An introduction to Information Theory

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Lecture #3A: Block to variable length coding-I: Prefix-free code

Outline of the lecture

- Coding a single random variable

Coding a single random variable Prefix-free code Kraft's inequality

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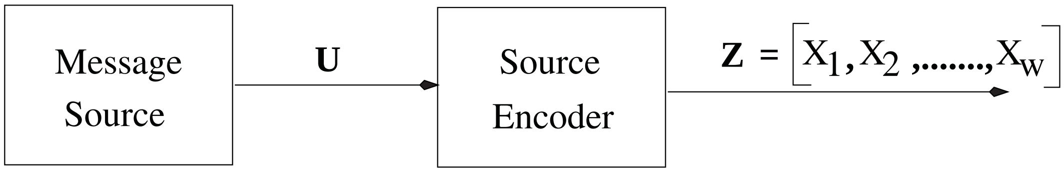
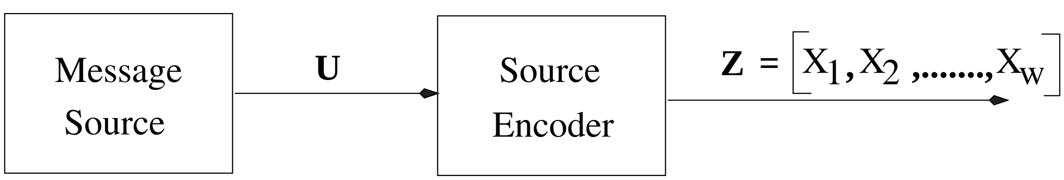


Variable length coding scheme

- **U** is a $K-$ary random variable.

Coding a single random variable



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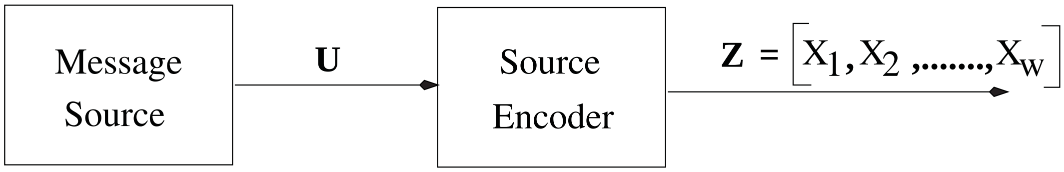

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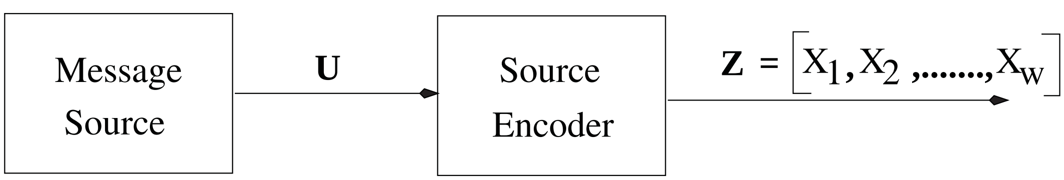

Variable length coding scheme

- **U** is a $K-$ary random variable.
- X_i takes on values in the $D-$ary alphabet.
- **W** is a random variable, i.e. **Z** is variable length.
- A list $(z_1, z_2, \cdots, z_K)$ of D-ary sequences is a codeword of $U = [u_1, u_2, \cdots, u_K]$.

Coding a single random variable

- If $\mathbf{z}_i = [x_{i1}, x_{i2}, \cdots, x_{iw_i}]$ is the codeword for u_i, and w_i is the length of this codeword, then average codeword length is defined as

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- Smallness of average codeword length is a measure of goodness of the code.

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- A D-ary sequence z is a prefix of another D-ary sequence z' if z has length n and the first n digits of z' form exactly the sequence z.

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- Can a non-prefix free code be uniquely decodeable?

U	Z
u_1	0
u_2	10
u_3	11

Prefix-free code

U	Z
u_1	1
u_2	00
u_3	11

Not a prefix-free code

Test for uniquely decodeability

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- In general to form $S_n, n > 1$, we compare S_0 and S_{n-1}. If a codeword in S_0 is a prefix of a codeword in S_{n-1} or vice-versa, we place the suffix $\in S_n$.

