# An introduction to Information Theory

Adrish Banerjee

Department of Electrical Engineering
Indian Institute of Technology Kanpur
Kanpur, Uttar Pradesh
India

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## Lecture #4: Block to variable length coding-III: Huffman Coding

# Outline of the lecture

- Huffman coding



Huffman coding Coding an information source

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# Outline







# Huffman code

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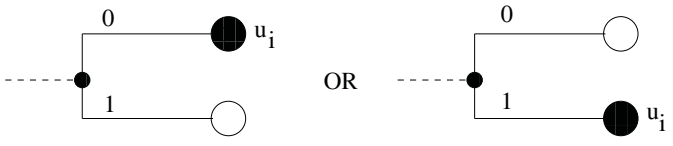

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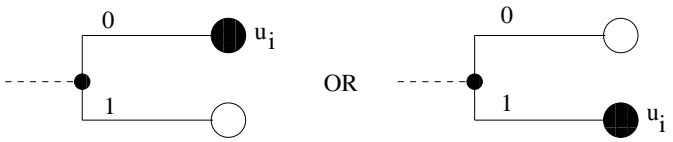


- In either case, we can delete the last digit of the codeword, and still have a prefix-free code. The new code has smaller E[W] and thus original code must not be optimal.





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- There is an optimal binary prefix-free code for U such that the two least likely codewords, say those for $u_{K-1}$ and $u_K$, differ only in their last digit.

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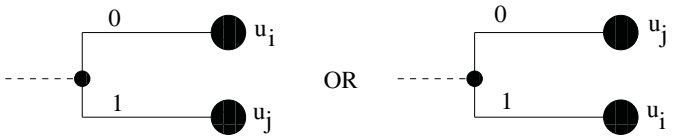






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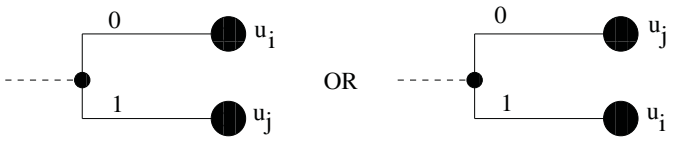


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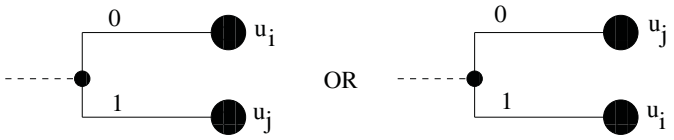


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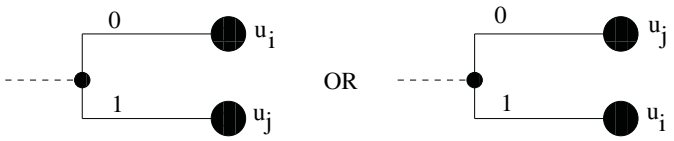


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- Similarly if $i \neq K-1$, we switch the leaves for $u_i$ and $u_{K-1}$ without increasing E[W].
- The new optimum code has its two least likely codewords differing only in their last digit.

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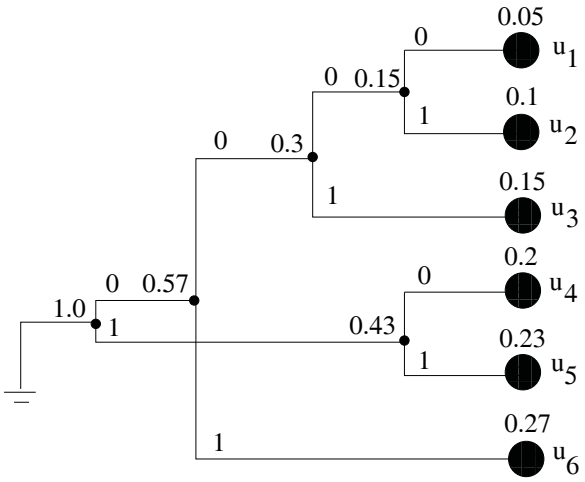
- If there is only one active vertex left. make it root and stop, otherwise go back to step 1.

# Huffman code

- Construct binary Huffman code for the following example: $P(u_1) = 0.05, P(u_2) = 0.1, P(u_3) = 0.15, P(u_4) = 0.2, P(u_5) = 0.23, P(u_6) = 0.27$




| U | Z |
|---|---|
| $u_1$ | 0000 |
| $u_2$ | 0001 |
| $u_3$ | 001 |
| $u_4$ | 10 |
| $u_5$ | 11 |
| $u_6$ | 01 |





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Huffman coding Coding an information source

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  Sketch of proof:
- In constructing a tree from root, we get D leaves initially.
- At each subsequent step, if we extend any leaf we get D new leaves and lose one old leaf.

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- If there are $D-1$ unused leaves in the tree of optimal prefix-free code, then we can shorten the codeword by removing its last digit.





