Lecture 50 – Wavelet Applications

Dr. Aditya Abhyankar

Wavelet Transform

- Decomposes signal into two separate series

- Single series to represent most coarse version
- Double series to represent refined version

- Scaling Function
- Wavelet Function

Framework

- Gave us power to move up or down the ladder
- We can now indeed zoom-in or zoom-out of any part of the signal
- This makes the entire analysis 'scalable'!!
- Scalability stems out of multi-resolution framework !

Framework

$$\phi(t) = \sqrt{2}\sum_{k} h_k \phi(2t - k)$$

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$$\phi(1t) = \phi(2^0 t) \in V_0$$

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$$V_0 \subset V_1$$

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$$V_1 = V_0 \oplus W_0$$

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$$V_{-1} \subset V_0$$

$$\phi\left(\frac{t}{2}\right) = \sqrt{2}\sum_k h_k \phi(t-k)$$

$$\phi\left(\frac{t}{2}\right) = \phi(2^{-1}t) \in V_{-1}$$

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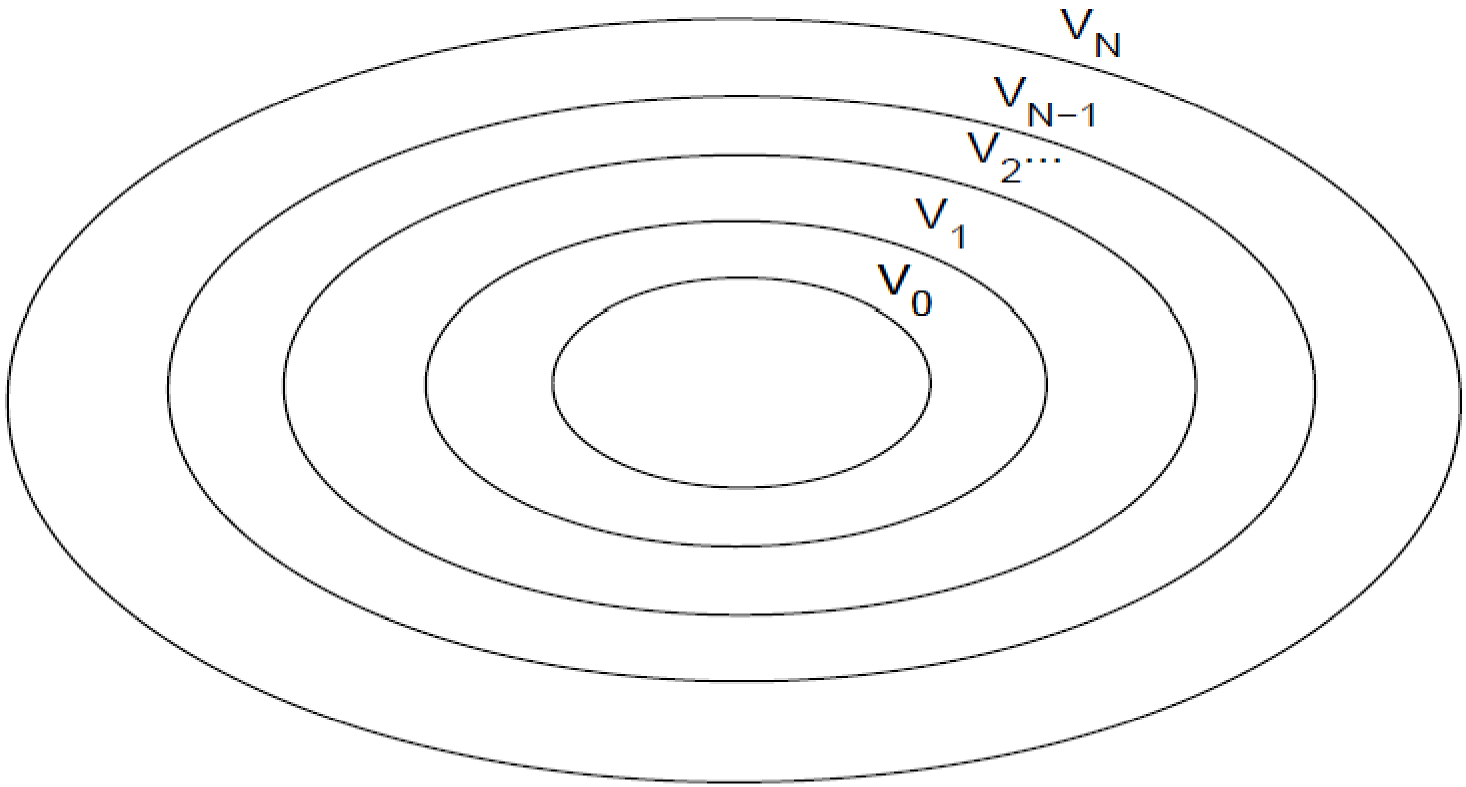

$$\cdots\cdots V_{-3} \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 \cdots\cdots$$

Framework

$$f_j(x) \in V_j, \ \text{Scale } \frac{1}{2^j}$$

$$\{2^{j/2}\phi(2^j x - k)\}_k$$

$$f_j(x) = \sum_k \alpha_{j,k} 2^{j/2}\phi(2^j x - k)$$

$$\alpha_{j,k} = \int_{-\infty}^{\infty} f_j(x) 2^{j/2}\phi(2^j x - k)dx$$

Framework

$$f_j(x) \in V_j, \ \text{Scale} \ \frac{1}{2^j}$$

Normalization

$$\{2^{j/2}\phi(2^j x - k)\}_k$$

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Normalization

Orthonormal basis

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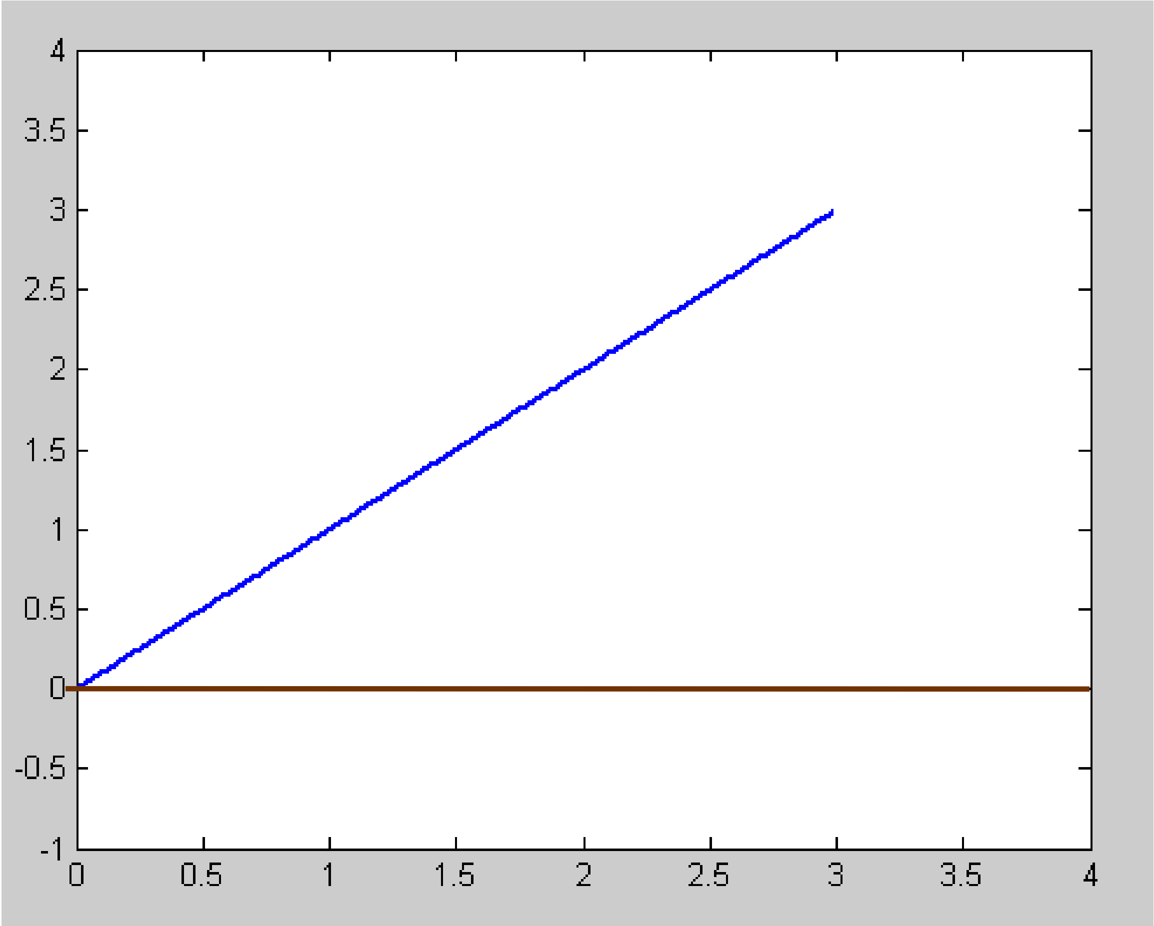
Test Signal

$$f(x) = \begin{cases} x, 0 \le x \le 3 \\ 0, \text{ otherwise} \end{cases}$$



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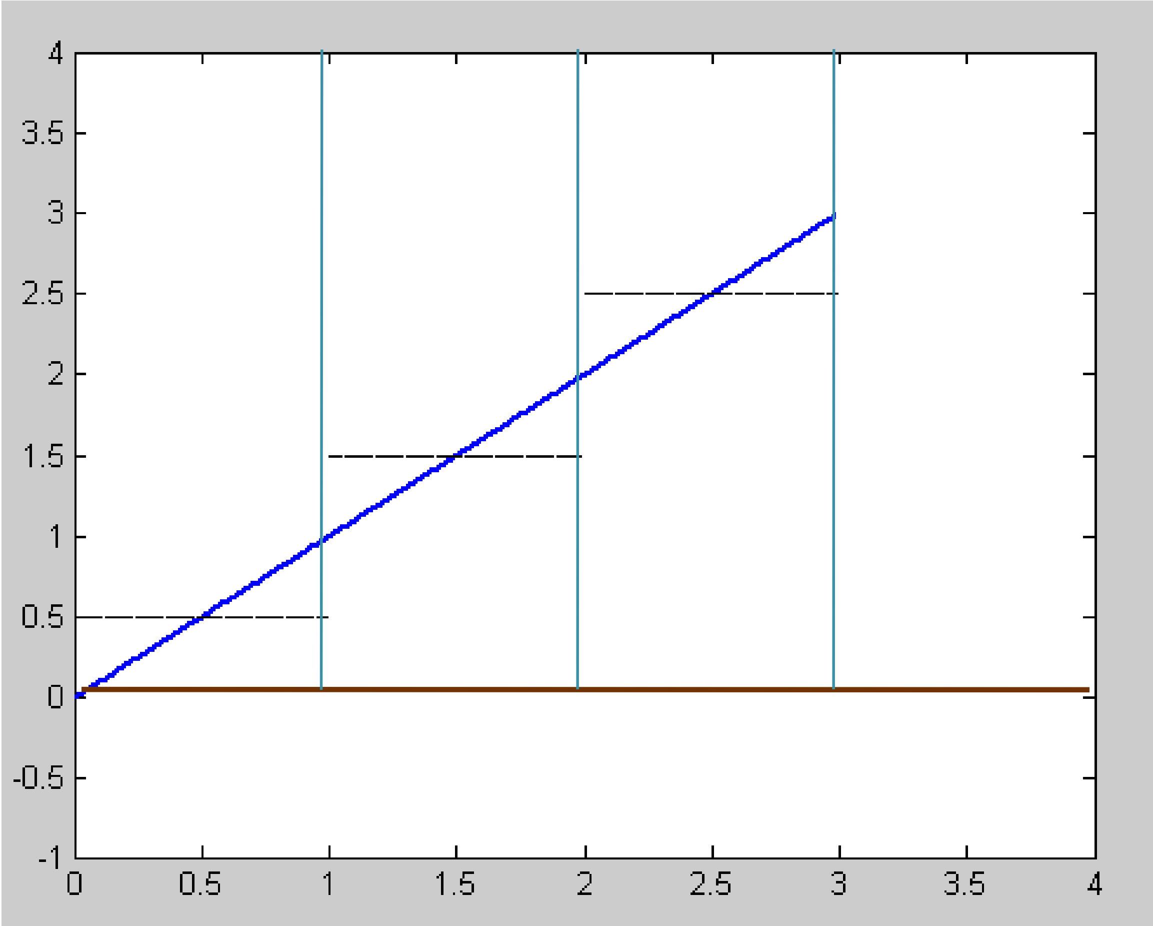

