Module 2 : Signals in Frequency Domain
Problem Set 2

Problem 1

Let $x(t)$ be a periodic signal with fundamental period T and Fourier series coefficients a_k. Derive the Fourier series coefficients of each of the following signals in terms of a_k :

(a) $x(t-t_0)+x(t+t_0)$

(b) $\varepsilon\upsilon\{x(t)\}$

(c) $\Re e\{x(t)\}$

(d) $\dfrac{d^2x(t)}{dt^2}$

(e) $x(3t-1)$ [for this part , first determine the period of $x(3t-1)$]

Solution 1

(a) $x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

where $\omega_0=\dfrac{2\pi}{t}$. Thus, we have

$$x(t-t_0)+x(t+t_0) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(t-t_0)} + \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(t+t_0)}$$

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jk\omega_0 t_0} + \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{jk\omega_0 t_0}$$

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}[e^{-jk\omega_0 t_0}+e^{jk\omega_0 t_0}]$$

$$\sum_{k=-\infty}^{\infty} 2a_k \cos(k\omega_0 t_0) e^{jk\omega_0 t}$$

Therefore the new Fourier coefficients are

$$\hat{a}_k = 2a_k\cos(k\omega_0 t_0)$$

(b) $\varepsilon\upsilon\{x(t)\} = \dfrac{x(t)+x(-t)}{2}$

$$= \frac{1}{2}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t}\right]$$

$$= \frac{1}{2}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} a_{-k} e^{jk\omega_0 t}\right]$$

$$= \sum_{k=-\infty}^{\infty} \left[\frac{a_k + a_{-k}}{2}\right] e^{jk\omega_0 t}$$

Hence the Fourier series coefficients of $\mathcal{E}v\{x(t)\}$ are given by

$$\hat{a}_k = \left[\frac{a_k + a_{-k}}{2}\right]$$

(c) $$\Re e\{x(t)\} = \frac{x(t) + x^*(t)}{2}$$

$$= \frac{1}{2}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} a^*{}_k e^{-jk\omega_0 t}\right]$$

$$= \frac{1}{2}\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} + \sum_{k=-\infty}^{\infty} a^*{}_{-k} e^{jk\omega_0 t}\right]$$

$$= \sum_{k=-\infty}^{\infty} \left[\frac{a_k + a^*{}_{-k}}{2}\right] e^{jk\omega_0 t}$$

Hence the Fourier series coefficients of $\Re e\{x(t)\}$ are given by

$$\hat{a}_k = \left[\frac{a_k + a^*{}_{-k}}{2}\right]$$

(d) $$\frac{d^2 x(t)}{dt^2} = \sum_{k=-\infty}^{\infty} \frac{d^2 (a_k e^{jk\omega_0 t})}{dt^2}$$

$$= \sum_{k=-\infty}^{\infty} a_k (jk\omega_0)^2 e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} -a_k k^2 \omega_0{}^2 e^{jk\omega_0 t}$$

Hence the Fourier series coefficients of $\frac{d^2 x(t)}{dt^2}$ are given by

$$\hat{a}_k = -a_k k^2 \omega_0{}^2$$

(e) $x(3t-1) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(3t-1)}$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j3k\omega_0 t} e^{-jk\omega_0}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0} e^{jk(3\omega_0)t} \quad \textbf{(1)}$$

Since $x(t)$ has fundamental period T, $x(3t-1)$ has fundamental period $\frac{T}{3}$. Therefore, from **(1)** the Fourier coefficients of $x(3t-1)$ are

$$\hat{a}_k = a_k e^{-jk\omega_0}$$

Problem 2

Suppose we are given the following information about a signal x(t) :

1. $x(t)$ is a real signal .
2. $x(t)$ is periodic with period T = 6 and has Fourier coefficients a_k .
3. a_k = 0 for k = 0 and k > 2 .
4. $x(t) = -x(t-3)$.
5. $\frac{1}{6}\int_{-3}^{3} |x(t)|^2 dt = \frac{1}{2}$.
6. a_1 is a positive real number .

Show that $x(t) = A\cos(Bt + C)$, and determine the values of the constants A , B , and C .

Solution 2

Since $x(t)$ is a real signal, $a_k = a^*_{-k}$. But from the given hypothesis, $a_k = 0$ for k > 2. This implies that $a_{-k} = a^*_k = 0$ for k > 2.

Also, it is given that $a_0 = 0$. Therefore the only non-zero Fourier coefficients are a_1, $a_{-1} = a_1^*$, a_2 and $a_{-2} = a_2^*$.

It is also given that a_1 is a positive real number. Therefore $a_{-1} = a_1$. Thus we have,

$$x(t) = a_1\left[e^{j\frac{2\pi}{T}t} + e^{-j\frac{2\pi}{T}t}\right] + a_2 e^{j\frac{4\pi}{T}t} + a_2^* e^{-j\frac{4\pi}{T}t}$$

$$= 2a_1\cos\left(\frac{2\pi}{T}t\right) + a_2 e^{j\frac{4\pi}{T}t} + a_2^* e^{-j\frac{4\pi}{T}t}$$

$$= 2a_1\cos\left(\frac{\pi}{3}t\right) + a_2 e^{j\frac{4\pi}{T}t} + a_2^* e^{-j\frac{4\pi}{T}t}$$

Since $e^{j\frac{4\pi}{T}t}$ and $e^{-j\frac{4\pi}{T}t}$ are both periodic with period 3, we have

$$x(t-3) = -2a_1\cos\left(\frac{\pi}{3}t\right) + a_2 e^{j\frac{4\pi}{T}t} + a_2^* e^{-j\frac{4\pi}{T}t}$$

But, by given hypothesis we have $x(t) = -x(t-3)$, which implies that

$$2\left[a_2 e^{j\frac{4\pi}{T}t} + a_2^* e^{-j\frac{4\pi}{T}t}\right] = 0$$

Therefore we have,

$$x(t) = 2a_1\cos\left(\frac{\pi}{3}t\right)$$

Finally, it is given that

$$\frac{1}{6}\int_{-3}^{3}|x(t)|^2\,dt = \frac{1}{2}$$

$$\Rightarrow \frac{4}{6}\int_{-3}^{3} a_1^2\cos^2\left(\frac{\pi}{3}\right)tdt = \frac{1}{2}$$

