An introduction to Information Theory

Adrish Banerjee

Department of Electrical Engineering
Indian Institute of Technology Kanpur
Kanpur, Uttar Pradesh
India

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Lecture #10B: Noisy channel coding theorem

Outline of the lecture

- Noisy channel coding theorem

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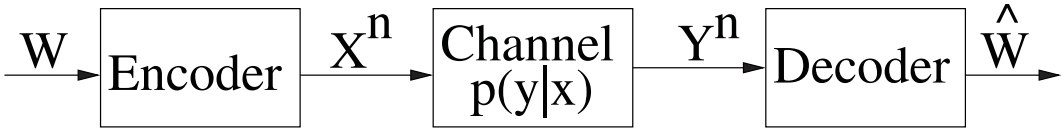
- Noisy channel coding theorem
- Converse to noisy channel coding theorem

Introduction



Communication Channel Model

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  3) A decoding function

$$g : Y^n \rightarrow \{1, 2, \cdots, M\}$$

  is a deterministic rule that assigns a guess to each possible received vector.

Definition

- Let

$$\lambda_i = Pr(g(Y^n) \neq i | X^n = X^n(i)) = \sum_{y^n} p(y^n|x^n(i)) I(g(y^n) \neq i)$$

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- The average probability of error $P_\epsilon^{(n)}$ for an (M,n) code is defined as

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- The capacity of a discrete memoryless channel is the supremum of all achievable rates.

Noisy channel coding theorem

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- We exhibit the 2^{nR} codewords as rows of the matrix

$$C = \begin{bmatrix} x_1(1) & x_2(1) & \cdots & x_n(1) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(2^{nR}) & x_2(2^{nR}) & \cdots & x_n(2^{nR}) \end{bmatrix}$$

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- Probability that we generate a particular code C is

$$Pr(C) = \prod_{w=1}^{2^{nR}} \prod_{i=1}^{n} p(x_i(w))$$

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- The wth codeword $X^n(w)$ corresponding to the wth row of C is sent over the channel.
- The receiver receives a sequence Y^n according to the distribution

$$P(y^n|x^n(w)) = \prod_{i=1}^{n} p(y_i|x_i(w))$$

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- The receiver declares that the index $\tilde{W}$ was sent if the following conditions are satisfied: $(X^n(\tilde{W}), Y^n)$ is jointly typical and there is no other index k, such that $(X^n(k), Y^n) \in A_\epsilon^{(n)}$.

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- If no such $\tilde{W}$ exists or if there is more than one such, an error is declared. If $\tilde{W} \neq W$, there is a decoding error.

Noisy channel coding theorem

- We will calculate the average probability of error, averaged over all codewords in the codebook and average over all codebooks.

$$\begin{aligned} Pr(E) &= \sum_{C} P(C) P_{\epsilon}^{(n)}(C) \\ &= \sum_{C} P(C) \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \lambda_w(C) \\ &= \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \sum_{C} P(C) \lambda_w(C) \end{aligned}$$

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- Due to symmetry of the code construction, the average probability of error averaged over all codes does not depend on particular index that was sent.

Noisy channel coding theorem

- We assume without loss of generality that the message $W = 1$ was sent. Average probability of error is gven by

$$\begin{aligned} Pr(E) &= \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \sum_{C} P(C)\lambda_w(C) \\ &= \sum_{C} P(C)\lambda_1(C) \\ &= Pr(E|W=1) \end{aligned}$$

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- Let E_i be the event that the ith codeword and Y^n are jointly typical.

$$E_i = \left\{(X^n(i), Y^n) \text{ is in } A_\epsilon^{(n)}\right\}, \quad i \in \left\{1, 2, \cdots, 2^{nR}\right\}$$

Noisy channel coding theorem

- We have $Pr(E) = Pr(E|W = 1)$ given by

$$\begin{aligned} Pr(E|W=1) &= P(E_1^c \cup E_2 \cup \cdots \cup E_{2^{nR}}|W=1) \\ &\leq P(E_1^c|W=1) + \sum_{i=2}^{2^{nR}} P(E_i|W=1) \end{aligned}$$

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- By joint AEP, we have $P(E_1^c|W = 1) \leq \epsilon$ for large n.
- $X^n(1)$ and $X^n(i)$ are independent, so are Y^n and $X^n(i)$, hence the probability that Y^n and $X^n(i)$ are jointly typical is given by $\leq 2^{-n(I(X;Y)-3\epsilon)}$.