$f_0(x) \in V_0$

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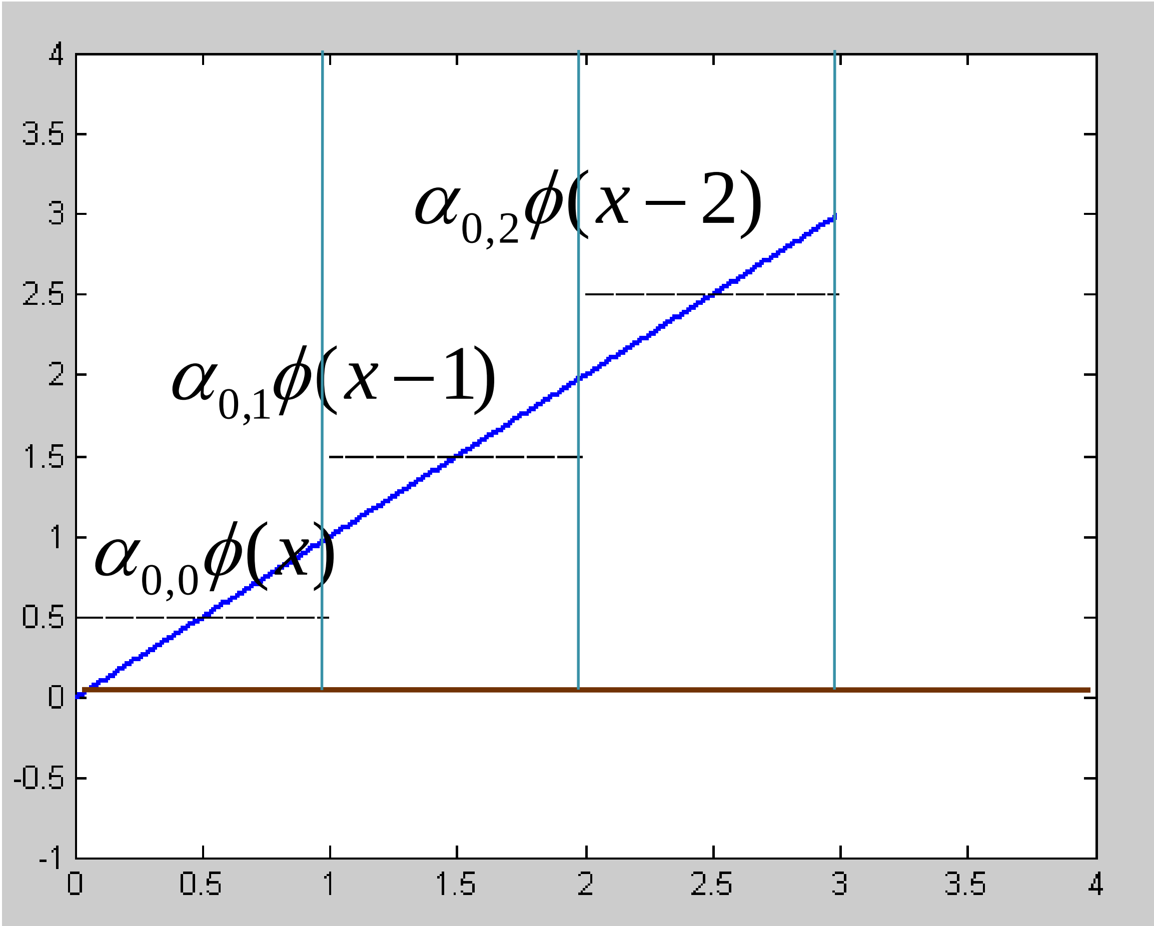

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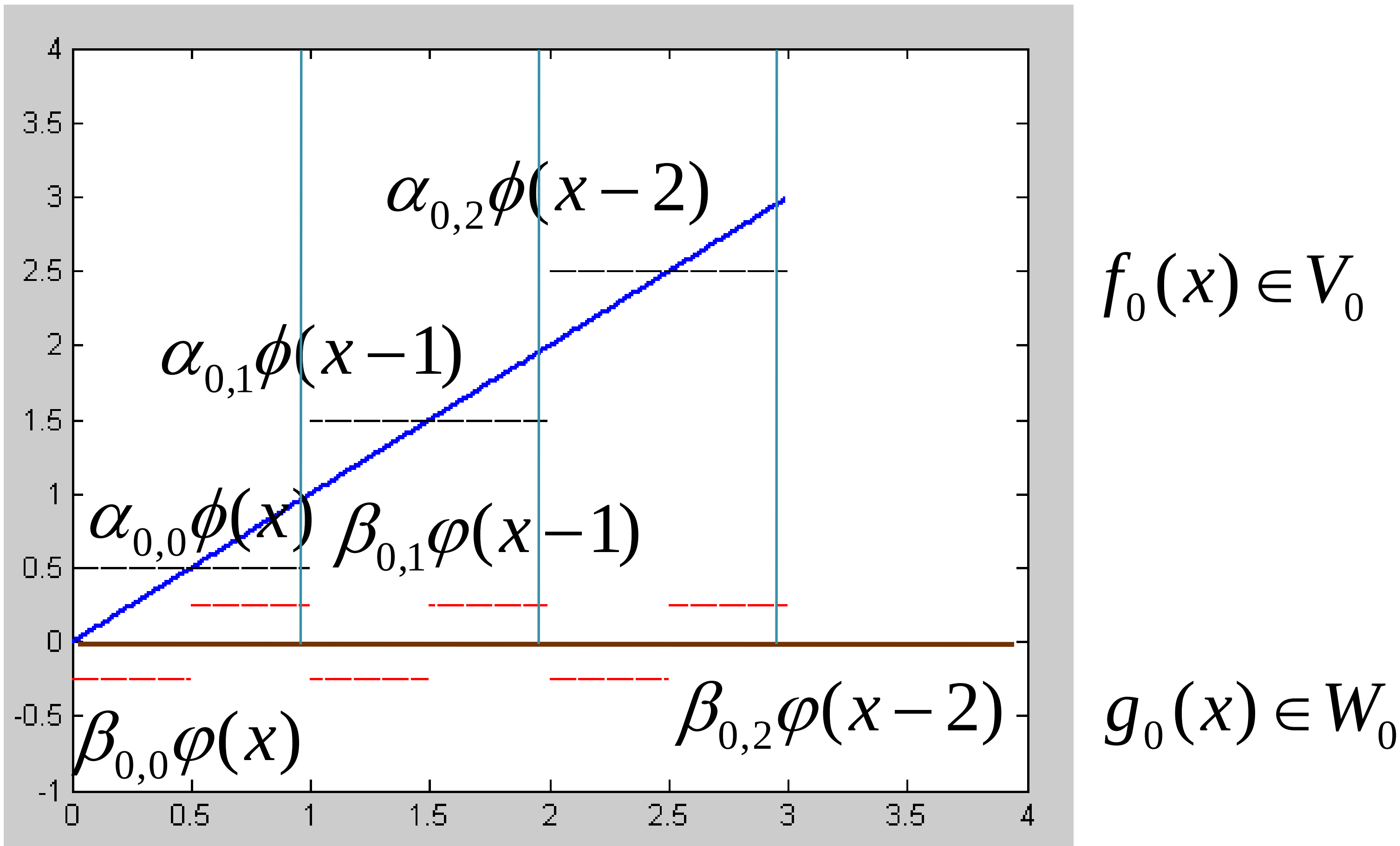

Test Signal

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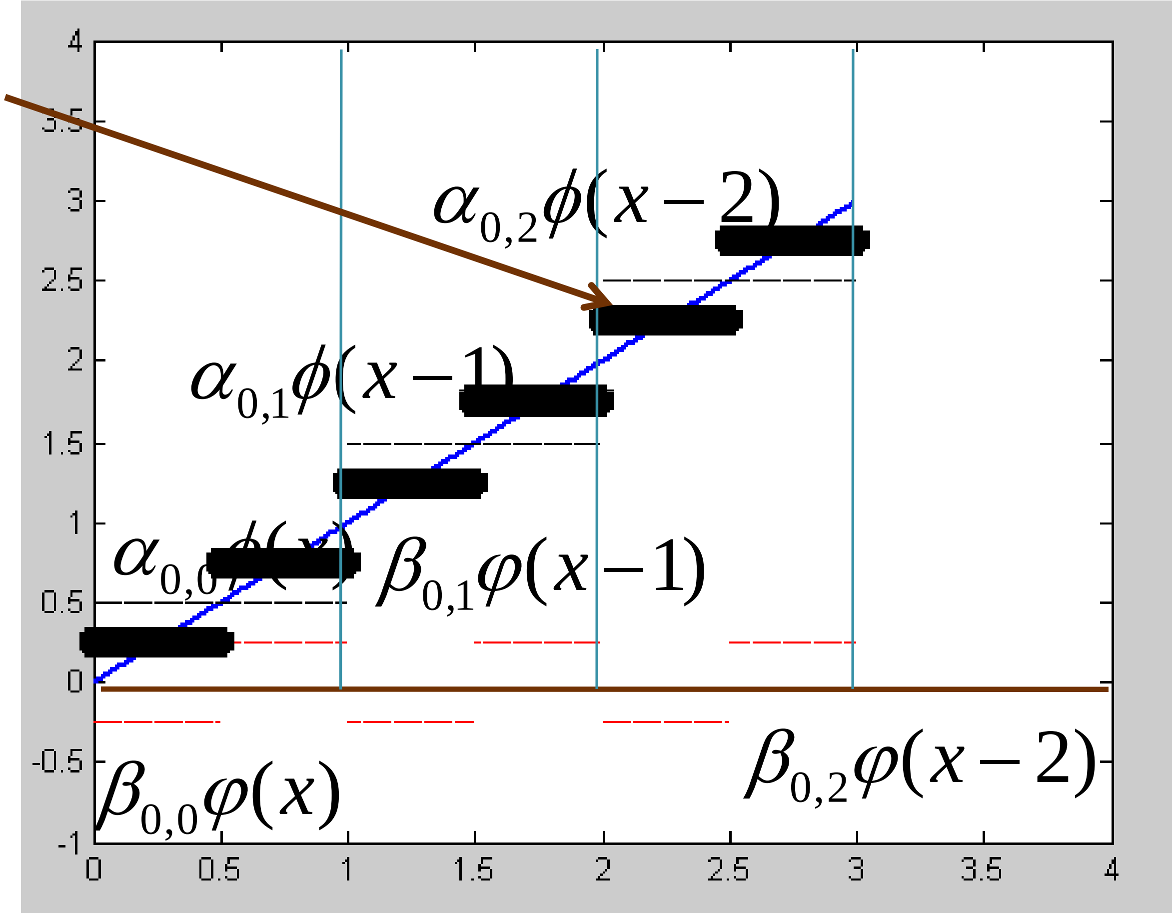
$f_1(x) \in V_1$



$f_0(x) \in V_0$

$\oplus$

$g_0(x) \in W_0$

Test Signal

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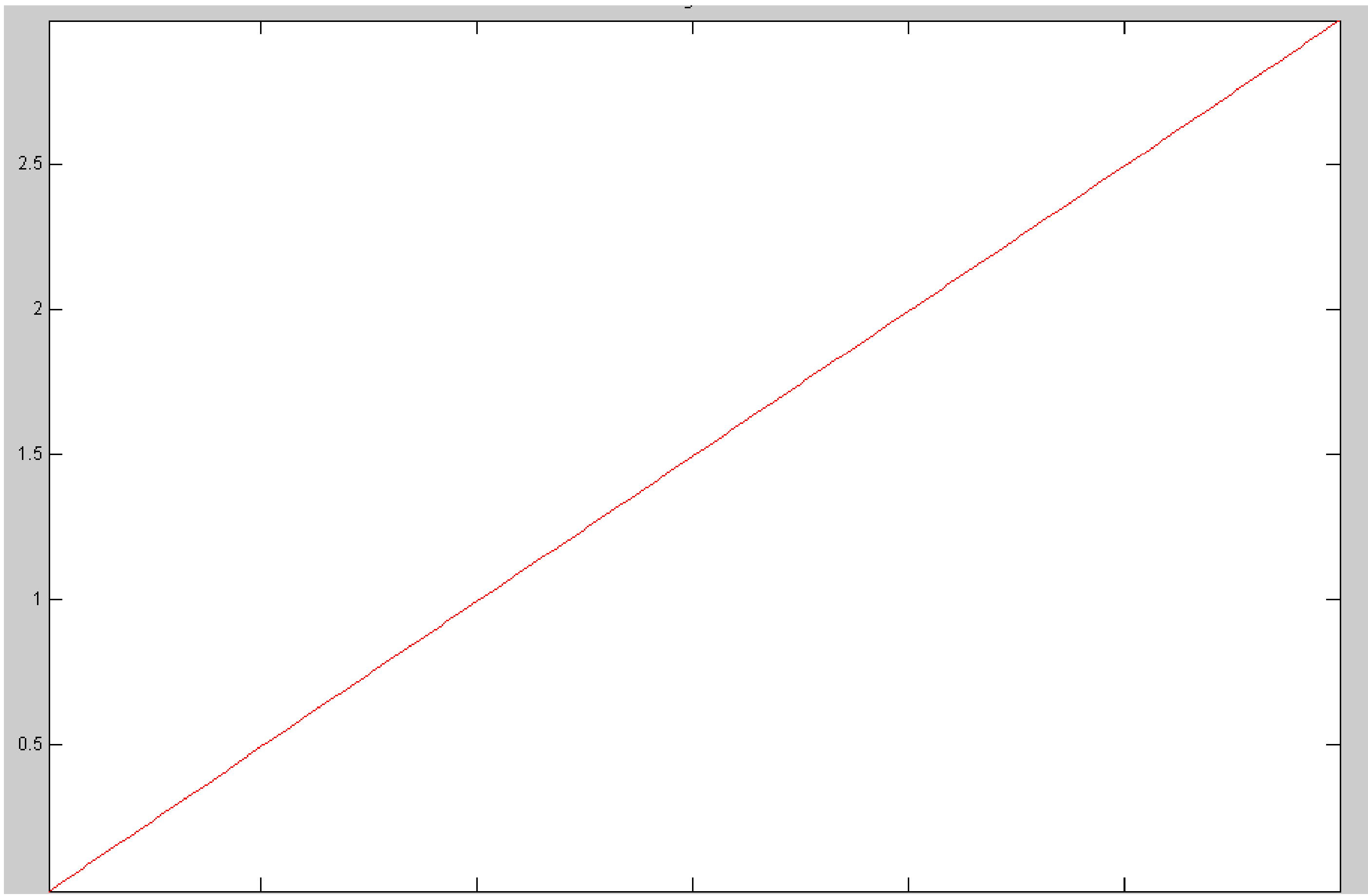