$$\Rightarrow a_1 = \frac{1}{2}$$

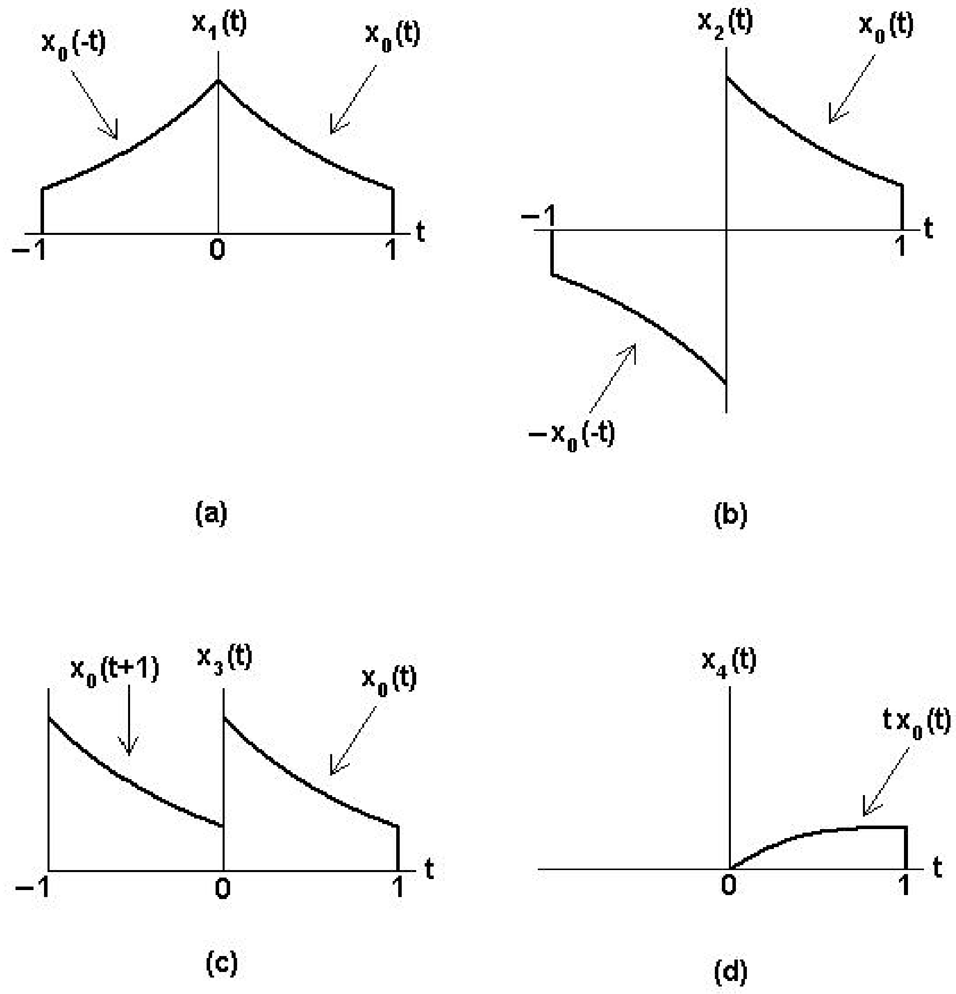
Therefore, $x(t) = \cos\left(\frac{\pi}{3}t\right)$ and the constants A = 1, B = $\frac{\pi}{3}$ and C = 0.

Problem 3

Consider the signal

$$x_0(t) = \begin{cases} e^{-t}, 0 \le t \le 1 \\ 0, elsewhere \end{cases}$$

Determine the Fourier transform of each of the signals shown in figure below . You should be able to do this by explicitly evaluating only the transform of $x_0(t)$ and then using properties of the Fourier transform.



Solution 3

$$X_0(j) = \int_{t=0}^{1} x(t)e^{-j\omega t}dt = \int_{t=0}^{1} e^{-t}e^{-j\omega t}dt = \int_{t=0}^{1} e^{-t(j\omega+1)}dt = \left.\frac{e^{-t(j\omega+1)}}{-(j\omega+1)}\right|_0^1$$

$$= \frac{e^{-(j\omega+1)}-1}{-(j\omega+1)}$$

(a) $x_1(t) = x_0(t) + x_0(-t)$

$$X_1(j) = X_0(j) + X_0(-j) = \frac{e^{-(j\omega+1)}-1}{-(j\omega+1)} + \frac{e^{(j\omega-1)}-1}{j\omega-1}$$

(b) $x_2(t) = x_0(t) - x_0(-t)$

$$X_2(j) = X_0(j) - X_0(-j) = \frac{e^{-(j\omega+1)} - 1}{-(j\omega+1)} - \frac{e^{(j\omega-1)} - 1}{j\omega - 1}$$

(c) $x_3(t) = x_0(t) + x_0(t+1)$

$$X_3(j) = X_0(j) + X_0(j)e^{+j\omega} = \frac{e^{-(j\omega+1)} - 1}{-(j\omega+1)}(1 + e^{+j\omega})$$

(d) $x_4(t) = tx_0(t)$

$$X_4(j) = j\frac{d}{d\omega}X_0(j) = \frac{e^{-(j\omega+1)}}{-(j\omega+1)} + \frac{1 - e^{-(j\omega+1)}}{(j\omega+1)^2}$$

Problem 4

A causal stable LTI system S has the frequency response $H(jw) = \dfrac{j\omega + 4}{6 - \omega^2 + 5j\omega}$

- Determine a differential equation relating the input x(t) and output y(t) of S .
- Determine the impulse response h(t) of S.
- What is the output of S when the input is $x(t) = e^{-4t}u(t) - te^{-4t}u(t)$?

Solution 4

$$H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega} = \frac{Y(j\omega)}{X(j\omega)}$$

(a) $Y(j\omega)(6 - \omega^2 + 5j\omega) = X(j\omega)(j\omega + 4)$

$$6y(t) + \frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) = \frac{d}{dt}x(t) + 4x(t)$$

(b) $H(j\omega) = \dfrac{j\omega + 4}{(2 + j\omega)(3 + j\omega)} = \dfrac{A}{(2 + j\omega)} + \dfrac{B}{(3 + j\omega)}$;

Multiply both sides by $(2 + j\omega)$

and set = 2j to get A =2

multiply both sides by $(3 + j\omega)$ and set = 3j to get B= -1

$$H(j\omega) = \frac{2}{(2 + j\omega)} - \frac{1}{(3 + j\omega)}$$

$$h(t) = F^{-1}\{H(j\omega)\} = (2e^{-2t} - e^{-3t})u(t)$$

(c) $Y(j\omega) = H(j\omega)X(j\omega) = (\frac{j\omega+4}{6-\omega^2+5j\omega})(\frac{1}{(4+j\omega)} - \frac{1}{(4+j\omega)^2})$

$$= \frac{(4+j\omega)(3+j\omega)}{(2+j\omega)(3+j\omega)(4+j\omega)^2} = \frac{1}{(2+j\omega)(4+j\omega)} = \frac{A}{(2+j\omega)} + \frac{B}{(4+j\omega)}$$

multiply both sides by $(2+j\omega)$ and set = 2j to get A=1/2

multiply both sides by $(4+j\omega)$ and set = -4j to get B= -1/2

$$Y(j\omega) = \frac{1/2}{(2+j\omega)} - \frac{1/2}{(4+j\omega)}$$

$$y(t) = F^{-1}\{Y(j\omega)\} = (0.5e^{-2t} - 0.5e^{-4t})u(t)$$

Problem 5

The output y(t) of a causal LTI system is related to the input x(t) by the equation

$$\frac{dy(t)}{dt} + 10y(t) = \int_{-\infty}^{+\infty} x(\tau)z(t-\tau)d\tau - x(t)$$

where $z(t) = e^{-t}u(t) + 3\delta(t)$.

- Find the frequency response $H(j\omega) = Y(j\omega)/X(j\omega)$ of this system.
- Determine the impulse response of the system.

Solution 5

$$\frac{dy(t)}{dt} + 10y(t) = \int_{-\infty}^{\infty} x(\tau)z(t-\tau)d\tau - x(t)$$

(a) $Y(j\omega)(10+j\omega) = X(j\omega)(Z(j\omega)-1)$

$$Z(j\omega) = \frac{1}{1+j\omega} + 3 = \frac{4+3j\omega}{1+j\omega}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{Z(j\omega)-1}{10+j\omega} = \frac{3+2j\omega}{(10+j\omega)(1+j\omega)}$$

(b) $H(j\omega) = \frac{3+2j\omega}{(10+j\omega)(1+j\omega)} = \frac{A}{(10+j\omega)} + \frac{B}{(1+j\omega)}$

Multiply both sides by $(10+j\omega)$ and set = 10j to get A = 17/9

Multiply both sides by $(1+j\omega)$ and set = j to get B= 1/9

$$H(j\omega) = \frac{17/9}{(10+j\omega)} + \frac{1/9}{(1+j\omega)}$$

$$h(t) = F^{-1}\{H(j\omega)\} = (\frac{17}{9}e^{-10t} + \frac{1}{9}e^{-t})u(t)$$

Problem 6

Consider the signal x(t) in the figure.

(a) Find the Fourier transform $X(j\omega)$ of $x(t)$.