# Huffman code

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- Adding (K-D)(D-1) to both sides of the above equation, we get

$$(K-D)(D-2) = (K-D-q)(D-1) + r \quad \text{where} \quad 0 \leq r < D-1$$





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- There is an optimal D-ary prefix-free code for a random variable U with K possible values such that $D-r$ least likely codewords differ only in their last digit, where r is the remainder when $(K-D)(D-2)$ is divided by $D-1$.

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- Arguments similar to binary case can be used to prove this.





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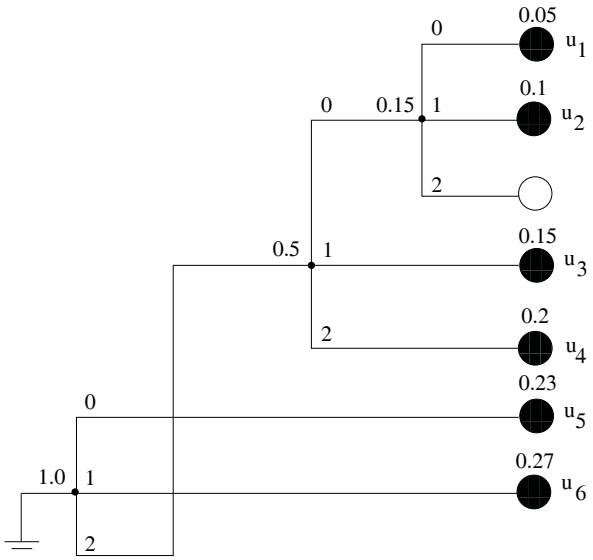



| U | Z |
|---|---|
| $u_1$ | 200 |
| $u_2$ | 201 |
| $u_3$ | 21 |
| $u_4$ | 22 |
| $u_5$ | 0 |
| $u_6$ | 1 |

# Outline

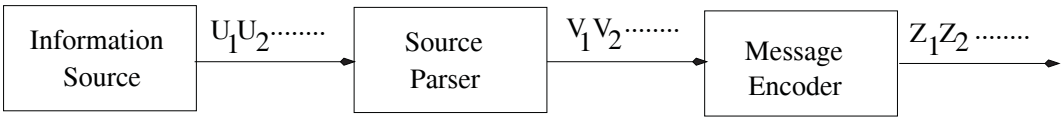






# Coding an Information source

- Parsing an information source



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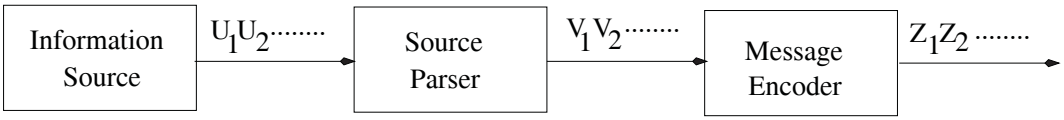


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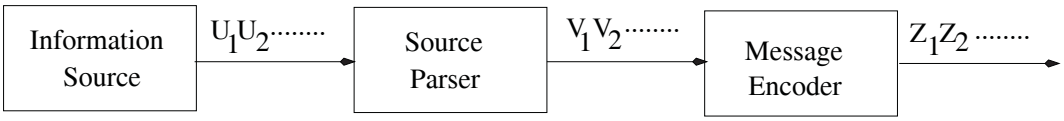


- The source parser divides the output sequences from the information source into messages that are encoded by the message encoder.
- We consider an L-block source parser, i.e.

$$\begin{aligned} V_1 &= [U_1, U_2, \cdots, U_L] \\ V_2 &= [U_{L+1}, U_{L+2}, \cdots, U_{2L}] \\ \vdots & \quad \vdots \end{aligned}$$

# Coding an Information source

- Block to variable length coding theorem for a DMS. There exists a D-ary prefix-free coding of a L-block message from DMS such that the average number of D-ary code digits per source satisfies

$$\frac{E[W]}{L} < \frac{H(U)}{\log D} + \frac{1}{L}$$

where H(U) is the uncertainty of a single source letter. Conversely for every D-ary prefix-free coding of an L-block message

$$\frac{E[W]}{L} \geq \frac{H(U)}{\log D}$$





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- Proof: Since the message $V = [U_1, U_2, \cdots, U_L]$ has L i.i.d. components, we have

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- For any D-ary prefix-free code for V, we have

$$\begin{aligned} \frac{H(V)}{\log D} &\leq E[W] < \frac{H(V)}{\log D} + 1 \\ \implies \frac{LH(U)}{\log D} &\leq E[W] < \frac{LH(U)}{\log D} + 1 \\ \implies \frac{H(U)}{\log D} &\leq \frac{E[W]}{L} < \frac{H(U)}{\log D} + \frac{1}{L} \end{aligned}$$