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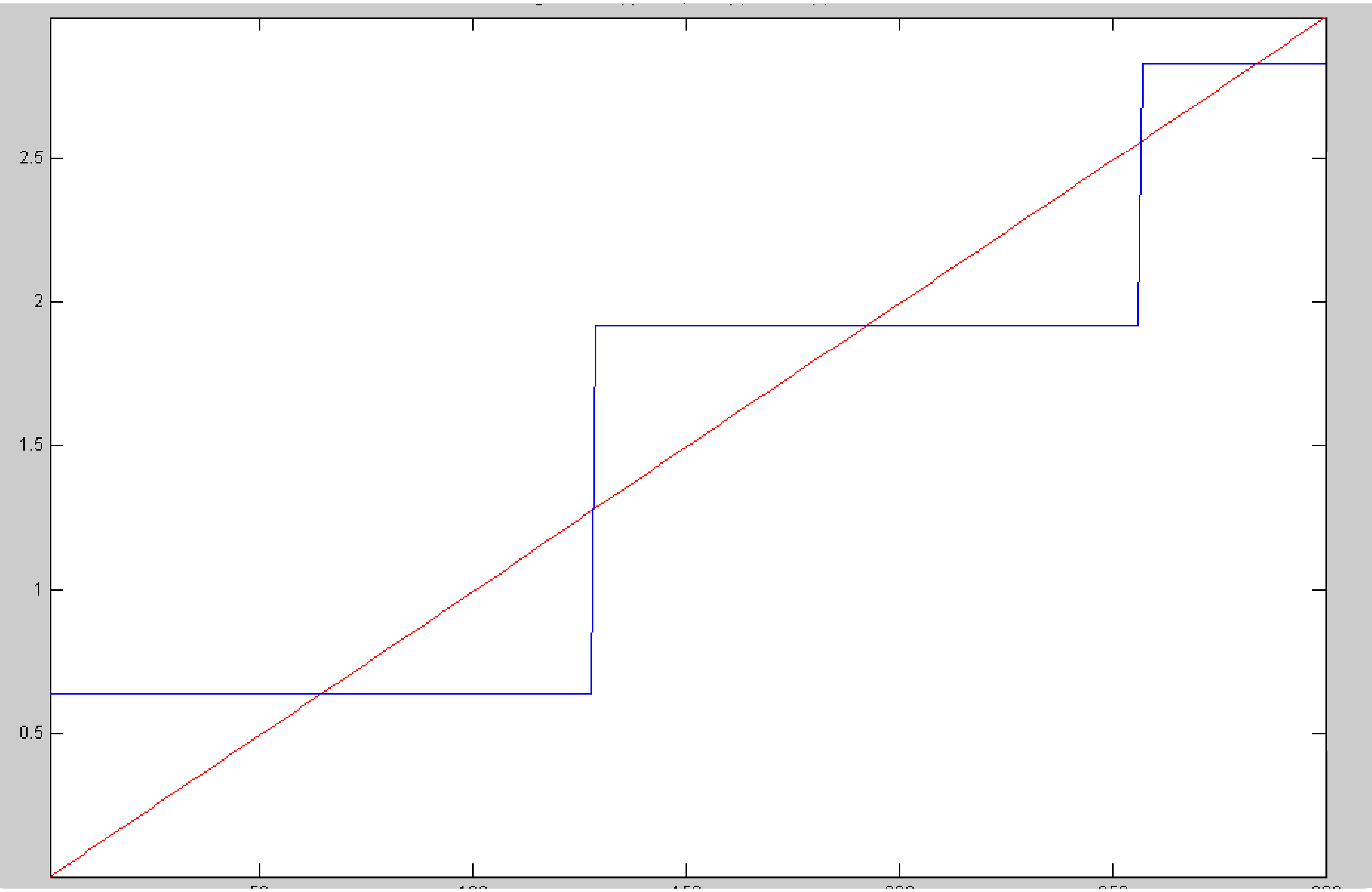

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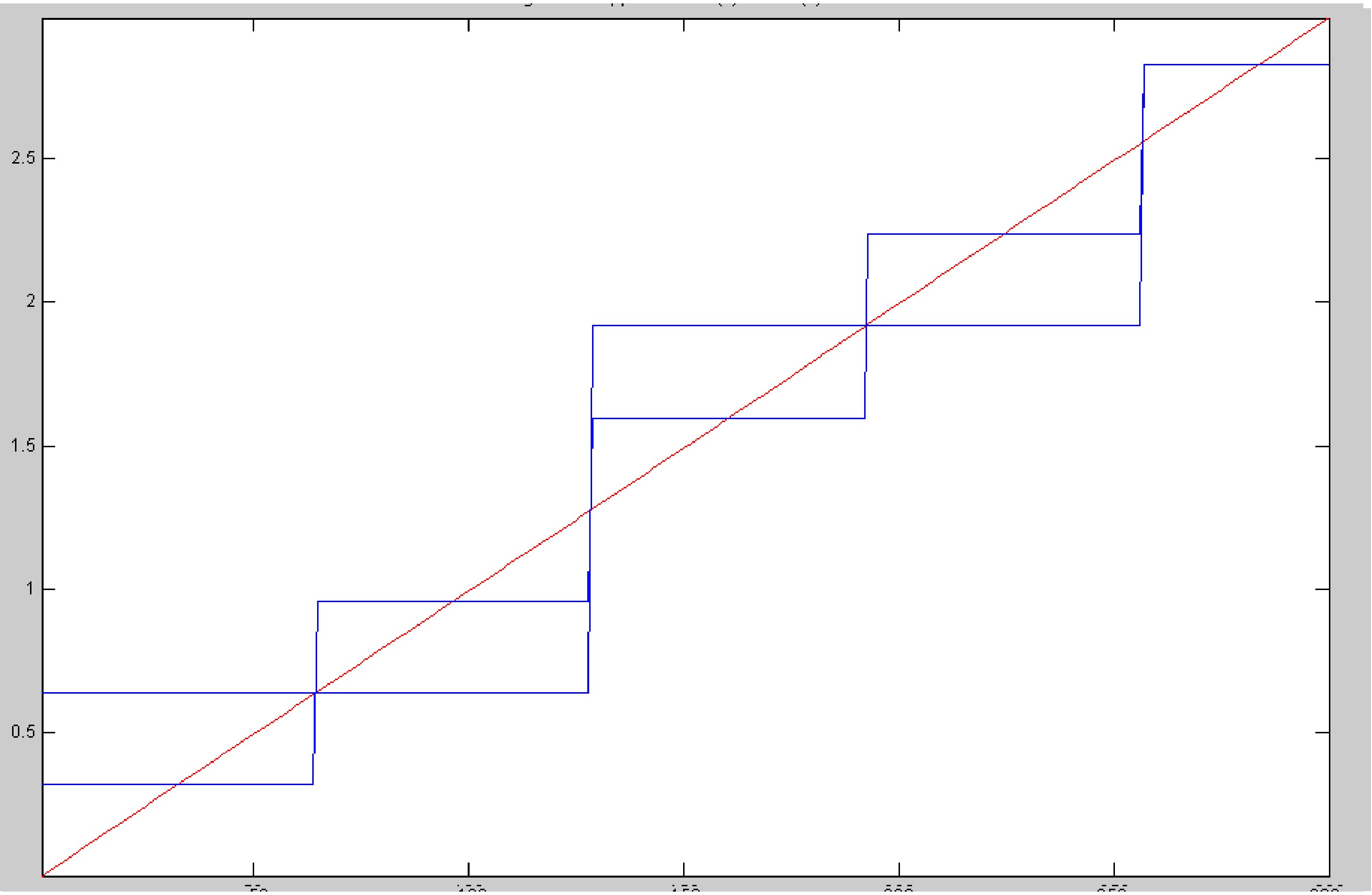

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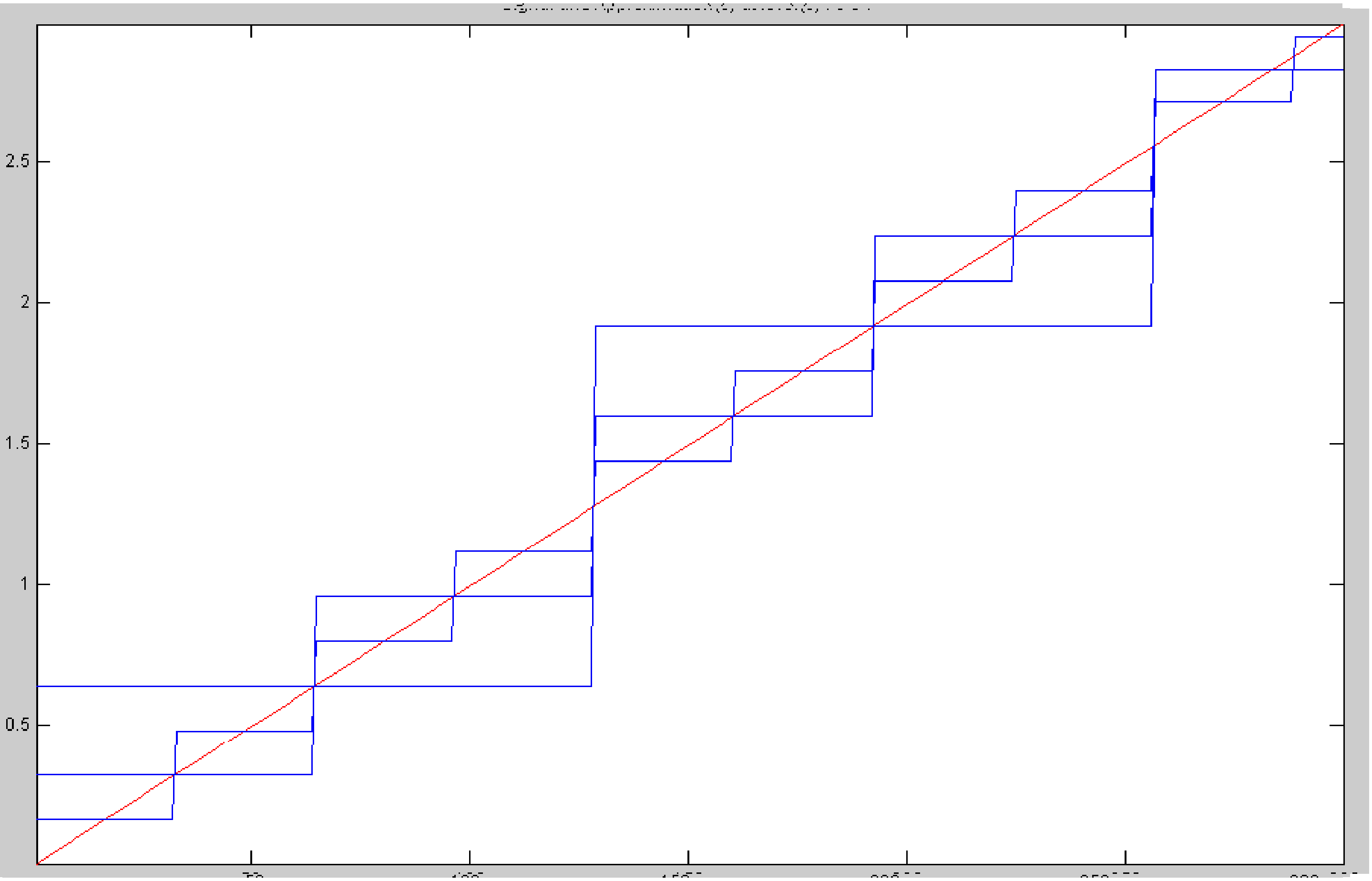