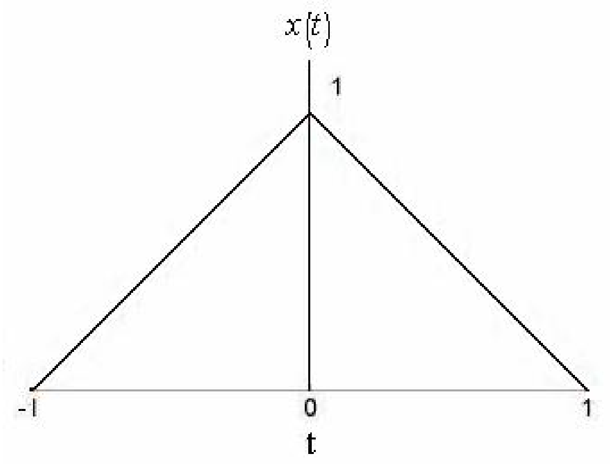
(b) Sketch the signal

$$\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t-4k)$$.

(c) Find another signal $g(t)$ such that $g(t)$ is not the same as $x(t)$ and

$$\tilde{x}(t) = g(t) * \sum_{k=-\infty}^{\infty} \delta(t-4k)$$.

(d) Argue that, although $G(j\omega)$ is different from $X(j\omega)$, $G\left(j\frac{\pi k}{2}\right) = X\left(j\frac{\pi k}{2}\right)$ for all integers k. You should not explicitly evaluate $G(j\omega)$ to answer this question.



Solution 6

(a) Consider the signal $y(t)$ shown below.

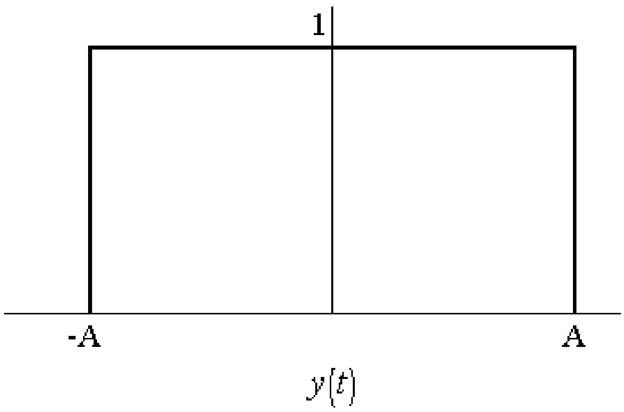


Fig. (a)

Now consider convolution of $y(t)$ with itself. Let the resultant signal be $z(t)$.

Then, $z(t) = y(t) * y(t)$.

$$= \int_{-\infty}^{\infty} y(k)\, y(t-k)\, dk$$

Now, consider $y(k)$ and $y(t-k)$.

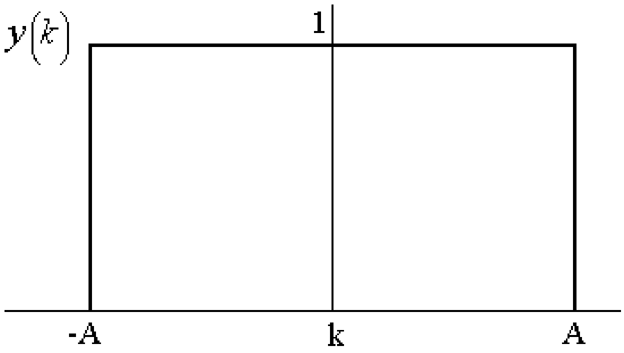


Fig (b)

$$X\left(j\frac{\pi k}{2}\right)e^{-j2m\pi} \quad m \in \mathbb{Z} \quad and \quad m = kn$$

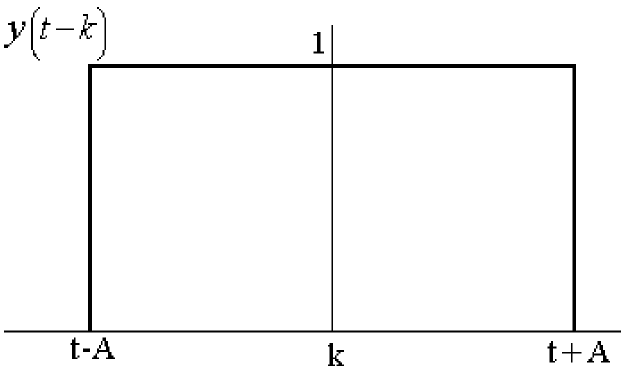


Fig (c)

So, when $t + A < -A, \quad \Rightarrow t < -2A$.

$z(t) = 0$ as there is no overlap between $y(k)$ and $y(t-k)$.

When $-A \le t + A < A, \quad \Rightarrow -2A \le t < 0$.

Then, $z(t) = \int_{-A}^{t+A} y(k)\, y(t-k)\, dk$.

Here, $y(k) = 1$ and $y(t-k) = 1$.

So, $z(t) = \int_{-A}^{t+A} 1 dk$.

$$= [k]_{-A}^{t+A} = t + 2A.$$

When $t - A > A \quad \Rightarrow t > 2A$.

Then, there is no overlap between $y(k)$ and $y(t-k)$.

So, the signal $z(t)$ is

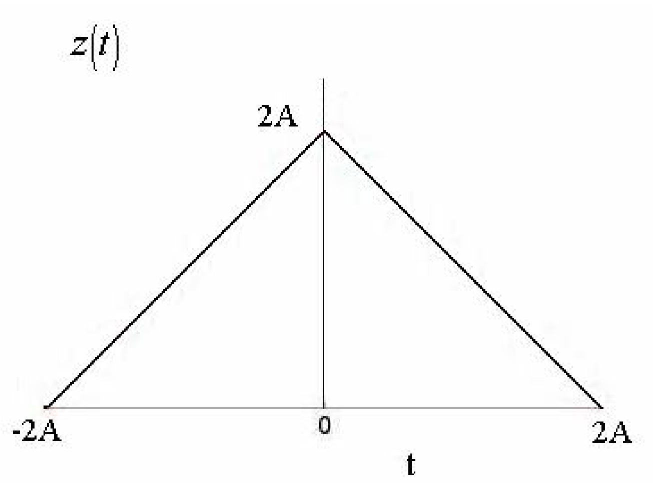


Fig. (d)

Now compare $z(t)$ with the given signal $x(t)$. Here $x(t)$ is

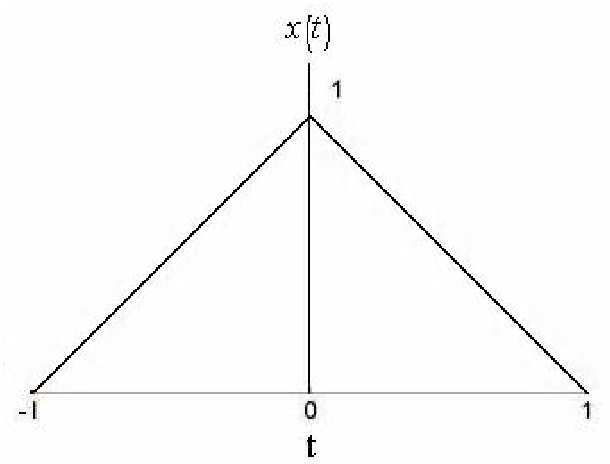


Fig. (e)

By comparing, we get $A = \frac{1}{2}$.

So, $x(t)$ can be thought of as a signal which is a convolution of the signal $v(t)$ with itself, where $v(t)$ is

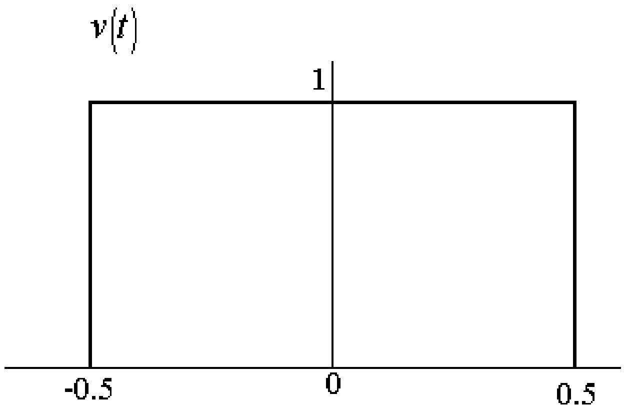


Fig. (f)

This implies $x(t) = v(t) * v(t)$.

