An introduction to Information Theory

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Lecture #14B: Problem solving session-IV

Measure of Information

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  - (Axiom 3:) $H(p_1, p_2, \cdots, p_M) = H(p_1 + \cdots + p_r, p_{r+1} + \cdots + p_M)$
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    $+(p_{r+1} + \cdots + p_M) H\left(\frac{p_{r+1}}{\sum_{i=r+1}^{M} p_i}, \cdots, \frac{p_M}{\sum_{l=r+1}^{M} p_i}\right)$

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  - (Axiom 4:) $H = H(p_1, p_2, \cdots, p_N)$ is a continuous function of the probability set p_i.

Problem # 1 (contd.)

- **Solutions:** We will prove that only function satisfying these four axioms is of the form

$$H(p_1, \cdots, p_M) = -C \sum_{i=1}^{M} p_i \log p_i$$

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- If M is fixed integer, then it is true for $k = 1$.
- Since $f(M^k) = f(M \cdot M^{k-1}) = f(M) + f(M^{k-1})$. Assume it is true for $k-1$, then $f(M^k) = f(M) + (k-1)f(M) = kf(M)$.

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- Let M be a positive integer greater than 1. Let r be an arbitrary positive integer, such that 2^r lies between two powers of M, i.e. $M^k \leq 2^r < M^{k+1}$. We have from Axiom 1, $f(M^k) \leq f(2^r) < f(M^{k+1})$.

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- Thus we have $kf(M) \leq rf(2) < (k+1)f(M)$ or $k/r \leq f(2)/f(M) < (k+1)/r$.

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- Thus we have $kf(M) \leq rf(2) < (k+1)f(M)$ or $k/r \leq f(2)/f(M) < (k+1)/r$.
- Logarithm is a monotone increasing function, hence $\log M^k \leq \log 2^r < \log M^{k+1}$ or we have $k \log M \leq r \log 2 < (k+1) \log M$, or $k/r \leq (\log 2)/(logM) < (k+1)/r$.

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- Since M is fixed and r is arbitrary, we may let $r \to \infty$ and we get

$$(\log 2)/(\log M) = f(2)/f(M)$$

or $f(M) = c \log M$, where $c = f(2)/\log 2$

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$$\begin{aligned} f(s) &= H\left(\frac{1}{s}, \cdots, \frac{1}{s}\right) \\ &= H\left(\frac{r}{s}, \frac{s-r}{s}\right) + \frac{r}{s} f(r) + \frac{(s-r)}{s} f(s-r) \end{aligned}$$

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- Thus we have

$$\begin{aligned} H(p, 1-p) &= -C\left[p \log r - \log s + (1-p)\log(s-r)\right] \\ &= -C\left[p \log \frac{r}{s} + (1-p)\log \frac{s-r}{s}\right] \\ &= -C\left[p \log p + (1-p)\log(1-p)\right] \end{aligned}$$

Problem # 1 (contd.)

d) $H(p, 1-p) = -C[p \log p + (1-p)\log(1-p)]$ for all p. This follows from continuity axiom.

e) $H(p_1, \cdots, p_M) = -C \sum_{i=1}^{M} p_i \log p_i$ $(M = 1, 2, \cdots)$.

- Using mathematical induction, we know the above equation is true for $M = 1, 2$.
- If $M > 2$, by Axiom 3, we get

$$H(p_1, \cdots, p_M) = H(p_1 + \cdots + p_{M-1}, p_M)$$
$$+(p_1 + \cdots + p_{M-1}) H\left(\frac{p_1}{\sum_{i=1}^{M} p_i}, \cdots, \frac{p_{M-1}}{\sum_{i=1}^{M} p_i}\right) + p_M H(1)$$

Problem # 1 (contd.)