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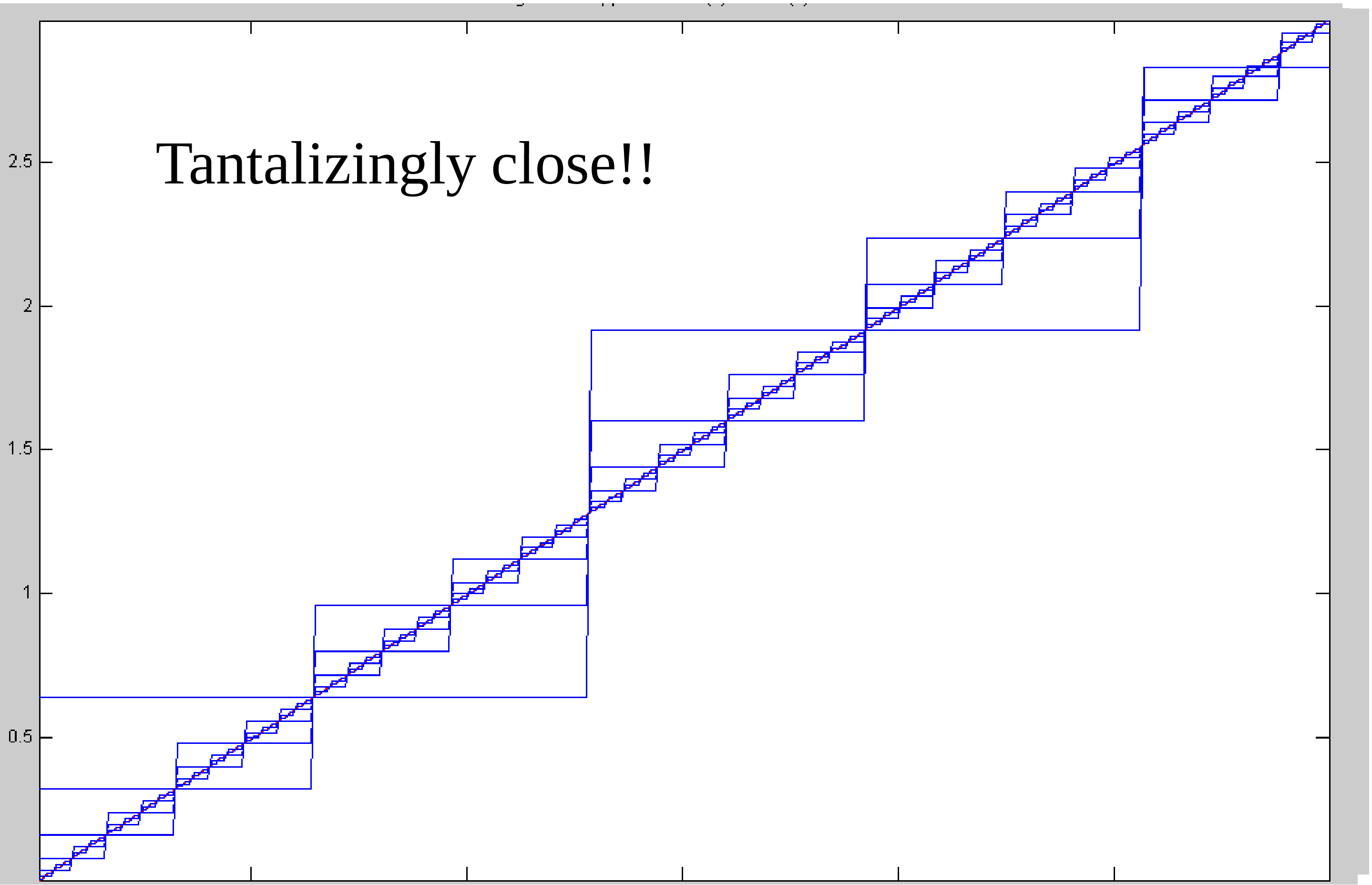

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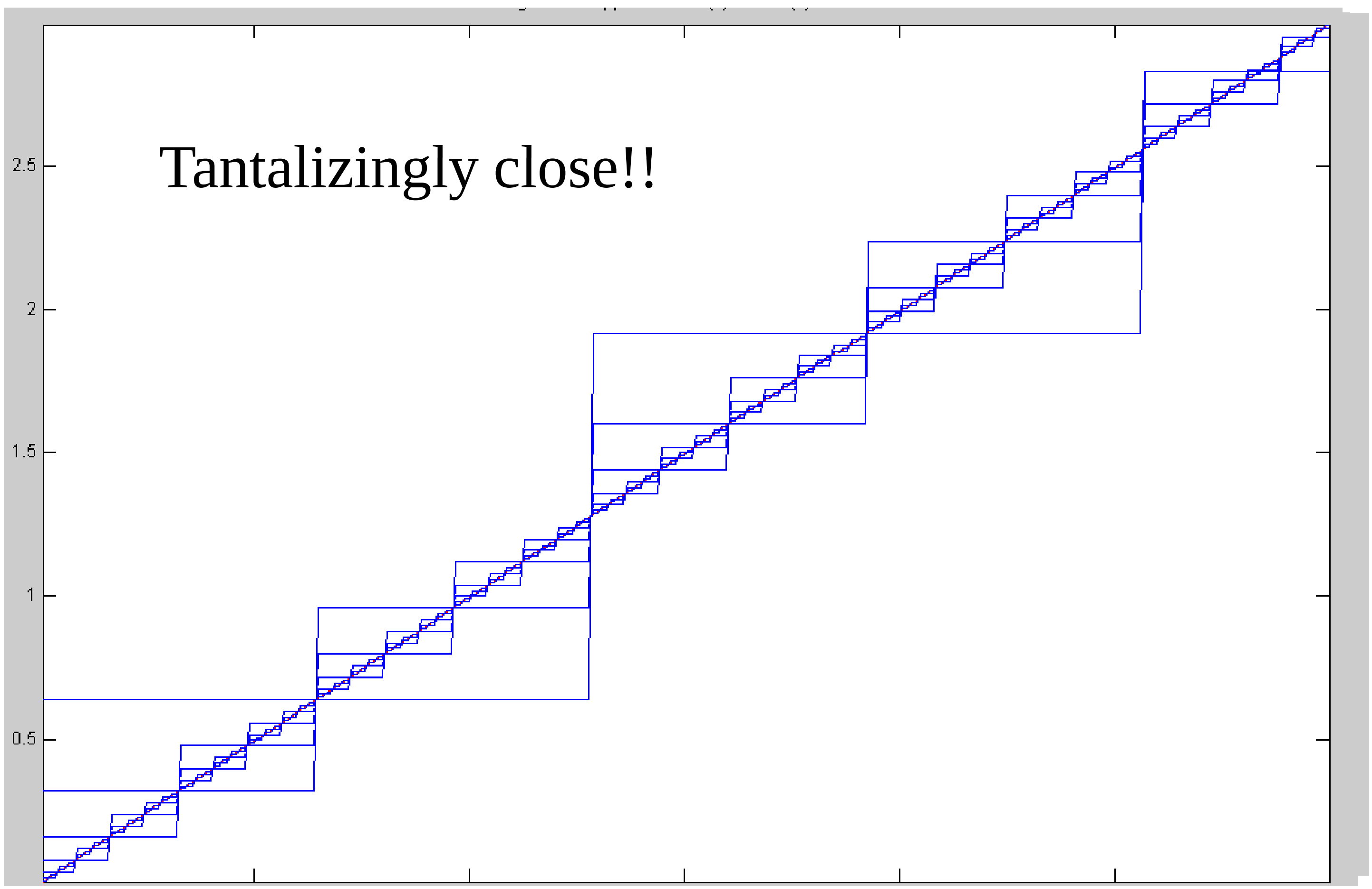

Framework

- Leads us to two questions

1) How do we go about selecting the mother wavelet and scale of analysis?
2) What is the procedure to calculate scaling and wavelet coefficients?

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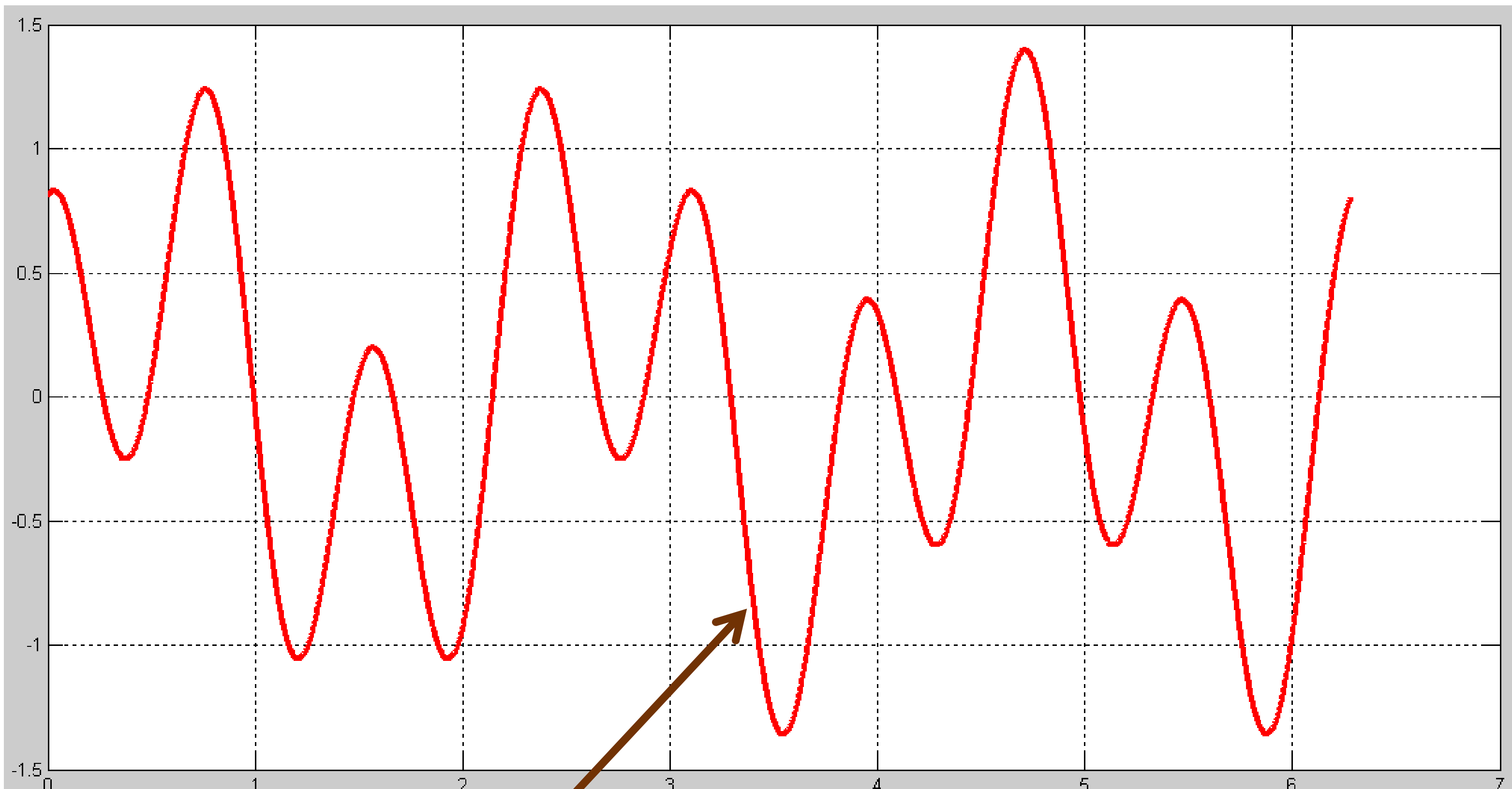
Vanishing moments