By Convolution Theorem, the Fourier Transform of $x(t)$ is square of the Fourier Transform of $v(t)$, i.e. $\log[X(j\omega)] = 2(\log[V(j\omega)])$.

Now, consider $v(t)$. Its Fourier Transform $V(j\omega)$ is

$$V(j\omega) = \int_{-\infty}^{\infty} v(t) e^{-j\omega t} dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t} dt$$

$$= 2\frac{Sin\left(\frac{\omega}{2}\right)}{\omega}.$$

So, $X(j\omega) = \left[V(j\omega)\right]^2$

$$X(j\omega) = 4\frac{Sin^2\left(\frac{\omega}{2}\right)}{\omega^2}.$$

(b) $\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t - 4k)$

Now, consider a signal $y_k(t)$, where

$$y_k(t) = x(t) * \delta(t-4k)$$

$$y_k(t) = \int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda-4k)\,d\lambda$$

$$= \int_{-\infty}^{\infty} x(\lambda)\delta((t-4k)-\lambda)\,d\lambda$$

$y_k(t) = x(t-4k)$ as $\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)\,d\lambda = x(t)$.

So, $y_k(t)$ is the shifted version of $x(t)$ by 4k along the t axis which is shown as

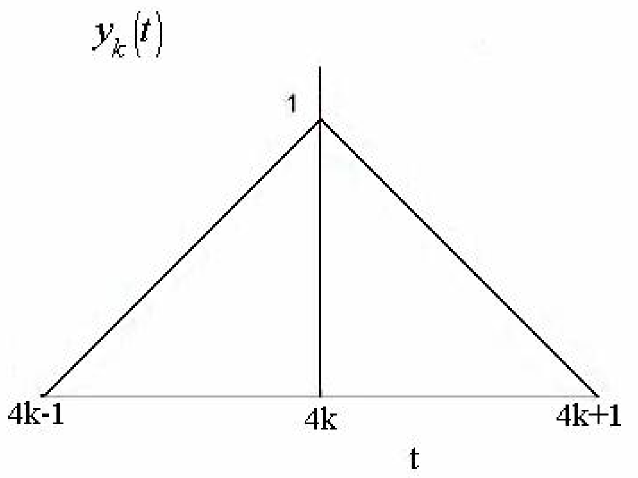


Fig. (g)

Now, $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} y_k(t)$

So, $\tilde{x}(t)$ is

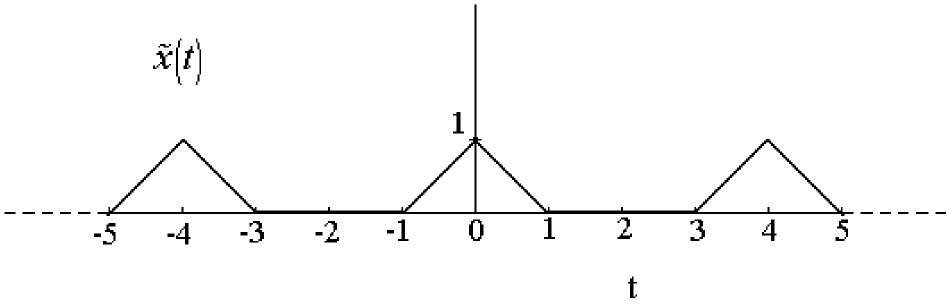


Fig. (h)

(c) Any signal which is the shifted version of $x(t)$ by 4k on the t axis can be taken as $g(t)$, which satisfies $g(t) = g(t) * \sum_{k=-\infty}^{\infty} \delta(t-4k)$ because by the last part we can say that when we convolve $g(t)$ with $\sum_{k=-\infty}^{\infty} \delta(t-4k)$, it will result in $\tilde{x}(t)$.

In this case, $g(t) = x(t-4k)$, where $k \in \mathbb{Z}$.

(d) By part **(c)** ,

$g(t) = x(t-4n)$, where $n \in \mathbb{Z}$.

$$\Rightarrow G(j\omega) = X(j\omega)e^{-j\omega 4n}$$

Now, put $\omega = \frac{\pi k}{2}$.

$$\Rightarrow G\left(j\frac{\pi k}{2}\right) = X\left(j\frac{\pi k}{2}\right)e^{-j\frac{\pi k}{2}4n}$$

$$X\left(j\frac{\pi k}{2}\right)e^{-j2m\pi} \quad m \in \mathbb{Z} \quad and \quad m = kn$$

$G\left(j\frac{\pi k}{2}\right) = X\left(j\frac{\pi k}{2}\right)$ as $e^{-j2m\pi} = 1$

Problem 7

Consider an LTI system whose response to the input

$$x(t) = \left[e^{-t} + e^{-3t}\right]u(t)$$

is $y(t) = \left[2e^{-t} - 2e^{-4t}\right]u(t)$

(a) Find the frequency response of this system.

(b) Determine the system's impulse response.

Solution 7

(a) Given:

$$x(t) = \left[e^{-t} + e^{-3t}\right]u(t)$$

$$y(t) = \left[2e^{-t} - 2e^{-4t}\right]u(t)$$

The Fourier transform of $e^{-at}u(t)$ [where a>0] is $\frac{1}{a+j\omega}$.

So Fourier transform of x(t) is

$$X(j\omega) = \frac{1}{1+j\omega} + \frac{1}{3+j\omega}$$

Similarly, the Fourier transform of output y[t] is

$$Y(j\omega) = \frac{2}{1+j\omega} - \frac{2}{4+j\omega}$$

Let the frequency response of the given LTI system be $H(j\omega)$.

So, by the convolution theorem,

As $y(t) = x(t) * h(t)$,

$$\Rightarrow Y(j\omega) = X(j\omega)\,H(j\omega)$$

$$\Rightarrow \frac{2}{1+j\omega} - \frac{2}{4+j\omega} = \left(\frac{1}{1+j\omega} + \frac{1}{3+j\omega}\right)H(j\omega)$$

By solving this, we get

$$H(j\omega) = 3\left[\frac{3+j\omega}{(4+j\omega)(2+j\omega)}\right]$$

$$H(j\omega) = \frac{3}{2(4+j\omega)} + \frac{3}{2(2+j\omega)}$$

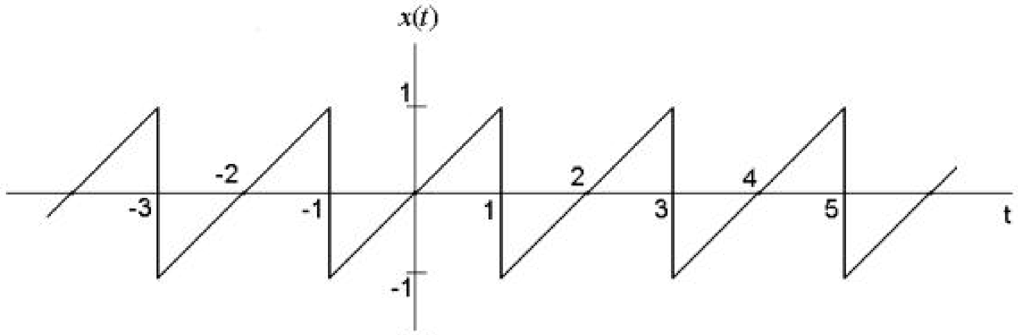
(b) By taking Inverse Fourier Transform of $H(j\omega)$, we get

$$h(t) = \frac{3}{2}\left(e^{-4t} + e^{-2t}\right)u(t)$$

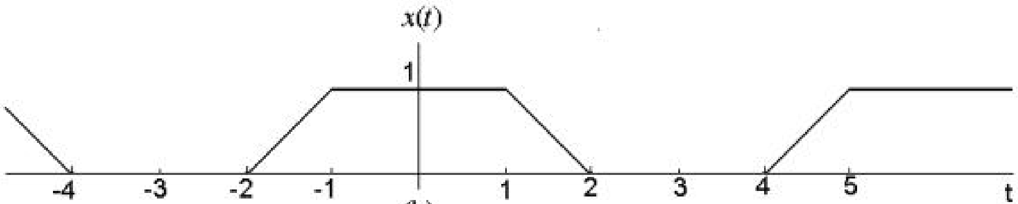
Problem 8

Determine the Fourier series representations for the following signals:

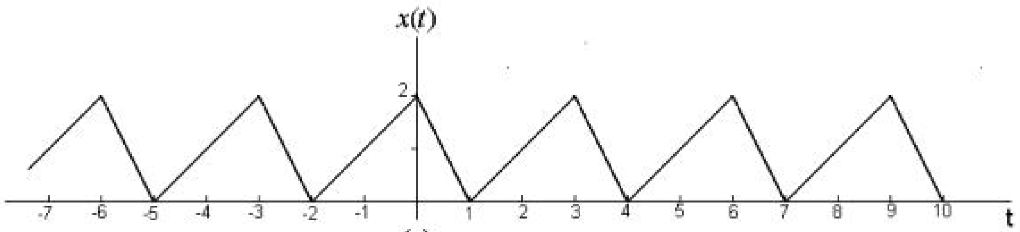
(1) Each $x(t)$ illustrated in the figures (a) - (f)



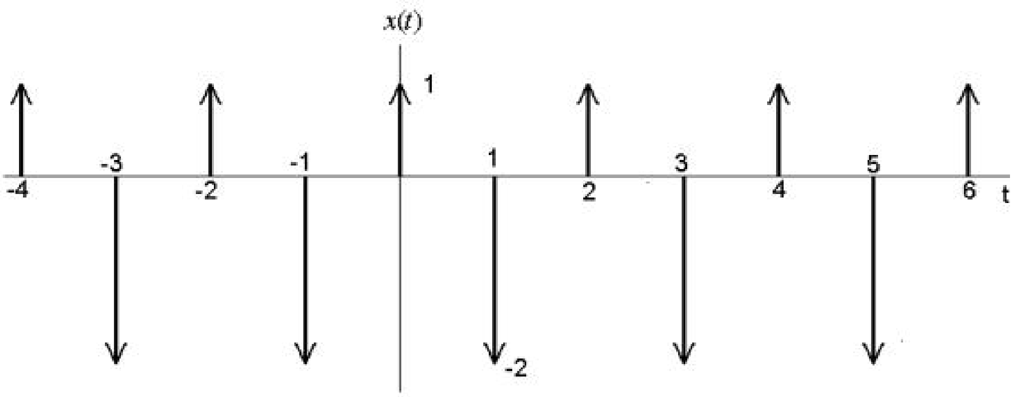
(a)



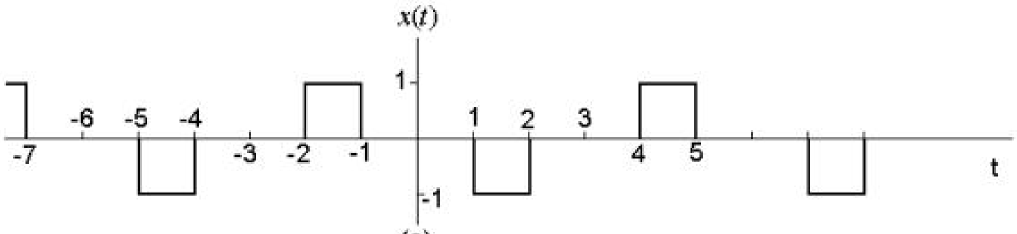
(b)



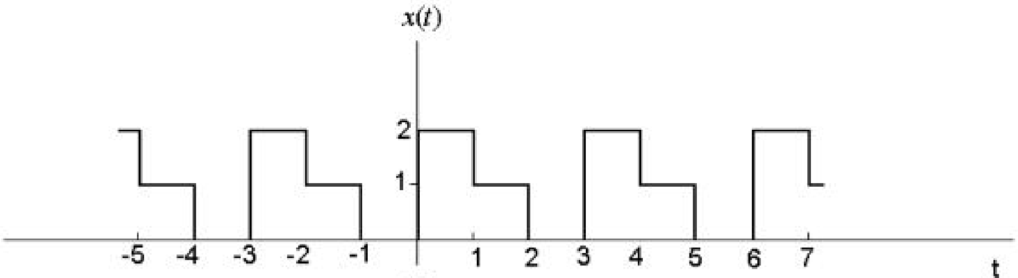
(c)



(d)



(e)



(f)

(2) $x(t)$ periodic with period 2 and $x(t)=e^{-t}$ for $-1<t<1$.

(3) $x(t)$ periodic with period 4 and

$$x(t)=\begin{cases} Sin\ \pi t, & 0\le t\le 2 \\ 0, & 2<t\le 4 \end{cases}$$

Solution 8

(1)

(a) It is periodic with period 2. So, consider segment between $t\in[-1,1]$.

Let

$$x(t)=\sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T}$$

be the Fourier expansion, where:

$$C_k=\frac{1}{T}\int_{(T)} x(t)e^{-j2\pi kt/T}dt$$

$$\Rightarrow C_0=\frac{1}{2}\int_{(T)} x(t)dt=0$$

For $k\neq 0$,

$$C_k = \frac{1}{2}\int_{(T)} x(t)e^{-j2\pi kt/T}dt$$

But, $x(t) = t$ in $[-1,1]$

$$\Rightarrow C_k = \frac{1}{2}\int_{(T)} te^{-j2\pi kt/T}dt$$

$$= \frac{1}{2}\left[t\frac{e^{-j\pi kt}}{-j\pi k}\right]_{-1}^{1} - \frac{1}{2}\left[\frac{e^{-j\pi kt}}{(-j\pi k)^2}\right]_{-1}^{1}$$

$$= \frac{(-1)^{k+1}}{j\pi k}$$

Substituting in x(t)

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T}$$

(b) It is periodic with period 6. So, consider segment between $t \in [-3,3]$.

The function in this interval is:

$$\begin{aligned} x(t) &= 0 && -3 \le t \le -2 \\ &= t+2 && -2 \le t \le -1 \\ &= 1 && -1 \le t \le 1 \\ &= -t+2 && 1 \le t \le 2 \\ &= 0 && 2 \le t \le 3 \end{aligned}$$

Let

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T}$$

be the Fourier expansion, where:

$$C_k = \frac{1}{T}\int_{(T)} x(t)e^{-j2\pi kt/T}dt$$

$$\Rightarrow C_0 = \frac{1}{6}\int_{(T)} x(t)dt = 1/2$$

For $k \neq 0$,

$$C_k = \frac{1}{6}\int_{-3}^{3} x(t)e^{-j2\pi kt/6}dt$$

$$\Rightarrow C_k = \frac{1}{6}\left[\int_{-2}^{-1}(t+2)e^{-j\pi kt/3}dt + \int_{-1}^{1}e^{-j\pi kt/3}dt + \int_{1}^{2}(2-t)e^{-j\pi kt/3}dt\right]$$

$$=\frac{1}{6}\left[\frac{18}{\pi^2k^2}\left\{Cos(\pi k/3)-Cos(2\pi k/3)-\frac{\pi k}{3}Sin(\pi k/3)\right\}+\frac{6}{\pi k}\sin(\pi k/3)\right]$$

$$=\frac{3}{\pi^2k^2}\left[Cos(\pi k/3)-Cos(2\pi k/3)\right]$$

Substituting in x(t)

$$x(t)=\sum_{k=-\infty}^{\infty}C_k e^{j2\pi kt/T}$$

(c) It is periodic with period 3, so take the segment $t\in[-2,1]$

The function in the interval is:

$$\begin{aligned} x(t) &= t+2 \qquad & -2\le t\le 0 \\ &= 2-2t \qquad & 0\le t\le 1 \end{aligned}$$