- Assuming the formula is valid for positive integers upto $M-1$, we obtain

$$\begin{aligned} H(p_1, \cdots, p_M) &= -C\left[(p_1 + \cdots + p_{M-1}) \log(p_1 + \cdots + p_{M-1}) + p_M \log p_M\right] \\ &-C(p_1 + \cdots + p_{M-1}) \left[\frac{p_1}{\sum_{i=1}^{M} p_i} \log \frac{p_1}{\sum_{i=1}^{M} p_i} + \cdots + \frac{p_{M-1}}{\sum_{i=1}^{M} p_i} \log \frac{p_{M-1}}{\sum_{i=1}^{M} p_i}\right] \\ &+p_M(0) \\ = \quad & -C\left[\left(\sum_{i=1}^{M-1} p_i\right) \log\left(\sum_{i=1}^{M-1} p_i\right) + p_M \log p_M\right] \\ & -C\left[\sum_{i=1}^{M} p_i \log p_i - \left(\sum_{i=1}^{M-1} p_i \log \sum_{i=1}^{M-1} p_i\right)\right] \\ = \quad & -C \sum_{i=1}^{M} p_i \log p_i \end{aligned}$$

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- **Problem # 2:** Suppose one has 12 coins, among which there may or may not be one counterfeit coin. If there is a counterfeit coin, it may be either heavier or lighter than the other coins. The coins are to be weighed by a balance. In three weightings, you will have to find the counterfeit coin (if any) and correctly declare it to be heavier or lighter.

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- **Solutions:** For 12 coins we have 23 possible outcomes, corresponding to the case when one of the 12 coins is heavier or one of the 12 coins is lighter or all coins are of same weight. We denotes numbers $\{-12, -11, \cdots, -1, 0, 1, \cdots, 11, 12\}$ in ternary number system with alphabets $\{-1, 0, 1\}$.

Problem # 2 (contd.)

- The representation of the positive numbers is shown below

	1	2	3	4	5	6	7	8	9	10	11	12	
3^0	1	-1	0	1	-1	0	1	-1	0	1	-1	0	$\sum_1 = 0$
3^1	0	1	1	1	-1	-1	-1	0	0	0	1	1	$\sum_2 = 2$
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- Row sum can be made zero by negating columns 7, 9, 11 and 12. Thus we have

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- If the expansion is same as the column of the matrix, the coin is heavier, if it is of opposite sign, the coin is lighter.

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- **Solutions:** For a Markov Source by data processing lemma, we have

$$I(X_0; X_n) \leq I(X_0; X_{n-1})$$

Hence we have

$$H(X_0) - H(X_0|X_n) \leq H(X_0) - H(X_0|X_{n-1})$$

or

$$H(X_0|X_n) \geq H(X_0|X_{n-1})$$

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- Hence $H(X|g(Y) = H(X|Y)$ iff $X \to g(Y) \to Y$.

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- From the derivation of the inequality, we have equality iff $X \to g(Y) \to Y$ forms a Markov chain.
- Hence $H(X|g(Y) = H(X|Y)$ iff $X \to g(Y) \to Y$.
- This condition includes many special cases, such as g being one-to-one, and X and Y being independent. However, these two special cases do not exhaust all the possibilities.

Convex Function

- **Problem # 5:** Let $P_0(j/k)$ and $P_1(j/k)$, $0 \leq k \leq K-1$, $0 \leq j \leq J-1$, be two arbitrary sets of transition probabilities, and let $P(j/k) = \theta P_0(j/k) + (1-\theta)P_1(j/k)$. for an arbitrary θ, $0 \leq \theta < 1$.

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- Let I_0, I_1, and I be the average mutual informations for these sets of transition probabilities, then to prove that the mutual information, $I(X;Y)$ is a convex function of $p(y/x)$ for fixed $p(x)$.

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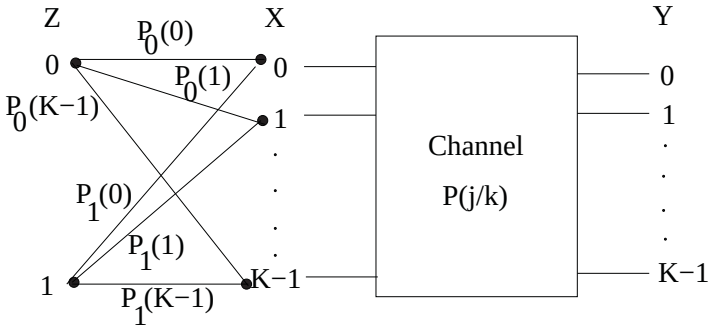
- Let I_0, I_1, and I be the average mutual informations for these sets of transition probabilities, then to prove that the mutual information, $I(X;Y)$ is a convex function of $p(y/x)$ for fixed $p(x)$.
- One has to prove that

$$\theta I_0 + (1-\theta)I_1 \geq I$$

Problem # 5 (contd.)