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- A code is uniquely decodeable if and only if none of the sets $S_1, S_2, \cdots$ contains a code word that is a member of S_0.

Test for uniquely decodeability

S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7
a	d	eb	de	b	ad	d	eb
c	bb	cde			bcde		
ad							
abb							
bad							
deb							
bbcde							

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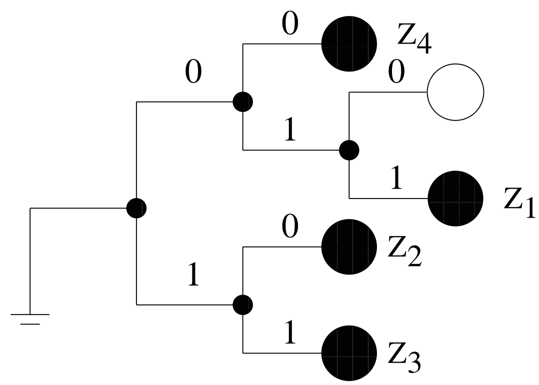
Prefix-free code

- A $D-$ary tree, is a finite rooted tree such that D branches stem outward from each node.
- The full $D-$ary tree of length N is the $D-$ary tree with D^N leaves each at depth N from the root.
- D-ary prefix-free code can be identified with a set of leaves in a D-ary tree.
- Ex. Draw binary tree for the following code $z_1 = [011], z_2 = [10], z_3 = [11],$ and $z_4 = [00]$

Prefix-free code



Binary tree for prefix-tree code

Outline

Kraft's inequality

- There exists a D-ary prefix-free code whose codeword lengths are the positive integers $w_1, w_2, \cdots, w_K$ if and only if

$$\sum_{i=1}^{K} D^{-w_i} \leq 1$$

Sketch of Proof:

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Sketch of Proof:

- In the full D-ary tree of length N, D^{N-w} leaves stem from each node at depth w where $w < N$.
- Suppose there exist a D-ary prefix-free code, construct a tree for the code by pruning the full D-ary tree of length $N = \max_i w_i$ at all vertices corresponding to codewords.

Kraft's inequality

- Due to prefix-free condition, if at depth w_i, we delete D^{N-w_i} leaves, none of these leaves could have been previously deleted.

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- Since there are only D^N leaves that can be deleted, we have

$$D^{N-w_1} + D^{N-w_2} + \cdots + D^{N-w_K} \leq D^N$$

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- Dividing the above equation by D^N we get the necessary condition for prefix-free code.

Kraft's inequality

- Sketch of converse proof: Suppose $w_1, \cdots, w_K$ are positive integers such that Kraft inequality is satisfied. Consider an ordered list of length $w_1 \leq w_2 \leq \cdots \leq w_K$. Consider the following algorithm

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- (1) $i \leftarrow 1$
- (2) Choose z_i as any surviving node or leaf at depth w_i, and prune the tree. Stop if there is no such surviving node or leaf.

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- (1) $i \leftarrow 1$
- (2) Choose z_i as any surviving node or leaf at depth w_i, and prune the tree. Stop if there is no such surviving node or leaf.
- (3) $i = K$ stop , otherwise $i \leftarrow i + 1$

Kraft's inequality

- If we are able to choose z_K in step (2), we are able to construct prefix-free code. Suppose $z_1, z_2, \cdots, z_{i-1}$ have been chosen, the number of surviving leaves at depth N not stemming from any codeword is

$$D^N - (D^{N-w_1} + D^{N-w_2} + \cdots + D^{N-w_{i-1}}) = D^N \left(1 - \sum_{j=1}^{i-1} D^{-w_j}\right)$$

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- Therefore the number of surviving leaves at depth N is greater than zero. If there is a surviving leaf at depth N, there must be some unused nodes at depth $w_i < N$ and no already chosen codeword can stem outward from such a surviving node. Thus the surviving node can be chosen as z_i.

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Prefix-free code

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- Since $\sum_{i=1}^{5} 3^{-w_i} = 1/3 + 1/9 + 1/9 + 1/27 + 1/81 = 49/81 < 1$, a prefix-free code exists.