Let

$$x(t)=\sum_{k=-\infty}^{\infty}C_k e^{j2\pi kt/T}$$

be the Fourier expansion, where:

$$C_k=\frac{1}{T}\int_{(T)}x(t)e^{-j2\pi kt/T}dt$$

$$\Rightarrow C_0=\frac{1}{3}\int_{-2}^{1}x(t)dt=3$$

For $k\neq 0$,

$$C_k=\frac{1}{3}\left[\int_{-2}^{0}(t+2)e^{-j2\pi kt/3}dt+\int_{0}^{1}(2-2t)e^{-j2\pi kt/3}dt\right]$$

$$=\frac{1}{3}\left[\frac{3j}{\pi k}+\frac{9\left(1+e^{j4\pi k/3}\right)}{4\pi^2k^2}-\frac{3j}{2\pi k}+\frac{9}{2\pi^2k^2}\left(1-e^{-j2\pi k/3}\right)\right]$$

$$=\frac{j}{2\pi k}+\frac{9}{4\pi^2k^2}\left(3-2e^{-j2\pi k/3}+e^{j4\pi k/3}\right)$$

Substituting in x(t)

$$x(t)=\sum_{k=-\infty}^{\infty}C_k e^{j2\pi kt/T}$$

(d) It is periodic with period 2. So take segment $t \in \left[\frac{-1}{2}, \frac{3}{2}\right]$

Here,

$$x(t) = \delta(t) - 2\delta(t-1)$$

Let

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T}$$

be the Fourier expansion, where:

$$C_k = \frac{1}{T}\int_{(T)} x(t)e^{-j2\pi kt/T}dt$$

$$\Rightarrow C_0 = \frac{1}{2}\int_{-1/2}^{3/2}\left(\delta(t) - 2\delta(t-1)\right)dt$$

$$= -\frac{1}{2}$$

For $k \neq 0$

$$C_k = \frac{1}{2}\int_{-1/2}^{3/2}\left(\delta(t) - 2\delta(t-1)\right)e^{-j2\pi kt/2}dt$$

$$= \frac{1 - 2e^{-j\pi k}}{2}$$

Substituting in x(t)

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T}$$

(e) It is periodic with period 6. So, take the segment $t \in [-3, 3]$

Here $x(t)$ will be

$$
\begin{aligned}
x(t) &= 1 \qquad -2 \le t \le -1 \\
&= -1 \qquad 1 \le t \le 2 \\
&= 0 \qquad elsewhere\ in\ [-3,3]
\end{aligned}
$$

Let

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T}$$

be the Fourier expansion, where:

$$C_k = \frac{1}{T}\int_{(T)} x(t)e^{-j2\pi kt/T}dt$$

$$\Rightarrow C_0 = \frac{1}{6}\int_{-3}^{3} x(t)dt = 0$$

For $k \neq 0$

$$C_k = \frac{1}{6}\left[\int_{-2}^{-1} e^{-j2\pi kt/6}dt + \int_{1}^{2}(-1)e^{-j2\pi kt/6}dt\right]$$

$$= \frac{1}{6}\left[\frac{6\left(Cos(\pi k/3) - Cos(2\pi k/3)\right)}{-j\pi k}\right]$$

$$= \frac{j}{\pi k}\left(Cos(\pi k/3) - Cos(2\pi k/3)\right)$$

Substituting in x(t)

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T}$$

(f) It is periodic with period 3. So, take the segment $t \in [0,3]$

Here $x(t)$ will be

$$\begin{aligned} x(t) &= 2 \qquad 0 \le t \le 1 \\ &= 1 \qquad 1 \le t \le 2 \\ &= 0 \qquad 2 \le t \le 3 \end{aligned}$$

Let

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T}$$

be the Fourier expansion, where:

$$C_k = \frac{1}{T}\int_{(T)} x(t)e^{-j2\pi kt/T}dt$$

$$\Rightarrow C_0 = \frac{1}{3}\int_{(T)} x(t)dt$$

$$= \frac{1}{3} \times 3 = 1$$

For $k \neq 0$

$$C_k = \frac{1}{3}\left[\int_{0}^{1} 2e^{-j2\pi kt/3}dt + \int_{1}^{2} e^{-j2\pi kt/3}dt\right]$$

$$= \frac{j}{2\pi k}\left(e^{-j4\pi k/3} + 3e^{-j2\pi k/3} - 1\right)$$

Substituting in x(t)

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T}$$

---

(2)

Let

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T}$$

be the Fourier expansion, where:

$$C_k = \frac{1}{T}\int_{(T)} x(t) e^{-j2\pi kt/T} dt$$

Given $T = 2$.

$$\Rightarrow C_0 = \frac{1}{2}\int_{-1}^{1} e^{-t} dt = \frac{1}{2}\left(e - e^{-1}\right)$$

For $k \neq 0$

$$C_k = \frac{1}{2}\int_{-1}^{1} e^{-t} e^{-j\pi kt} dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-t-j\pi kt} dt$$

$$= \frac{e^{1+j\pi k} - e^{-1-j\pi k}}{2\left(1 + j\pi k\right)}$$

Substituting in x(t)

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T}$$

---

(3)

Let

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T}$$

be the Fourier expansion, where:

$$C_k = \frac{1}{T}\int_{(T)} x(t) e^{-j2\pi kt/T} dt$$

Given $T = 4$.

$$\Rightarrow C_0 = \frac{1}{4}\int_{0}^{2} Sin\left(\pi t\right) dt = 0$$

For $k \neq 0$

$$C_k = \frac{1}{4}\int_0^2 Sin(\pi t)e^{-j\pi kt/2}dt$$

$$= \frac{1}{8j}\int_0^2 \left(e^{j\pi t} - e^{-j\pi t}\right)e^{-j\pi kt/2}$$

$$= \frac{1}{8j}\left[\frac{e^{j\pi(2-k)}-1}{j\pi(1-k/2)} + \frac{e^{-j\pi(2+k)}-1}{j\pi(1+k/2)}\right]$$

$$= \frac{e^{-j\pi k}}{\pi(k^2-4)} = \frac{(-1)^k}{\pi\left(k^2-4\right)}$$

Substituting in x(t)

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi kt/T}$$

Problem 9

A continuous-time periodic signal $x(t)$ is real valued and has a fundamental period T=8 **.** The NON ZERO Fourier series coefficients for $x(t)$ are specified as

$a_1 = a_{-1} = j$, $a_5 = a_{-5} = 2$

Express $x(t)$ in the form

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(w_k t + \phi_k)$$

Solution 9

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$$

Given that the non zero coefficients are:

$a_1 = j$; $a_{-1} = -j$; $a_5 = 2$; $a_{-5} = 2$

=> $x(t) = 2(e^{j2\pi(5)t/8} + e^{-j2\pi(5)t/8}) + j(e^{j2\pi(1)t/8} + e^{-j2\pi(1)t/8})$

$= 4\cos(5\pi t/4 + 0) + 2\cos(\pi t/4 - \pi/2)$

$w_k = 2\pi k/8 = \pi k/4$

Comparing with

$$x(t) = \sum_{k=0}^{\infty} A_k \cos(w_k t + \phi_k)$$

We get the non-zero coefficients as:

$A_1 = 2$; $\phi_1 = -\pi / 2$

$A_5 = 4$; $\phi_5 = 0$

Problem 10

In this problem, we provide examples of the effects of nonlinear changes in phase.

(a) Consider the continuous time LTI system with frequency response

$$H(j\omega) = \frac{a - j\omega}{a + j\omega} \qquad \text{where } a>0.$$

what is the magnitude of $H(j\omega)$? What is $\sphericalangle H(j\omega)$? What is the impulse response of this system ?