- Consider P_0 and P_1 as conditional on a binary variable Z

$$P_0(j/k) = P_{Y/XZ}(j|k,0) \quad P_1(j/k) = P_{Y/XZ}(j|k,1)$$



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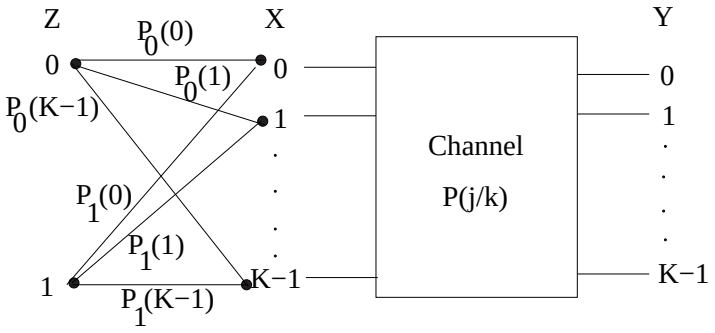

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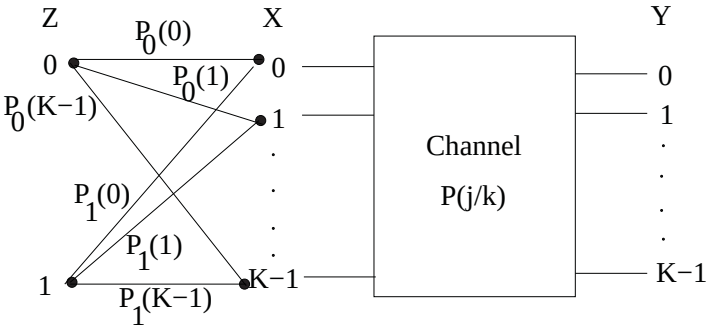

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- Let $P_Z(0) = \theta$, $P_Z(1) = 1 - \theta$, and we define Z to be statistically independent of X.
- Use the above formulation to prove that the mutual information, $I(X;Y)$ is a convex function of $p(y/x)$ for fixed $p(x)$.

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- **Solutions:** Let $P_0(j/k)$ and $P_1(j/k)$, $0 \leq k \leq K-1$, $0 \leq j \leq J-1$, be two arbitrary sets of transition probabilities, and let $P(j/k) = \theta P_0(j/k) + (1-\theta)P_1(j/k)$. for an arbitrary θ, $0 \leq \theta < 1$.

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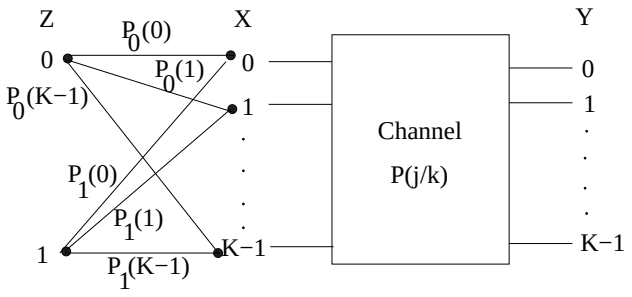

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$$\begin{aligned} I(X; Y|Z) &= I(X; Y) + I(Z; X|Y) \\ I(X; Y|Z) &\geq I(X; Y) \end{aligned}$$

Uniquely Decodable Codes

- **Problem # 6:** Consider a source with source alphabet $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ in which the symbol probabilities are as follows:

$$p_1 = 0.27, p_2 = 0.09, p_3 = 0.23, p_4 = 0.11, p_5 = 0.15, p_6 = 0.15$$

Find an optimal uniquely decodable code for this source that is not a Huffman code.