Noisy channel coding theorem

- Hence, we have

$$\begin{aligned} P(E) = P(E|W=1) &\leq P(E_1^c|W=1) + \sum_{i=2}^{2^{nR}} P(E_i|W=1) \\ &\leq \epsilon + \sum_{i=2}^{2^{nR}} 2^{-n(I(X;Y)-3\epsilon)} \\ &= \epsilon + (2^{nR}-1)2^{-n(I(X;Y)-3\epsilon)} \\ &= \epsilon + 2^{3n\epsilon}2^{-n(I(X;Y)-R)} \\ &\leq 2\epsilon \end{aligned}$$

if n is sufficiently large and $R < I(X;Y) - 3\epsilon$.

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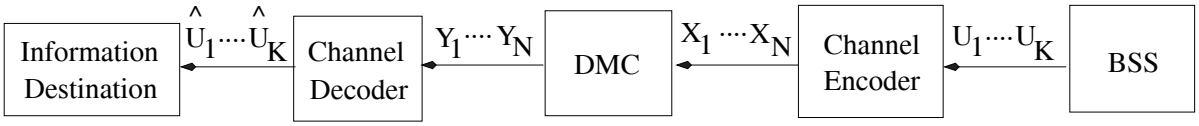
- If we reindex these codewords, we have 2^{nR-1} codewords.
- Thus we have constructed a code of rate $R' = R - \frac{1}{n}$ with maximal probability of error $\lambda^{(n)} \leq 4\epsilon$.

Converse to noisy channel coding theorem

- If information bits from a binary symmetric source (BSS) are sent at a rate R via a DMC of capacity C without feedback, then the bit error probability at the destination satisfies

$$P_b \geq H^{-1}\left(1 - \frac{C}{R}\right), \quad \text{if } R > C$$

where H^{-1} denotes the inverse binary entropy function defined by $H^{-1}(x) = \min\{p : H(p) = x\}$.



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$$\implies H(Y_1 \cdots Y_N | X_1 \cdots X_N) = \sum_{i=1}^{N} H(Y_i|X_i)$$

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- Applying data processing lemma, we get

$$I(U_1 \cdots U_K; \hat{U}_1 \cdots \hat{U}_K) \leq I(X_1 \cdots X_N; \hat{U}_1 \cdots \hat{U}_K)$$

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- We can write $I(X_1 \cdots X_N; Y_1 \cdots Y_N)$ as

$$\begin{aligned} I(X_1 \cdots X_N; Y_1 \cdots Y_N) &= H(Y_1 \cdots Y_N) - H(Y_1 \cdots Y_N | X_1 \cdots X_N) \\ &= H(Y_1 \cdots Y_N) - \sum_{i=1}^{N} H(Y_i|X_i) \\ &\leq \sum_{i=1}^{N} [H(Y_i) - H(Y_i|X_i)] \\ &= \sum_{i=1}^{N} I(X_i; Y_i) \leq NC \end{aligned}$$

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- We can $H(U_1 \cdots U_K | \hat{U}_1 \cdots \hat{U}_K)$ as
$$\begin{aligned} H(U_1 \cdots U_K | \hat{U}_1 \cdots \hat{U}_K) &= H(U_1 \cdots U_K) - I(U_1 \cdots U_K; \hat{U}_1 \cdots \hat{U}_K) \\ &= K - I(U_1 \cdots U_K; \hat{U}_1 \cdots \hat{U}_K) \\ &\geq K - NC \\ &= N(R - C) \end{aligned}$$

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- Also,
$$\begin{aligned} H(U_1 \cdots U_K | \hat{U}_1 \cdots \hat{U}_K) &= \sum_{i=1}^{K} H(U_i | \hat{U}_1 \cdots \hat{U}_K U_1 \cdots U_{i-1}) \\ &\leq \sum_{i=1}^{K} H(U_i | \hat{U}_i) \end{aligned}$$

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- Using Fano's lemma we get

$$\sum_{i=1}^{K} H(U_i | \hat{U}_i) \leq \sum_{i=1}^{K} H(P_{ei})$$

Converse to noisy channel coding theorem

- Combining the previous results we get

$$\frac{1}{K}\sum_{i=1}^{K} H(P_{ei}) \geq \frac{N}{K}(R-C) = 1 - \frac{C}{R}$$

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- Since H(p) is a concave function, we get

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