Correlation

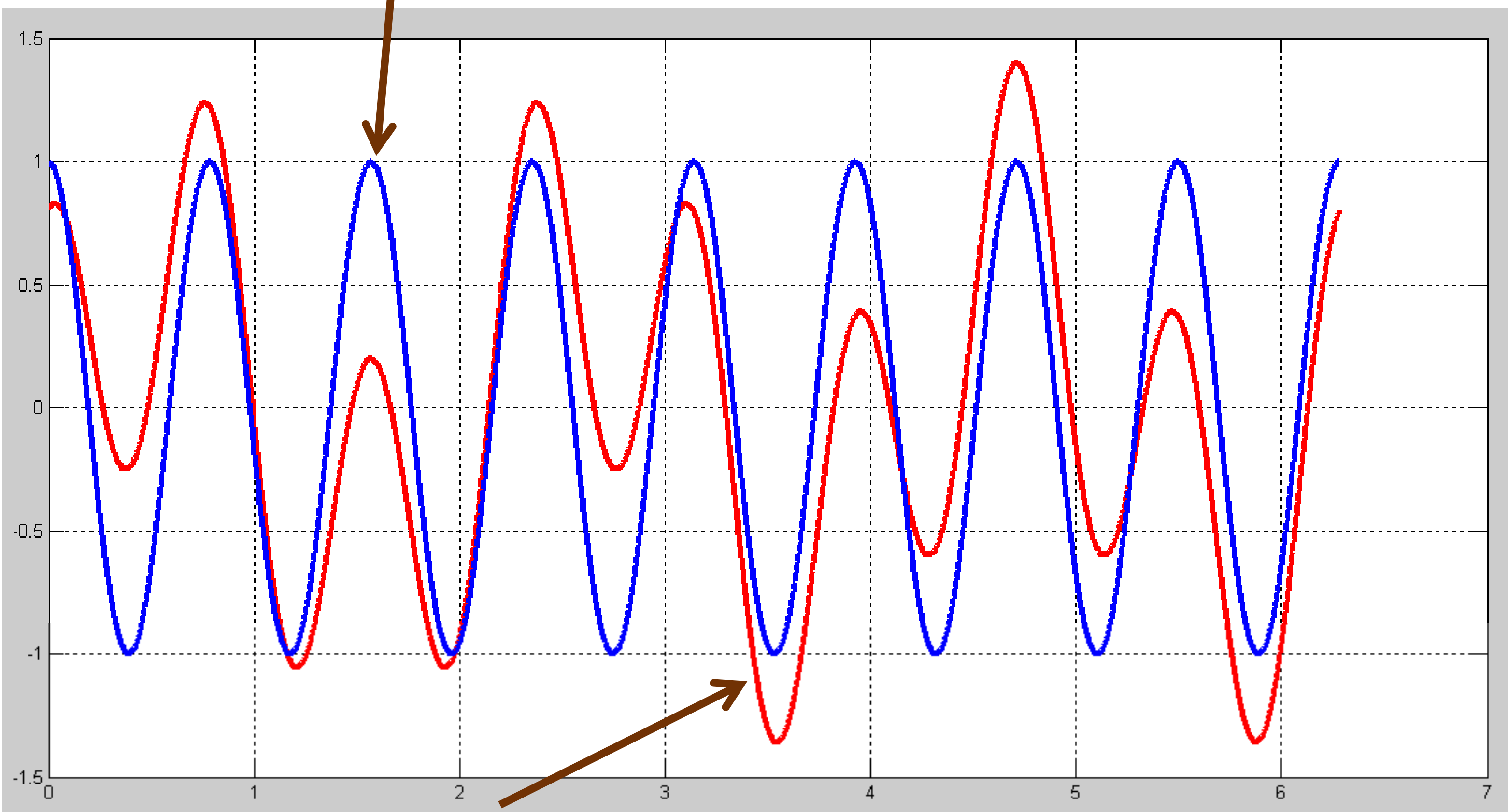
How Fourier Works!!



$$y[n] = 0.6\sin(2\pi 3n) + 0.8\cos(2\pi 8n)$$

How Fourier Works!!

$\cos(2\pi 8n)$



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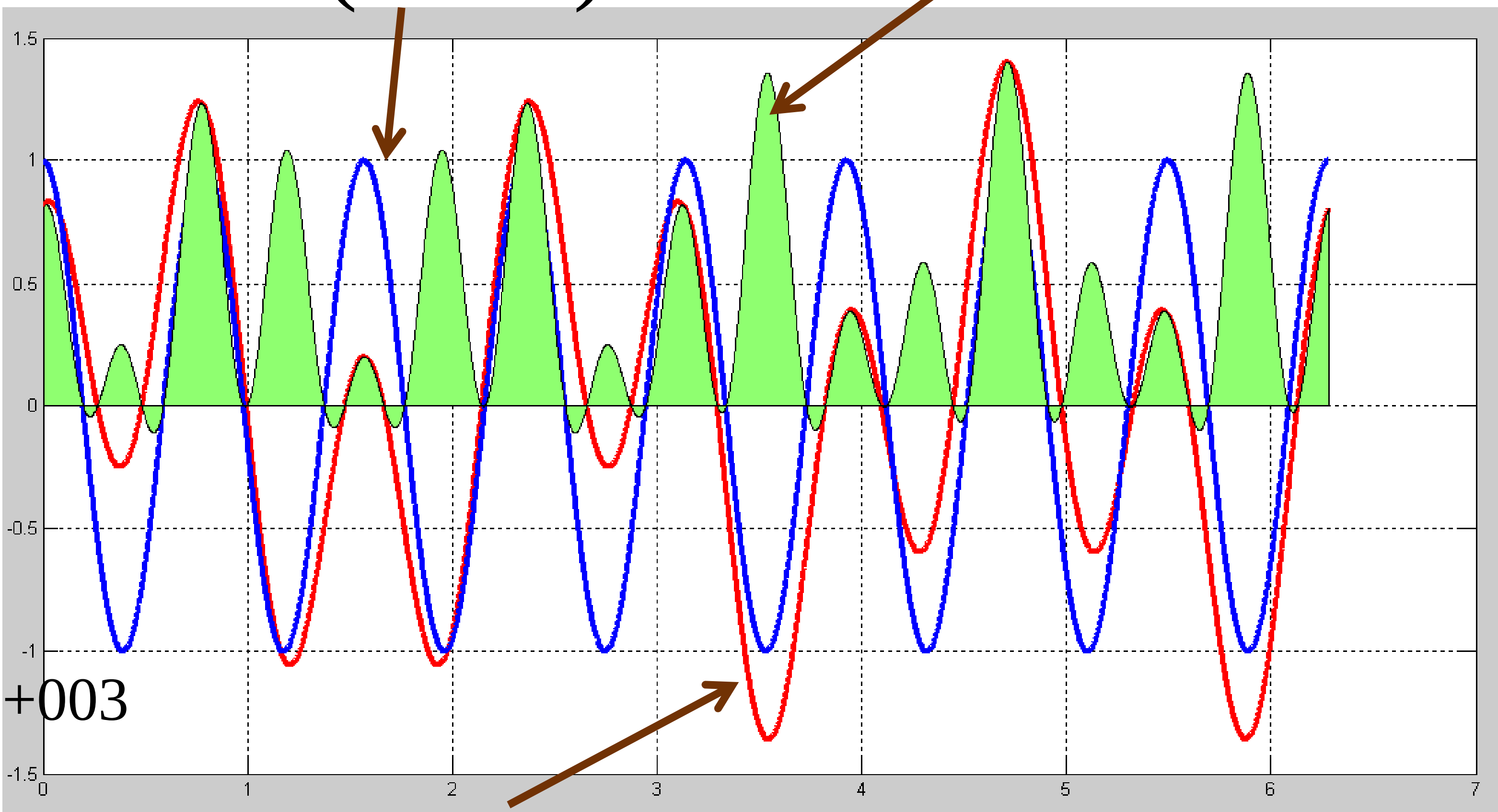
$$\cos(2\pi 8n) \qquad < y[n], \cos(2\pi 8n) >$$



2.0004e+003

$$y[n] = 0.6\sin(2\pi 3n) + 0.8\cos(2\pi 8n)$$

Moments

- The moment of order *m*, of function f(x) on (a,b) can be given as

$$M_m = \int_a^b x^m f(x)dx$$

Application

- Detecting hidden jump discontinuity
- Consider function

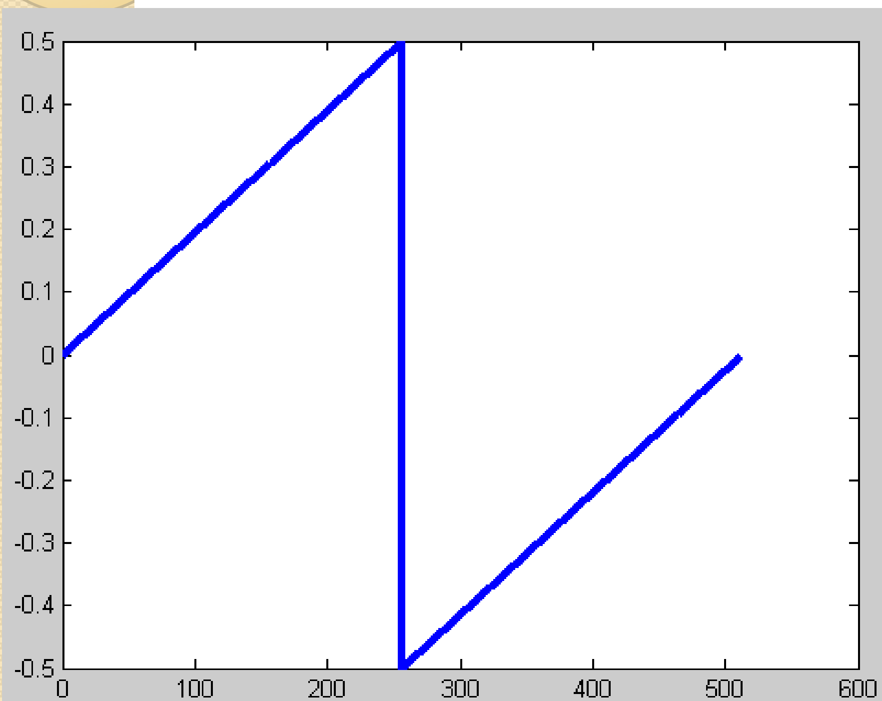
$$g(t) = \begin{cases} t, 0 \le t < \dfrac{1}{2} \\ t-1, \dfrac{1}{2} \le t < 1 \end{cases}$$

- Clear jump at *t=0.5*

Application

- Detecting hidden jump discontinuity

n



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- Detecting hidden jump discontinuity
- Let's integrate

$$h(t)=\int g(t)dt=\begin{cases}\dfrac{t^2}{2}, 0\le t<\dfrac{1}{2}\\ \dfrac{t^2}{2}-t+\dfrac{1}{2}, \dfrac{1}{2}\le t<1\end{cases}$$

- Cusp jump at *t=0.5*

Application

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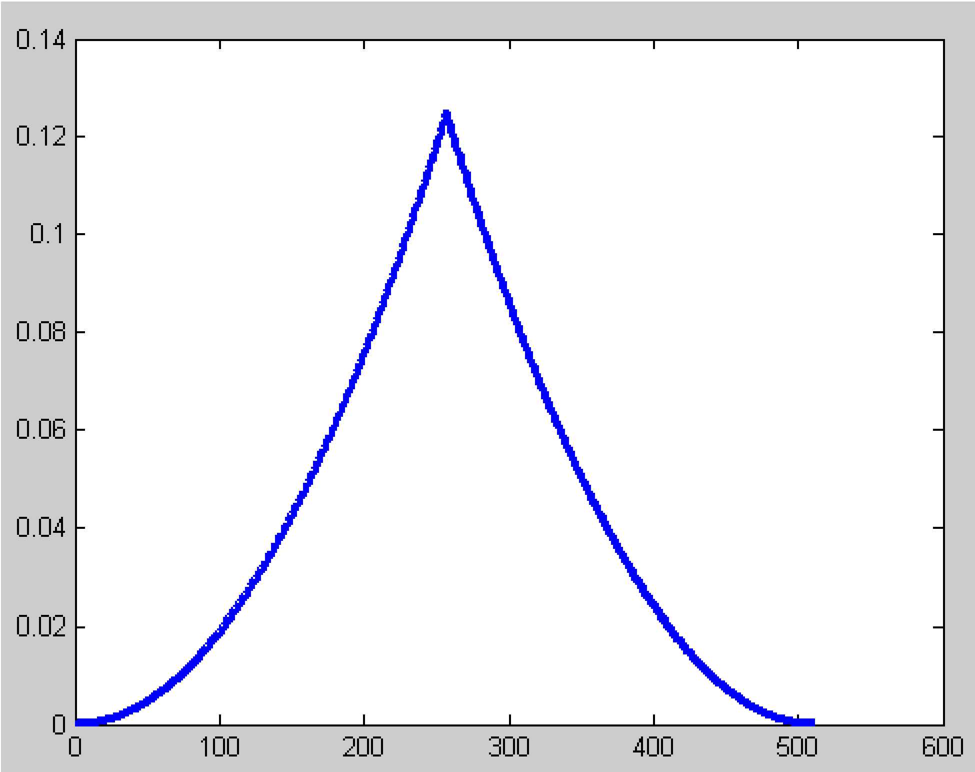


$$h(t) = \int g(t)dt = \begin{cases} \dfrac{t^2}{2}, 0 \le t < \dfrac{1}{2} \\ \dfrac{t^2}{2} - t + \dfrac{1}{2}, \dfrac{1}{2} \le t < 1 \end{cases}$$