(b) Determine the output of the system of part (a) with a=1 when the input is

$$\mathrm{Cos}\frac{t}{\sqrt{3}} + \mathrm{Cos}\, t + \mathrm{Cos}\sqrt{3}t$$

Roughly sketch both the input and the output.

Solution 10 :

(a) $H(j\omega) = \frac{a-j\omega}{a+j\omega}$

$$|H(j\omega)| = \left|\frac{a-j\omega}{a+j\omega}\right|$$

$$= \frac{|a-j\omega|}{|a+j\omega|}$$

$$= \frac{\sqrt{a^2+\omega^2}}{\sqrt{a^2+\omega^2}}$$

$$=1$$

As a is real and > 0 => $\mathrm{Arg}(a-j\omega) = -\tan^{-1}(\omega / a)$ (similarly $\mathrm{Arg}(a+j\omega)= \tan^{-1}(\omega / a)$)

=> $\sphericalangle H(j\omega) = -2\tan^{-1}(\omega / a)$

(b) $H(j\omega) = \frac{a-j\omega}{a+j\omega}$

$$x(t) = \cos(\frac{t}{\sqrt{3}}) + \cos t + \cos \sqrt{3}\, t$$

Taking Fourier Transform of x(t)

$$X(j\omega) = \frac{1}{2}\left\{\delta(\omega - \frac{1}{\sqrt{3}}) + \delta(\omega + \frac{1}{\sqrt{3}}) + \delta(\omega+1) + \delta(\omega-1) + \delta(\omega - \sqrt{3}) + \delta(\omega + \sqrt{3})\right\}$$

Output response will be the convolution of x(t) with h(t).

$$Y(j\omega) = X(j\omega).\ H(j\omega)$$

$$Y(j\omega) = \frac{1}{2}\left\{\delta(\omega - \frac{1}{\sqrt{3}}) + \delta(\omega + \frac{1}{\sqrt{3}}) + \delta(\omega+1) + \delta(\omega-1) + \delta(\omega - \sqrt{3}) + \delta(\omega + \sqrt{3})\right\}(\frac{a-j\omega}{a+j\omega})$$

$$Y(j\omega) = \frac{1}{2}\left\{\delta(\omega - \frac{1}{\sqrt{3}})\frac{\sqrt{3}a-j}{\sqrt{3}a+j} + \delta(\omega + \frac{1}{\sqrt{3}})\frac{\sqrt{3}a+j}{\sqrt{3}a-j} + \delta(\omega+1)\frac{a+j}{a-j} + \delta(\omega-1)\frac{a-j}{a+j} + \delta(\omega - \sqrt{3})\frac{a-\sqrt{3}j}{a+\sqrt{3}j} + \delta(\omega + \sqrt{3})\frac{a+j\sqrt{3}}{a-j\sqrt{3}}\right\}$$

Taking the Fourier transform Inverse of $Y(j\omega)$ we get y(t)

$$y(t) = \frac{1}{2}\left\{ e^{\frac{jt}{\sqrt{3}}}\frac{\sqrt{3}a-j}{\sqrt{3}a+j} + e^{-\frac{jt}{\sqrt{3}}}\frac{\sqrt{3}a+j}{\sqrt{3}a-j} + e^{-jt}\frac{a+j}{a-j} + e^{jt}\frac{a-j}{a+j} + e^{j\sqrt{3}t}\frac{a-\sqrt{3}j}{a+\sqrt{3}j} + e^{-j\sqrt{3}t}\frac{a+j\sqrt{3}}{a-j\sqrt{3}} \right\}$$

$$y(t) = \frac{1}{2}\left\{ e^{\frac{jt}{\sqrt{3}}} e^{-j2\tan^{-1}(\frac{1}{a\sqrt{3}})} + e^{-\frac{jt}{\sqrt{3}}} e^{j2\tan^{-1}(\frac{1}{a\sqrt{3}})} + e^{-jt}e^{j2\tan^{-1}(\frac{1}{a})} + e^{jt}e^{-j2\tan^{-1}(\frac{1}{a})} + e^{j\sqrt{3}t}e^{-j2\tan^{-1}(\frac{\sqrt{3}}{a})} + e^{-j\sqrt{3}t} e^{j2\tan^{-1}(\frac{\sqrt{3}}{a})} \right\}$$

$$y(t) = \frac{1}{2}\left\{ \cos(\frac{t}{\sqrt{3}} - 2\tan^{-1}(\frac{1}{a\sqrt{3}})) + \cos(t - 2\tan^{-1}(\frac{1}{a})) + \cos(\sqrt{3}\,t - 2\tan^{-1}(\frac{\sqrt{3}}{a})) \right\}$$

The input $x(t)$ and the output $y(t)$ are shown below (for $a=1$) : -

Problem 11 :

Consider an LTI system whose response to the input

$$x(t) = (e^{-t} + e^{-3t})u(t)$$

is

$$y(t) = (2\,e^{-t} - 2\,e^{-4t})u(t)$$

(a) Find the frequency response of this system.

(b) Determine the system's impulse response.

Solution 11 :

(a) Given :

$$x(t) = [e^{-t} + e^{-3t}]u(t)$$

$$y(t) = [\,2e^{-t} - 2e^{-4t}\,]u(t)$$

The Fourier of $e^{-at}u(t)$ [where $a>0$] is $\frac{1}{a+j\omega}$

So Fourier tranform of $x(t)$ is

$$X(j\omega) = \frac{1}{1+j\omega} + \frac{1}{3+j\omega}$$

Like this the fourier transform of output $y(t)$ is

$$Y(j\omega) = \frac{2}{1+j\omega} - \frac{2}{4+j\omega}$$

Let the frequency response of the given LSI system is $H(j\omega)$

So by the Convolution Theorem :-

as $y(t) = x(t) * h(t)$

$\Rightarrow Y(j\omega) = X(j\omega)\,H(j\omega)$

$$\Rightarrow \frac{2}{1+j\omega} - \frac{2}{4+j\omega} = \left(\frac{1}{1+j\omega} + \frac{1}{3+j\omega}\right)H(j\omega)$$

By solving this we get:-

$$H(j\omega) = 3\left[\frac{3+j\omega}{(4+j\omega)(2+j\omega)}\right]$$

$$H(j\omega) = \frac{3}{2(4+j\omega)} + \frac{3}{2(2+j\omega)}$$

(b) By taking inverse fourier transform of $H(j\omega)$, we get:-

$$h(t) = \frac{3}{2}(e^{-4t} + e^{-2t})u(t)$$

Problem 12 :

Consider the signal x(t) in **figure below :**

(a) Find the Fourier transform $X(j\omega)$ of x(t).

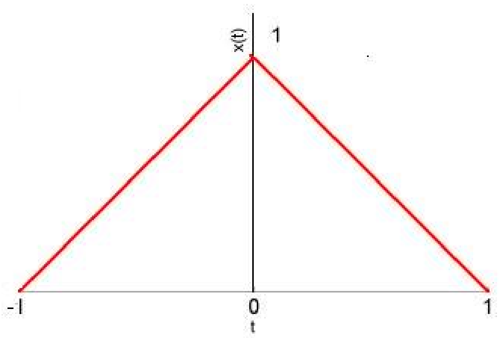
(b) Sketch the signal

$$\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t-4k)$$

(c) Find another signal g(t) such that g(t) is not the same as x(t) and

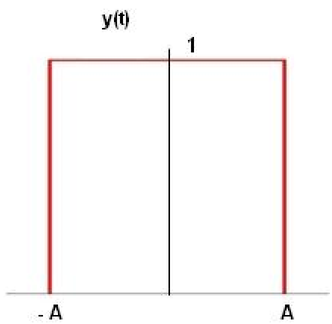
$$\tilde{x}(t) = g(t) * \sum_{k=-\infty}^{\infty} \delta(t-4k)$$

(d) Argue that, although $G(j\omega)$ is different from $X(j\omega)$, $G(j\frac{\pi k}{2})=X(j\frac{\pi k}{2})$ for all integers k. You should not explicitly evaluate $G(j\omega)$ to answer this question.



