v]) + 2\log_2(\log_2(E[\Delta_i|V_i = v]) + 1) + 1$$

Elias-Willems Source Coding

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$$E[W_i|V_i = v] \leq \log_2(E[\Delta_i|V_i = v]) + 2\log_2(\log_2(E[\Delta_i|V_i = v])+1)+1$$

- Substituting the value of $E[\Delta_i|V_i = v]$ in the above expression, we get

$$E[W_i|V_i = v] \leq -\log_2(P_V(v)) + 2\log_2(-\log_2(P_V(v))+1)+1$$

Elias-Willems Source Coding

- Codeword length W_i can be upper bounded by

$$W_i = L_2(N_i) \leq L_2(\Delta_i) \leq \log_2(\Delta_i) + 2\log_2(\log_2(\Delta_i)+1)+1$$

- Taking the conditional expectation, given that $V_i = v$ and applying Jensen's inequality, we get

$$E[W_i|V_i = v] \leq \log_2(E[\Delta_i|V_i = v]) + 2\log_2(\log_2(E[\Delta_i|V_i = v])+1)+1$$

- Substituting the value of $E[\Delta_i|V_i = v]$ in the above expression, we get

$$E[W_i|V_i = v] \leq -\log_2(P_V(v)) + 2\log_2(-\log_2(P_V(v))+1)+1$$

- Multiplying the above expression by $P_V(v)$, summing over v and using Jensen's inequality, we get

$$E[W_i] \leq H(V) + 2\log_2(H(V)+1)+1$$

Adrish Banerjee Department of Electrical Engineering Indian Institute of Technology Kanpur Kanpur, Uttar Pradesh India
An introduction to Information Theory

Elias-Willems Source Coding

- Also

$$\begin{aligned} H(V) &= H(U_1 U_2 \cdots U_L) \\ &= L H_L(U). \end{aligned}$$

Elias-Willems Source Coding

- Also

$$\begin{aligned} H(V) &= H(U_1 U_2 \cdots U_L) \\ &= L H_L(U). \end{aligned}$$

- Substituting this value of $H(V)$ in the previous expression and dividing by L, we get

$$\frac{E[W_i]}{L} \leq H_L(U) + \frac{2}{L}\log_2(LH_L(U) + 1) + \frac{1}{L}$$