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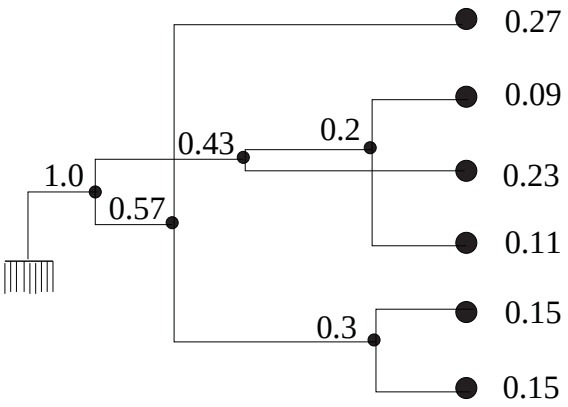
- **Solutions:** We will first create a Huffman code for the source distribution.



Problem #6 (contd.)

- Huffman code can thus be written as

Source alphabet probability	Codeword
0.27	10
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- In Huffman code, two least likely codewords differ at the last bit. If we interchange the codeword for one of the source alphabet with probability 0.15 with the codeword for source alphabet with probability 0.11, we still get a uniquely decodable code, but it is not a Huffman code.

Problem #6 (contd.)

- Thus one example of uniquely decodable code that is not Huffman code for this source distribution is given by

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Huffman Coding

- **Problem # 7:** Prove that for any binary Huffman code, if the most probable message symbol has probability $p_1 > 2/5$ and it is the only message symbol with this probability, then that symbol must be assigned a codeword of length 1.

Huffman Coding

- **Problem # 7:** Prove that for any binary Huffman code, if the most probable message symbol has probability $p_1 > 2/5$ and it is the only message symbol with this probability, then that symbol must be assigned a codeword of length 1.
- **Solutions:** Let's assume that the codeword length of the most probable message symbol is larger than one.

Huffman Coding

- **Problem # 7:** Prove that for any binary Huffman code, if the most probable message symbol has probability $p_1 > 2/5$ and it is the only message symbol with this probability, then that symbol must be assigned a codeword of length 1.
- **Solutions:** Let's assume that the codeword length of the most probable message symbol is larger than one.
- Then, since the most probable codeword must have the shortest codeword length, the codeword tree can be reduced to

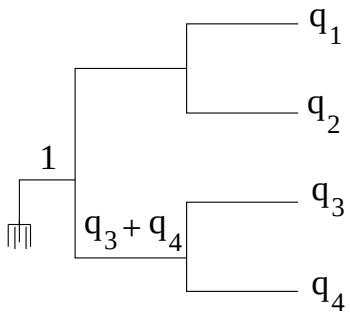

Figure: Huffman code tree

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- Thus the original assumption is false and the codeword length of the most probable message must be one.

Rate Distortion Theory

- **Problem # 8:** Let $X \sim N(0, \sigma^2)$ and let the distortion measure be squared error. Show that the optimum reproduction points for 1 bit quantization are $\pm\sqrt{\frac{2}{\pi}}\sigma$, and that the expected distortion for 1 bit quantization is $\frac{\pi-2}{\pi}\sigma^2$.

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- **Solutions:** Let X $\sim \mathcal{N}(0, \sigma^2)$ and let the distortion measure be squared error.
- With one bit quantization, the obvious reconstruction regions are positive and negative real axes.

Rate Distortion Theory

- The reconstruction point is the centroid of each region. for example, for the positive real line, the centroid a is

$$\begin{aligned} a &= \int_0^\infty x \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= \int_0^\infty \sigma\sqrt{\frac{2}{\pi}} e^{-y} dy = \sigma\sqrt{\frac{2}{\pi}}, \end{aligned}$$

using the substitution $y = x^2/2\sigma^2$.

Rate Distortion Theory

- The expected distortion for one bit quantization is

$$\begin{aligned} D &= \int_{-\infty}^0 \left(x + \sigma\sqrt{\frac{2}{\pi}}\right)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx + \int_0^\infty \left(x - \sigma\sqrt{\frac{2}{\pi}}\right)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= 2\int_{-\infty}^\infty \left(x^2 + \sigma^2\frac{2}{\pi}\right) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx - 2\int_0^\infty \left(-2x\sigma\sqrt{\frac{2}{\pi}}\right) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= \sigma^2 + \frac{2}{\pi}\sigma^2 - 4\frac{1}{\sqrt{2\pi}}\sigma^2\sqrt{\frac{2}{\pi}} \\ &= \sigma^2\frac{\pi - 2}{\pi}. \end{aligned}$$