An introduction to Information Theory

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Lecture #13: Rate Distortion Theory

Outline of the lecture

- Introduction

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- Introduction
- Calculation of rate distortion function

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- Calculation of rate distortion function
- Simultaneous description of independent Gaussian RVs.

Introduction

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- A distortion measure is said to be bounded if the maximum value of the distortion is finite

$$d_{\max} = \max_{x \in X, \hat{x} \in \hat{X}} d(x, \hat{x}) < \infty$$

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- We are interested to know: given a source distribution and a distortion measure, what is the minimum expected distortion achievable at a particular rate?

Definition

- A $(2^{nR}, n)$ rate distortion code consists of following encoding function

$$f_n : X^n \rightarrow \{1, 2, \cdots, 2^{nR}\}$$

and following decoding function

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$$g_n : \{1, 2, \cdots, 2^{nR}\} \to \hat{X}^n$$
- Distortion for the $(2^{nR}, n)$ rate distortion code is defined as
$$D = \sum_{x^n} p(x^n) d(x^n, g_n(f_n(x^n)))$$

Definition

- A rate distortion pair (R, D) is said to be achievable if there exists sequence of $(2^{nR}, n)$ rate distortion codes (f_n, g_n) with $\lim_{n\to\infty} Ed(X^n, g_n(f_n(X^n))) \leq D$.

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- The distortion rate function D(R) is the infimum of all distortions D such that (R,D) is in the rate distortion region of the source for a given rate R.

Definition

- The information rate distortion function $R^I(D)$ for a source X with distortion measure $d(x, \hat{x})$ is defined as

$$R^I(D) = \min_{p(\hat{x}|x): \sum_{(x,\hat{x})} p(x)p(\hat{x}|x)d(x,\hat{x}) \leq D} I(X; \hat{X})$$

where minimization is over all conditional distributions $p(\hat{x}|x)$ for which the joint distribution $p(x, \hat{x}) = p(x)p(\hat{x}|x)$ satisfies the expected distortion constraint.

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- The rate distortion function for an i.i.d. source X with distribution p(x) and bounded distortion function $d(x, \hat{x})$ is equal to the associated information rate distortion function. Thus

$$R(D) = R^I(D) = \min_{p(\hat{x}|x): \sum_{(x,\hat{x})} p(x)p(\hat{x}|x)d(x,\hat{x}) \leq D} I(X; \hat{X})$$

is the minimum achievable rate at distortion D.

Rate Distortion Function

- Let X be $N(0, \sigma^2)$, By rate distortion theorem, we have

$$R(D) = \min_{f(\hat{x}|x): E(\hat{X}-X)^2 \leq D} I(X, \hat{X})$$

Rate Distortion Function

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$$R(D) = \min_{f(\hat{x}|x):E(\hat{X}-X)^2 \le D} I(X, \hat{X})$$

- We know that

$$\begin{aligned} I(X, \hat{X}) &= h(X) - h(X|\hat{X}) \\ &= \frac{1}{2}\log(2\pi e)\sigma^2 - h(X - \hat{X}|\hat{X}) \\ &\ge \frac{1}{2}\log(2\pi e)\sigma^2 - h(X - \hat{X}) \\ &\ge \frac{1}{2}\log(2\pi e)\sigma^2 - h(N(0, E(X - \hat{X})^2)) \\ &= \frac{1}{2}\log(2\pi e)\sigma^2 - \frac{1}{2}\log(2\pi e)E(X - \hat{X})^2 \\ &\ge \frac{1}{2}\log(2\pi e)\sigma^2 - \frac{1}{2}\log(2\pi e)D \\ &= \frac{1}{2}\log\frac{\sigma^2}{D} \end{aligned}$$

Rate Distortion Function

- If $D \le \sigma^2$, we choose $X = \hat{X} + Z$, $\hat{X} \sim N(0, \sigma^2 - D), Z \sim N(0, D)$, where $\hat{X}$ and Z are independent. For this we have

$$I(X, \hat{X}) = \frac{1}{2}\log\frac{\sigma^2}{D}$$

and $E(X - \hat{X})^2 = D$.

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  and $E(X - \hat{X})^2 = D$.
- If $D > \sigma^2$, we chose X=0 with probability 1, achieving R(D)=0.

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  and $E(X - \hat{X})^2 = D$.
- If $D > \sigma^2$, we chose X=0 with probability 1, achieving R(D)=0.
- We can rewrite the distortion in terms of the rate

$$D(R) = \sigma^2 2^{-2R}$$

Rate Distortion Function

- The rate distortion function for a Bernoulli(p) source with Hamming distortion is given by

$$R(D) = \begin{cases} H(p) - H(D) & 0 \leq D \leq \min\{p, 1-p\} \\ 0 & D > \min\{p, 1-p\} \end{cases}$$

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- X $\sim$ Bernoulli(p). Without loss of generality, we consider $p < \frac{1}{2}$. $X \oplus \hat{X} = 1$ is equivalent to $X \neq \hat{X}$.