- Cusp jump at *t=0.5*

Application

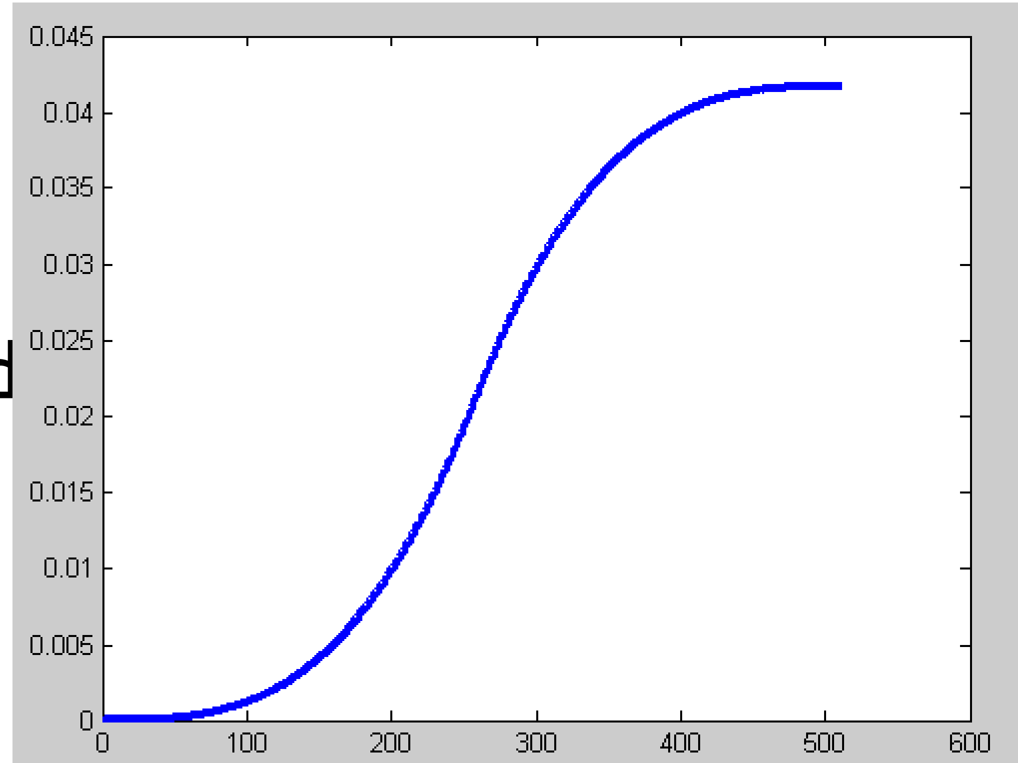
- Detecting hidden jump discontinuity
- Let's integrate again

$$f(t) = \int h(t)dt = \begin{cases} \dfrac{t^3}{6}, 0 \le t < \dfrac{1}{2} \\ \dfrac{t^3}{6} - \dfrac{t^2}{2} + \dfrac{t}{2} - \dfrac{1}{8}, \dfrac{1}{2} \le t < 1 \end{cases}$$

- Appears smooth to eye

Application

- Detecting hidd
- Let's integrate



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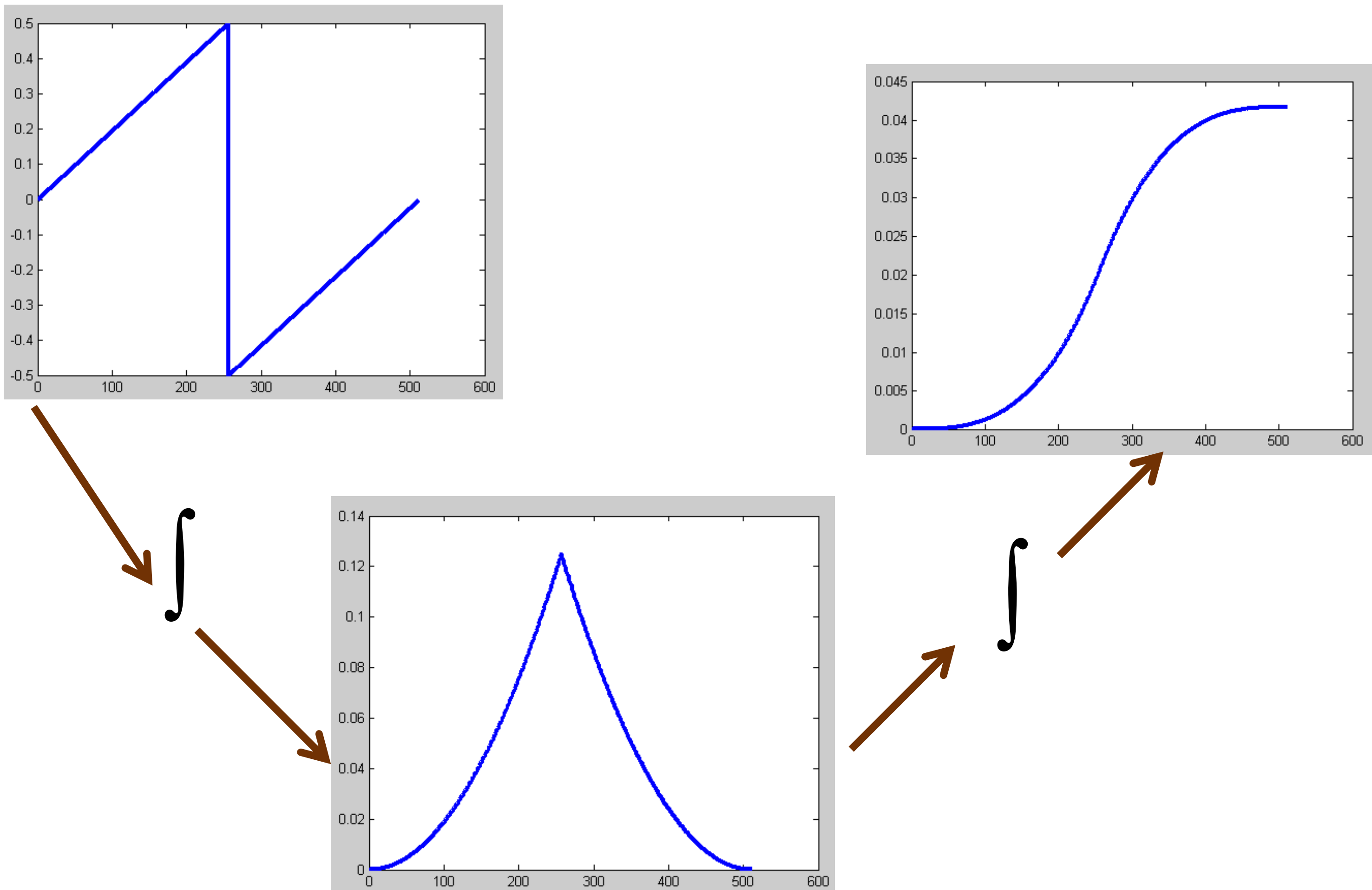

Application

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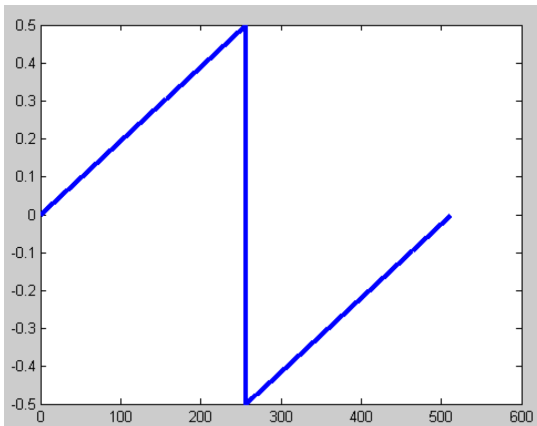
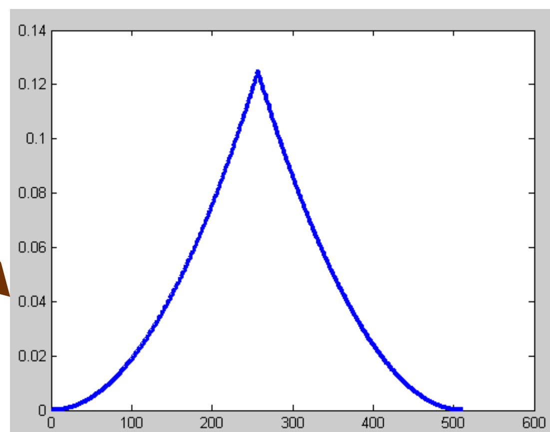
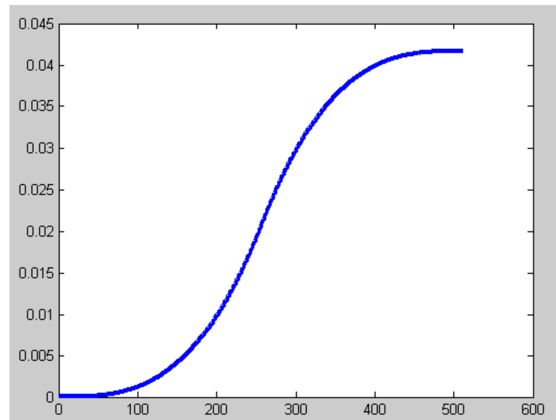