Solution 12 :

(a) Consider the signal y (t) shown below :-

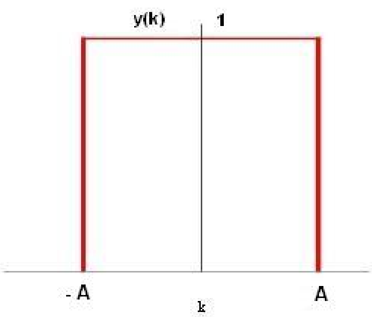


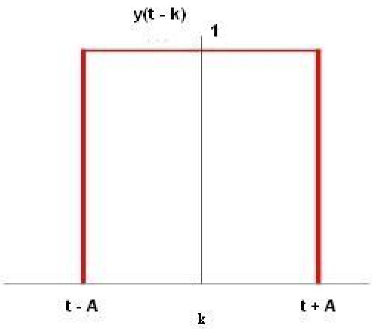
Now consider the convolution of y (t) with itself . Let the resultant figure be z (t)

$$z(t) = y(t) * x(t)$$

$$= \int_{-\infty}^{\infty} y(k)y(t-k)dk$$

Now consider y (k) and y (t-k) :





So when $t + A < -A \Rightarrow t < -2A$,

$z(t) = 0$ as there is no overlap between y(k) and y(t-k).

When $-A \leq t + A < A \Rightarrow -2A \leq t < 0$ then

$$z(t) = \int_{-A}^{t+A} y(k)\, y(t-k)\, dk$$

here $y(k) = 1$ and $y(t-k) = 1$, so

$$z(t) = \int_{-A}^{t+A} 1\, dk$$

$$= [k]_{-A}^{t+A} = t + 2A$$

When $-A \leq t - A \leq A \Rightarrow 0 \leq t \leq 2A$, then

$$z(t) = \int_{t-A}^{A} y(k)\, y(t-k)\, dk$$

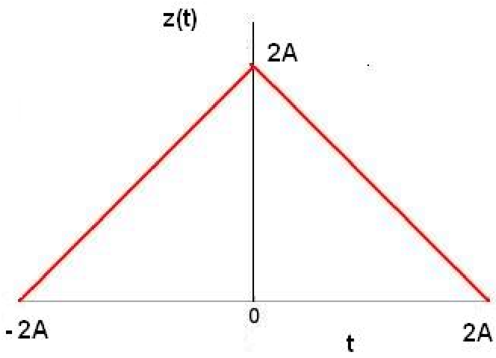
here $y(k) = 1$ and $y(t-k) = 1$

$$\Rightarrow z(t) = \int_{t-A}^{A} 1\, dk = [k]_{t-A}^{A} = 2A - t.$$

When $t - A > A \Rightarrow t > 2A$, then there is no overlap between y(k) and y(t-k). So,

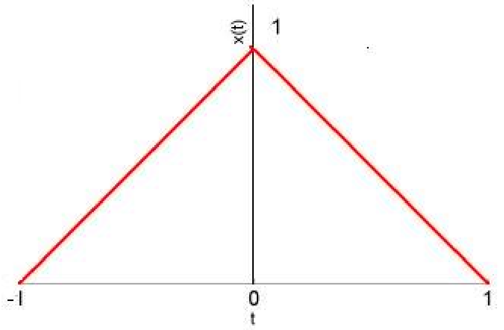
$z(t) = 0$.

So the signal z(t) is:-



Now compare z (t) with the given signal x (t) .

Here x (t) is :



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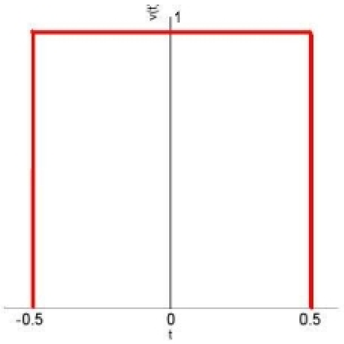
By comparing, we get

A= 1/2

So x(t) can be thought of a signal , which is the convolution of signal v(t) with itself where v(t) is :



$$\Rightarrow x(t) = v(t) * v(t)$$

By Convolution Theorem, the fourier transform of x(t) is square of the fourier transform of v(t), i.e.

$$X(j\omega) = [V(j\omega)]^2$$

Now consider v(t)

So its Fourier transform $V(j\omega)$ is :-

$$V(j\omega) = \int_{-\infty}^{\infty} v(t)\, e^{-j\omega t} dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t}\, dt$$

$$= 2\,\frac{\mathrm{Sin}\left(\frac{\omega}{2}\right)}{\omega}$$

So $$X(j\omega) = [V(j\omega)]^2$$

$$X(j\omega) = 4\,\frac{\mathrm{Sin}^2\left(\frac{\omega}{2}\right)}{\omega^2}$$

(b) $$\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t-4k)$$

Now consider a signal $y_k(t)$ where

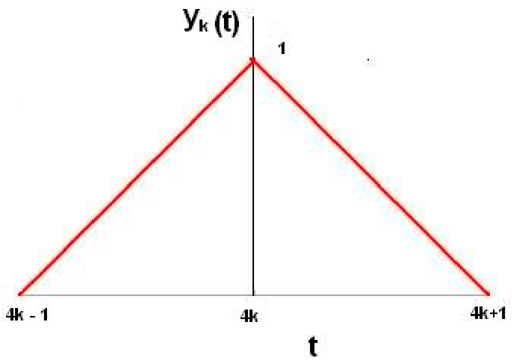
$$y_k(t) = x(t) * \delta(t-4k)$$

$$y_k(t) = \int_{-\infty}^{\infty} x(\lambda)\, \delta(t - \lambda - 4k)\, d\lambda$$

$$= \int_{-\infty}^{\infty} x(\lambda)\, \delta((t-4k) - \lambda)\, d\lambda$$

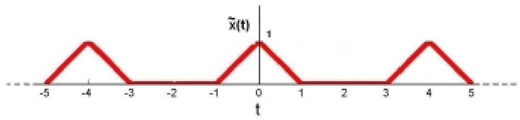
$$y_k(t) = x(t-4k) \qquad \text{as } \int_{-\infty}^{\infty} x(\lambda)\, \delta(t-\lambda)\, d\lambda = x(t)$$

So $y_k(t)$ is the shifted version of x(t) by 4k along the t axis which is shown as:-



Now $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} y_k(t)$

So $\tilde{x}(t)$ is :-



(c) Any signal which is the shifted version of x(t) by 4k on t axis can be taken as g(t) which satisfies

$$\tilde{x}(t) = g(t) * \sum_{k=-\infty}^{\infty} \delta(t - 4k)$$

because by the last part we can say that when we convolve g(t) with $\sum_{k=-\infty}^{\infty} \delta(t - 4k)$, it will result in $\tilde{x}(t)$. So,

in this case :-

$$g(t) = x(t - 4k) \quad \text{where } k \in Z$$

(d) By part (c)

$$g(t) = x(t - 4n) \quad \text{where } n \in Z$$

$$\Rightarrow G(j\omega) = X(j\omega)\, e^{-j\omega 4n}$$

Now put $\omega = \frac{\pi k}{2}$

$$\Rightarrow G(j\frac{\pi k}{2}) = X(j\frac{\pi k}{2})\, e^{-j\frac{\pi k}{2}4n}$$

$$= X(j\frac{\pi k}{2})\, e^{-j2m\pi} \quad m \in Z \text{ and } m = k.n$$

$$G(j\frac{\pi k}{2}) = X(j\frac{\pi k}{2}) \quad \text{as } e^{-j2m\pi} = 1$$