Rate Distortion Function

- We first will find a lower bound on $I(X,\hat{X})$ and then show that this lower bound is achievable.

$$\begin{aligned} I(X;\hat{X}) &= H(X) - H(X|\hat{X}) \\ &= H(p) - H(X \oplus \hat{X}|\hat{X}) \\ &\geq H(p) - H(X \oplus \hat{X}) \\ &\geq H(p) - H(D) \end{aligned}$$

since $Pr(X \neq \hat{X}) \leq D$ and H(D) increases with D for $D \leq \frac{1}{2}$.

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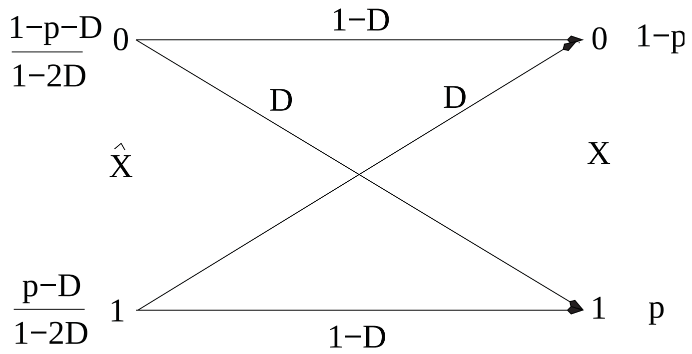
since $Pr(X \neq \hat{X}) \leq D$ and H(D) increases with D for $D \leq \frac{1}{2}$.

- Thus

$$R(D) \leq H(p) - H(D)$$

Rate Distortion Function

- For $0 \leq D \leq p$, we can achieve the value of the rate distortion function by choosing $(X, \hat{X})$ to have joint distribution given by the binary symmetric channel as shown below



Rate Distortion Function

- We choose the distribution of $\hat{X}$ at the input of the channel so that the output distribution of X is the specified distribution. Let $r = Pr(\hat{X} = 1)$. We choose r such that $r(1 - D) + (1 - r)D = p$ or $r = \frac{p-D}{1-2D}$.

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- If $D \leq p \leq \frac{1}{2}$, then $Pr(\hat{X} = 1) \geq 0$ and $Pr(\hat{X} = 0) \geq 0$. We have

$$I(X;\hat{X}) = H(X) - H(X|\hat{X}) = H(p) - H(D).$$

  and expected distortion is $P(X \neq \hat{X}) = D$.

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- If $D \geq p$, we achieve $R(D) = 0$ by letting $\hat{X} = 0$ with probability 1. In this case $I(X;\hat{X}) = 0$ and $D = p$.

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  and expected distortion is $P(X \neq \hat{X}) = D$.
- If $D \geq p$, we achieve $R(D) = 0$ by letting $\hat{X} = 0$ with probability 1. In this case $I(X;\hat{X}) = 0$ and $D = p$.
- Similarly, if $D \geq 1-p$, we can achieve $R(D) = 0$ by setting $\hat{X} = 1$ with probability 1.

Rate Distortion Function

- Consider a source X uniformly distributed on the set $\{1, 2, \cdots, m\}$, Find the rate distortion function for this source with Hamming distortion, i.e.

$$d(x,\hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x} \\ 1 & \text{if } x \neq \hat{x} \end{cases}$$

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- Since $D = Pr\{X \neq \hat{X}\}$, we have by Fano's inequality

$$H(X|\hat{X}) \leq H(D) + D\log(m-1)$$

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- Thus, we have

$$\begin{aligned} I(X;\hat{X}) &= H(X) - H(X|\hat{X}) \\ &\geq \log m - H(D) - D\log(m-1) \end{aligned}$$

Rate Distortion Function

- We can achieve this lower bound by choosing $p(\hat{x})$ to be uniform distribution and conditional distribution of $p(\hat{x}|x)$ to be

$$p(\hat{x}|x) = \begin{cases} 1 - D & \text{if } \hat{x} = x \\ D/(m-1) & \text{if } \hat{x} \neq x \end{cases}$$

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- The bound is satisfied with equality for $D \leq 1 - \frac{1}{m}$, hence

$$R(D) = \begin{cases} \log m - H(D) - D\log(m-1) & \text{if } 0 \leq D \leq 1 - \frac{1}{m} \\ 0 & \text{if } D > 1 - \frac{1}{m} \end{cases}$$

Rate Distortion Function

- Information rate distortion function is defined as

$$R(D) = \min_{q(\hat{x}|x): \sum_{(x,\hat{x})} p(x)q(\hat{x}|x)d(x,\hat{x}) \leq D} I(X;\hat{X})$$

where the minimization is over all conditional distributions $q(\hat{x}|x)$ for which the joint distribution $p(x)q(\hat{x}|x)$ satisfies the expected distortion constraint.