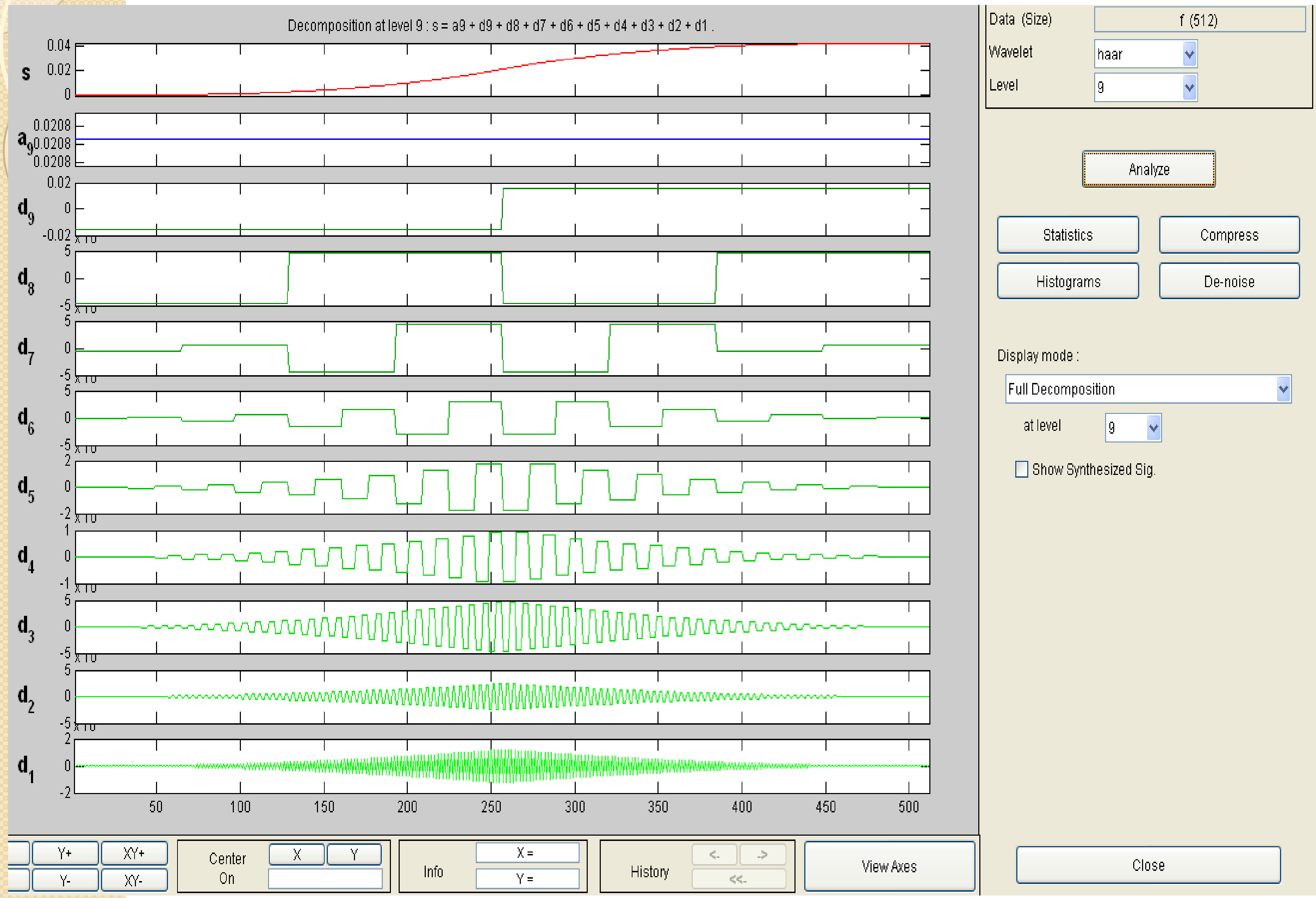




$\int$ $\int$

While selecting mother wavelet
At least M0=M1=0

Application



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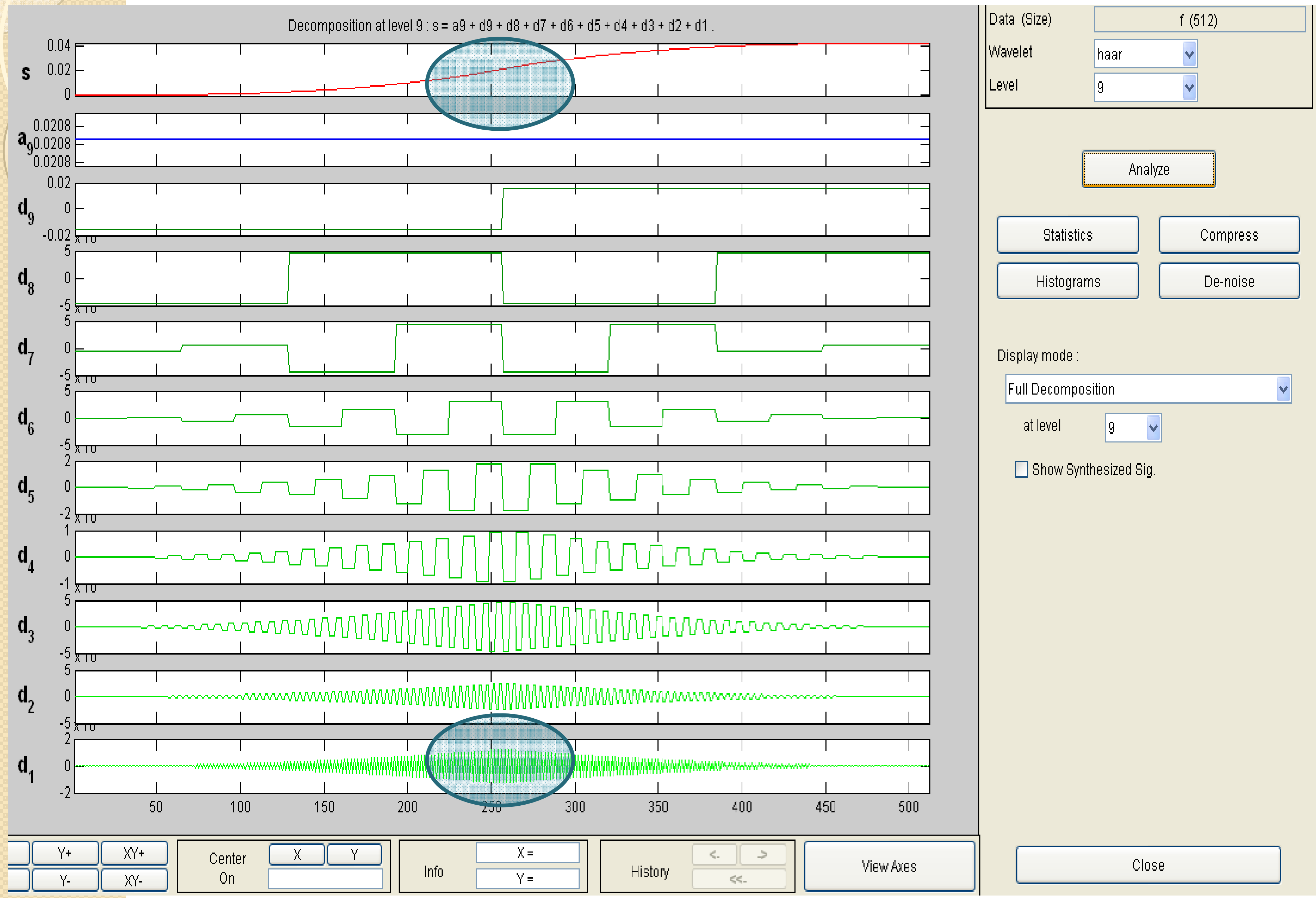

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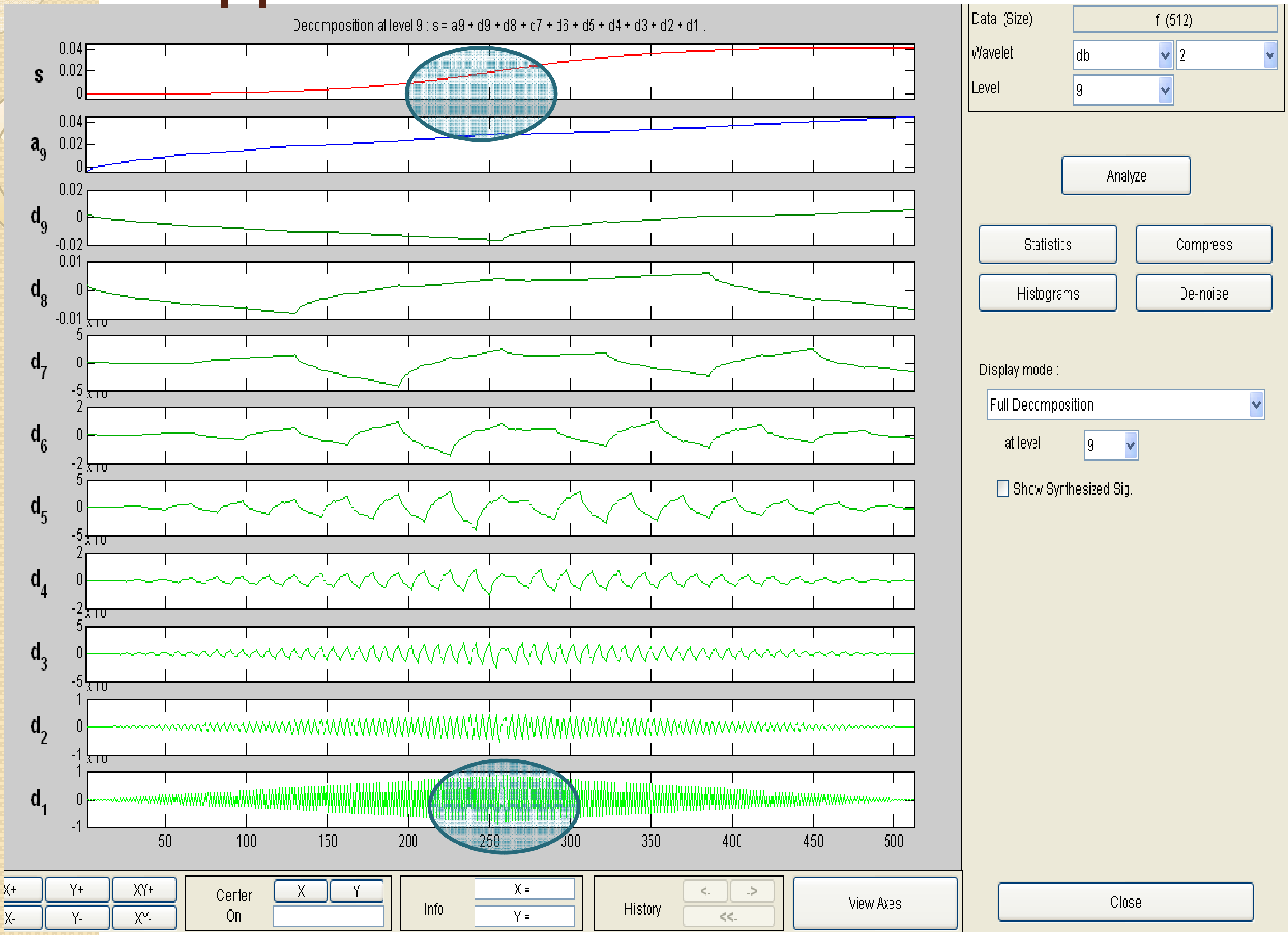

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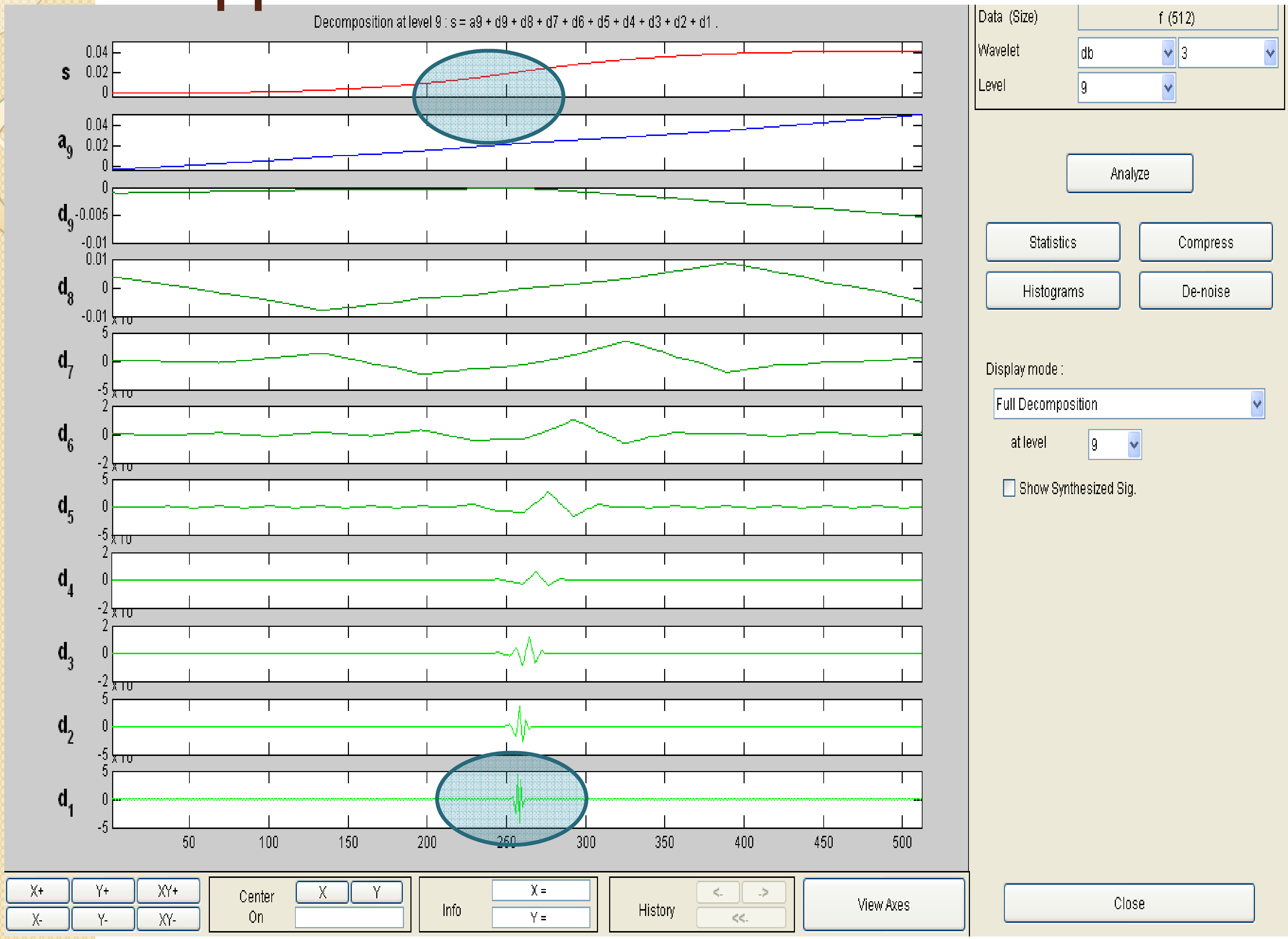

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Coefficients

- Who gives us coefficients of scaling equation?
- Haar

$$\psi(t) = \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ -1, & \frac{1}{2} \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\phi(t) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

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In search of coefficients

- In search of scaling equation coefficients!!!

$$\phi(t) = \sqrt{2}\sum_k h_k \phi(2t-k)$$

$$\varphi(t) = \sqrt{2}\sum_k (-1)^k h_{1-k} \phi(2t-k)$$

In search of coefficients

- In search of scaling equation coefficients!!!
- We can thinks of using three guiding theorems !

In search of coefficients

- We can thinks of using three guiding theorems !
- Theorem 1:

For the scaling equation $\phi(x) = \sum_k h_k \sqrt{2}\phi(2x-k)$, with

non-vanishing coefficients $\{h_k\}_{k=N}^{M}$ only for $N \le k \le M$,

its $\phi(x)$ is with a compact support contained in interval $[N, M]$

In search of coefficients

- We can thinks of using three guiding theorems !
- Theorem 2:

If the scaling function $\phi(x)$ has compact support on $0 \leq x \leq N - 1$ and if, $\{\phi(x-k)\}$ are linearly independent, then $h_n = h(n) = 0,$ for n<0 and $n > N - 1$.