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  where the minimization is over all conditional distributions $q(\hat{x}|x)$ for which the joint distribution $p(x)q(\hat{x}|x)$ satisfies the expected distortion constraint.
- This is a minimization of a convex function over the convex set of all $q(\hat{x}|x) \geq 0$ satisfying $\sum_{\hat{x}} q(\hat{x}|x) = 1$ for all x and $\sum q(\hat{x}|x)p(x)d(x,\hat{x}) \leq D$

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- This is a minimization of a convex function over the convex set of all $q(\hat{x}|x) \geq 0$ satisfying $\sum_{\hat{x}} q(\hat{x}|x) = 1$ for all x and $\sum q(\hat{x}|x)p(x)d(x,\hat{x}) \leq D$
- Using Lagrange multiplier method, we setup the functional

$$\begin{aligned} J(q) &= \sum_{x}\sum_{\hat{x}} p(x)q(\hat{x}|x)\log\frac{q(\hat{x}|x)}{\sum_x p(x)q(\hat{x}|x)} \\ &+ \lambda\sum_{x}\sum_{\hat{x}} p(x)q(\hat{x}|x)d(x,\hat{x}) + \sum_{x}\nu(x)\sum_{\hat{x}} q(\hat{x}|x) \end{aligned}$$

Rate Distortion Function

- If we let $q(\hat{x}) = \sum_x p(x)q(\hat{x}|x)$ be the distribution on $\hat{X}$ induced by $q(\hat{x}|x)$, we can write J(q) as

$$\begin{aligned} J(q) &= \sum_x \sum_{\hat{x}} p(x)q(\hat{x}|x) \log \frac{q(\hat{x}|x)}{q(\hat{x})} \\ &+ \lambda \sum_x \sum_{\hat{x}} p(x)q(\hat{x}|x)d(x,\hat{x}) + \sum_x \nu(x) \sum_{\hat{x}} q(\hat{x}|x) \end{aligned}$$

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- Differentiating with respect to $q(\hat{x}|x)$, we have

$$\begin{aligned} \frac{\partial J}{\partial q(\hat{x}|x)} &= p(x) \log \frac{q(\hat{x}|x)}{q(\hat{x})} + p(x) - \sum_{x'} p(x')q(\hat{x}|x') \frac{1}{q(\hat{x})} p(x) \\ &+ \lambda p(x)d(x,\hat{x}) + \nu(x) = 0 \end{aligned}$$

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- Setting $\log\mu(x) = \nu(x)/p(x)$, we obtain

$$p(x)\left[\log\frac{q(\hat{x}|x)}{q(\hat{x})} + \lambda d(x,\hat{x}) + \log\mu(x)\right] = 0$$

Rate Distortion Function

- Hence

$$q(\hat{x}|x) = \frac{q(\hat{x})e^{-\lambda d(x,\hat{x})}}{\mu(x)}$$

item Since $\sum_{\hat{x}} q(\hat{x}|x) = 1$, we have

$$\mu(x) = \sum_{\hat{x}} q(\hat{x})e^{-\lambda d(x,\hat{x})}$$

or

$$q(\hat{x}|x) = \frac{q(\hat{x})e^{-\lambda d(x,\hat{x})}}{\sum_{\hat{x}} q(\hat{x})e^{-\lambda d(x,\hat{x})}}$$

Rate Distortion Function

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$$q(\hat{x}|x) = \frac{q(\hat{x})e^{-\lambda d(x,\hat{x})}}{\sum_{\hat{x}} q(\hat{x})e^{-\lambda d(x,\hat{x})}}$$

- Multiplying both sides by p(x) and summing over all x, we get

$$q(\hat{x}) = q(\hat{x}) \sum_{x} \frac{p(x)e^{-\lambda d(x,\hat{x})}}{\sum_{\hat{x}} q(\hat{x})e^{-\lambda d(x,\hat{x})}}$$

Rate Distortion Function

- If $q(\hat{x}) > 0$, we obtain

$$\sum_{x} \frac{p(x)e^{-\lambda d(x,\hat{x})}}{\sum_{\hat{x}} q(\hat{x})e^{-\lambda d(x,\hat{x})}} = 1$$

Rate Distortion Function

- If $q(\hat{x}) > 0$, we obtain

$$\sum_x \frac{p(x)e^{-\lambda d(x,\hat{x})}}{\sum_{\hat{x}} q(\hat{x})e^{-\lambda d(x,\hat{x})}} = 1$$

- Applying Kuhn Tucker conditions, we have

$$\frac{\partial J}{\partial q(\hat{x}|x)} \begin{cases} = 0 & \text{if} q(\hat{x}|x) > 0 \\ \geq 0 & \text{if} q(\hat{x}|x) = 0 \end{cases}$$

Rate Distortion Function

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- Substituting the value of the derivative we get

$$\begin{aligned} \sum_x \frac{p(x)e^{-\lambda d(x,\hat{x})}}{\sum_{\hat{x}} q(\hat{x})e^{-\lambda d(x,\hat{x})}} &= 1 \quad if \quad q(\hat{x}) > 0 \\ \sum_x \frac{p(x)e^{-\lambda d(x,\hat{x})}}{\sum_{\hat{x}} q(\hat{x})e^{-\lambda d(x,\hat{x})}} &\leq 1 \quad if \quad q(\hat{x}) = 0 \end{aligned}$$

Simultaneous description of independent Gaussian RVs.

- Consider m independent normal random sources $X_1, \cdots, X_m$, where X_i are $\sim N(0, \sigma_i^2)$ with squared error distortion.

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- Consider m independent normal random sources $X_1, \cdots, X_m$, where X_i are $\sim N(0, \sigma_i^2)$ with squared error distortion.
- Assume that we are given R bits with which to represent this random vector. How should the bits be alloted to various components to minimize the total distortion.

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- Assume that we are given R bits with which to represent this random vector. How should the bits be alloted to various components to minimize the total distortion.
- We have

$$R(D) = \min_{f(\hat{x}^m|x^m):Ed(X^m,\hat{X}^m)\leq D} I(X^m;\hat{X}^m)$$

where $d(x^m, \hat{x}^m) = \sum_{i=1}^m (x_i - \hat{x}_i)^2$

Simultaneous description of independent Gaussian RVs.

- We have

$$\begin{aligned}
I(X^m;\hat{X}^m) &= h(X^m) - h(X^m|\hat{X}^m) \\
&= \sum_{i=1}^m h(X_i) - \sum_{i=1}^m h(X_i|X^{i-1},\hat{X}^m) \\
&\geq \sum_{i=1}^m h(X_i) - \sum_{i=1}^m h(X_i|\hat{X}_i) \\
&= \sum_{i=1}^m I(X_i;\hat{X}_i) \\
&\geq \sum_{i=1}^m R(D_i) \\
&= \sum_{i=1}^m \left(\frac{1}{2}\log\frac{\sigma_i^2}{D_i}\right)^+
\end{aligned}$$

where $D_i = E(X_i - \hat{X}_i)^2$

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Simultaneous description of independent Gaussian RVs.

- Hence the problem of finding the rate distortion function can be reduced to

$$R(D) = \min_{D_i = D} \sum_{i=1}^{m} \max \left\{ \frac{1}{2} \log \frac{\sigma_i^2}{D_i}, 0 \right\}$$

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- Differentiating with respect to D_i, and equating to 0, we get

$$\frac{\partial J}{\partial D_i} = -\frac{1}{2}\frac{1}{D_i} + \lambda = 0 \implies D_i = \lambda'$$

Simultaneous description of independent Gaussian RVs.

- Let $X_i \sim N(0, \sigma_i^2), i = 1, 2, \cdots, m$ be independent Gaussian random variables, and let the distortion measure be $d(x^m, \hat{x}^m) = \sum_{i=1}^{m}(x_i - \hat{x}_i)^2$.

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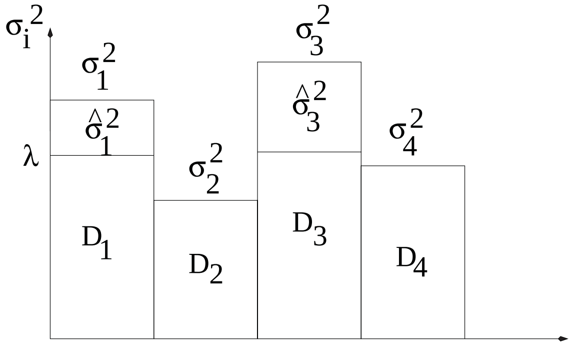
- This gives rise to a kind of reverse waterfilling.

Simultaneous description of independent Gaussian RVs.

- We choose a constant λ and only describe those random variables with variances greater than λ.

Simultaneous description of independent Gaussian RVs.

- We choose a constant λ and only describe those random variables with variances greater than λ.
- No bits are used to describe random variables with variance less than λ.



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