Hence N is the length of the sequence.

In search of coefficients

- We can thinks of using three guiding theorems !
- Theorem 3:

If the scaling coefficieents $\{h_k\}$

satisfy the condition for existence and

orthogonality of $\phi(x)$, then

$$\varphi(x) = \sum_k g_k \sqrt{2}\phi(2x - k)$$

where, $g_k = \pm(-1)^k h_{N-k}$

and, $\int_{-\infty}^{\infty} \varphi(x - l)\phi(x - k)dx = \delta_{l,k} = 0, l \neq k$

Properties of scaling coefficients

1. $\sum h_k = \sqrt{2}$

2. $\sum h_{2k} = \dfrac{1}{\sqrt{2}}$

3. $\sum h_{2k+1} = \dfrac{1}{\sqrt{2}}$

Properties of scaling coefficients

4. $\sum |h_k|^2 = 1$

5. $\sum h_{k-2l} h_k = \delta_{l,0}$

6. $\sum 2h_{k-2l} h_{k-2j} = \delta_{l,j}$

Lets derive P1

$$1. \quad \sum h_k = \sqrt{2}$$

$$\phi(t) = \sum_k h_k \sqrt{2} \phi(2t - k)$$

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(t) dt &= \int_{-\infty}^{\infty} [\sum_k h_k \sqrt{2} \phi(2t - k)] dt \\ &= \sum_k h_k \sqrt{2} \int_{-\infty}^{\infty} \phi(2t - k) dt \end{aligned}$$

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$$\int_{-\infty}^{\infty} \phi(2t - k) dt = \frac{1}{2} \int_{-\infty}^{\infty} \phi(x) dx$$

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$$\int_{-\infty}^{\infty} \phi(t) dt = \sum_k \frac{1}{2} h_k \sqrt{2} \int_{-\infty}^{\infty} \phi(x) dx$$

$\int_{-\infty}^{\infty} \phi(t) dt = 1$ **Normalization !!**

Lets derive P1

$$1. \quad \sum h_k = \sqrt{2}$$

$$\phi(t) = \sum_k h_k \sqrt{2} \phi(2t - k)$$

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$\int_{-\infty}^{\infty} \phi(t) dt = 1$ **Normalization !!**

$$\boxed{1. \quad \sum h_k = \sqrt{2}}$$

Lets derive P5 $5.\quad \sum h_{k-2l}h_k = \delta_{l,0}$

$$\phi(t) = \sum_k h_k\sqrt{2}\phi(2t-k)$$

change of the variable t in $\phi(t)$ to $x = 2^{j-1}t - l$,

$$\phi(2^{j-1}t-l) = \sum_{k=-\infty}^{\infty} h_k\sqrt{2}\phi(2(2^{j-1}t-l)-k)$$

change in the index k to $m = k+2l$,

$$\begin{aligned}\phi(2^{j-1}t-l) &= \sum_{m=-\infty}^{\infty} h_{m-2l}\sqrt{2}\phi(2^jt-2l-m+2l)\\ &= \sum_{m=-\infty}^{\infty} h_{m-2l}\sqrt{2}\phi(2^jt-m)\end{aligned}$$

Lets derive P5

$$5. \quad \sum h_{k-2l} h_k = \delta_{l,0}$$

$$\begin{aligned}\phi(2^{j-1}t - l) &= \sum_{m=-\infty}^{\infty} h_{m-2l}\sqrt{2}\phi(2^j t - 2l - m + 2l) \\ &= \sum_{m=-\infty}^{\infty} h_{m-2l}\sqrt{2}\phi(2^j t - m)\end{aligned}$$

$$\phi(2^{j-1}t - l) = \sum_{m=-\infty}^{\infty} h_{k-2l}\sqrt{2}\phi(2^j t - k)$$

do this we multiply both sides of the above equation by $\phi(t)$, and integrate from $-\infty$ to ∞, allowing the exchange of the integration

$$\int_{-\infty}^{\infty} \phi(t)\phi(2^{j-1}t - l)dt = \sum_{k=-\infty}^{\infty} h_{k-2l} \int_{-\infty}^{\infty} \sqrt{2}\phi(t)\phi(2^j t - k)dt$$

Lets derive P5 $5. \quad \sum h_{k-2l} h_k = \delta_{l,0}$

$$\int_{-\infty}^{\infty} \phi(t)\phi(t-l)dt = \delta_{0,l} = \sum_{k=-\infty}^{\infty} h_{k-2} h_k$$

$$\boxed{\sum_{k=-\infty}^{\infty} h_{k-2l} h_k = \delta_{l,0}}$$

Lets derive P4

$$4. \quad \sum |h_k|^2 = 1$$

special case of $l = 0$ gives the property 4

$$\boxed{\sum_{k=-\infty}^{\infty} h_{k-2l} h_k = \delta_{l,0}}$$

$$\sum_{k=-\infty}^{\infty} h_k \overline{h_k} = \sum_{k=-\infty}^{\infty} |h_k|^2 = \delta_{0,0} 1$$

Lets derive P3

3. $\sum h_{2k+1} = \frac{1}{\sqrt{2}}$

$$\delta_{l,0} = \sum_{k=-\infty}^{\infty} 2h_{k-2l}\overline{h_k} = 2\sum_{k=-\infty}^{\infty} h_{k+2l}\overline{h_k}$$

$$\delta_{l,0} = 2\sum_{k=-\infty}^{\infty} h_{2k+2l}\overline{h_{2k}} + 2\sum_{k=-\infty}^{\infty} h_{2k+1+2l}\overline{h_{2k+1}}$$

$$\sum_{l=-\infty}^{\infty} \delta_{l,0} = 1 = 2\sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} [h_{2h+2l}h_{2k} + h_{2k+1+2l}h_{2k+1}]$$

Lets derive P3

$$3. \quad \sum h_{2k+1} = \frac{1}{\sqrt{2}}$$

$$1 = \sum_{k=-\infty}^{\infty} \overline{h_{2k}} [\sum_{l=-\infty}^{\infty} h_{2k+2l}] + \sum_{k=-\infty}^{\infty} \overline{h_{2k+1}} [\sum_{l=-\infty}^{\infty} h_{2k+1+2l}]$$

$$\sum_{l=-\infty}^{\infty} h_{2k+2l} = \sum_{l=-\infty}^{\infty} h_{2l} = \sum_{l=-\infty}^{\infty} h_{2k} \equiv A$$

$$\sum_{l=-\infty}^{\infty} h_{2k+2l+1} = \sum_{l=-\infty}^{\infty} h_{2l+1} = \sum_{l=-\infty}^{\infty} h_{2k+1} \equiv B$$

Lets derive P3 $3. \quad \sum h_{2k+1} = \frac{1}{\sqrt{2}}$

$$1 = \sum_{k=-\infty}^{\infty} \overline{h_{2k}} \sum_{k=-\infty}^{\infty} h_{2k} + \sum_{k=-\infty}^{\infty} \overline{h_{2k+1}} \sum_{k=-\infty}^{\infty} h_{2k+1}$$

$$1 = \overline{A}A + \overline{B}B = |A|^2 + |B|^2$$

Lets derive P3

$$3. \quad \sum h_{2k+1} = \frac{1}{\sqrt{2}}$$

$$\sum_k h_k = \sqrt{2}$$

$$\sqrt{2} = \sum_{k=-\infty}^{\infty} h_k = \sum_{k=-\infty}^{\infty} h_{2k} + \sum_{k=-\infty}^{\infty} h_{2k+1} = A + B$$

$$A + B = \sqrt{2}$$

Lets derive P3

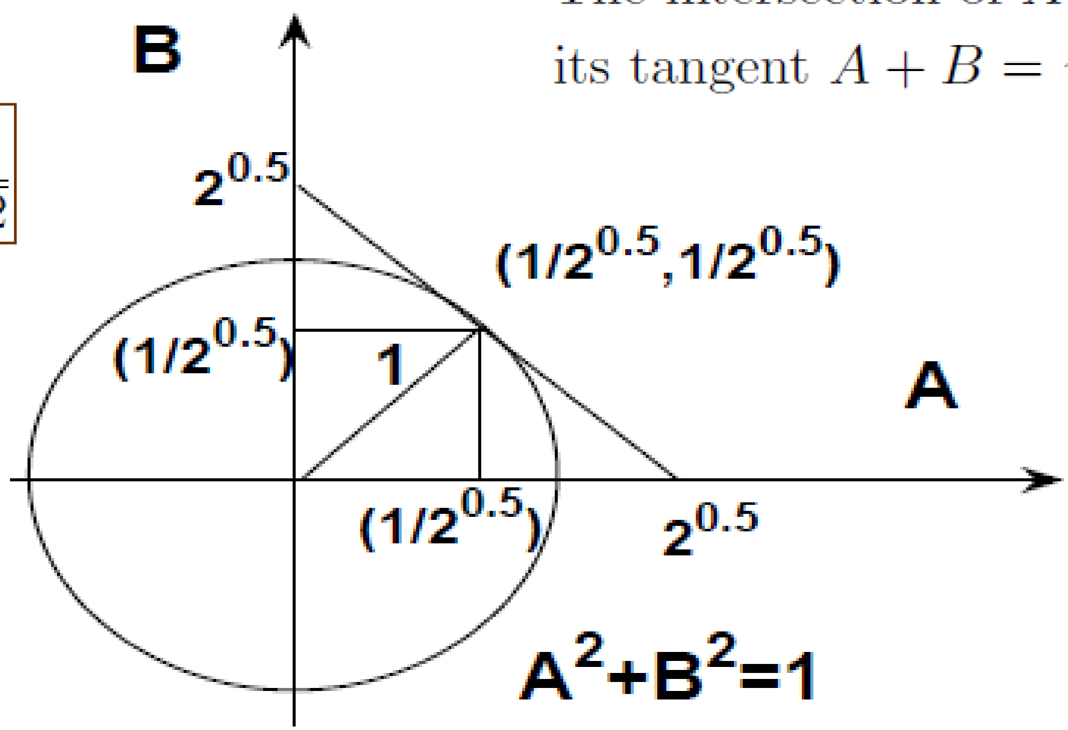
$$3. \quad \sum h_{2k+1} = \frac{1}{\sqrt{2}}$$

$$1 = \overline{A}A + \overline{B}B = |A|^2 + |B|^2$$

$$A + B = \sqrt{2}$$

$$\boxed{A = B = \frac{1}{\sqrt{2}}}$$

The intersection of $A^2 + B^2 = 1$ and its tangent $A + B = \sqrt{2}$ at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.



Lets derive P3 $3.\quad \sum h_{2k+1} = \frac{1}{\sqrt{2}}$

$$\boxed{A = B = \tfrac{1}{\sqrt{2}}}$$

$$\sum_{l=-\infty}^{\infty} h_{2k+2l+1} = \sum_{l=-\infty}^{\infty} h_{2l+1} = \sum_{l=-\infty}^{\infty} h_{2k+1} \equiv B$$

$$3.\quad \sum h_{2k+1} = \frac{1}{\sqrt{2}}$$

Lets derive P2

$$2. \quad \sum h_{2k} = \frac{1}{\sqrt{2}}$$

$$\boxed{A = B = \frac{1}{\sqrt{2}}}$$

$$\sum_{l=-\infty}^{\infty} h_{2k+2l} = \sum_{l=-\infty}^{\infty} h_{2l} = \sum_{l=-\infty}^{\infty} h_{2k} \equiv A$$

$$2. \quad \sum h_{2k} = \frac{1}{\sqrt{2}}$$

Verification for Haar

$$h_0 = h_1 = \frac{1}{\sqrt{2}}, h_k = 0 \text{ for } k \neq 0, 1,$$

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2. $\sum_{k=0} h_{2k} = h_0 = \frac{1}{\sqrt{2}}$

3. $\sum_{k=1} h_{2k+1} = h_1 = \frac{1}{\sqrt{2}}$

Verification for Haar

$$h_0 = h_1 = \tfrac{1}{\sqrt{2}},\ h_k = 0 \text{ for } k \neq 0, 1,$$

$$4. \quad \sum_{k=0,1} |h_k|^2 = |h_0|^2 + |h_1|^2 = \frac{1}{2} + \frac{1}{2} = 1$$

Verification for Haar

$$h_0 = h_1 = \frac{1}{\sqrt{2}},\, h_k = 0 \text{ for } k \neq 0, 1,$$

$$5. \quad \sum_k 2h_{k-2l}h_k$$

$$\begin{aligned}
\sum_{k=0,1;l=0} 2k_{k-2l}\overline{h_k} &= 2[h_0\overline{h_0} + h_1\overline{h_1}] \\
&= 2[\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}] = 2[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}] = 1
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\end{aligned}$$

Discrete Orthonormality

$$\sum_k 2h_{k-2l}\overline{h_k} = \delta_{l,0}$$

Not all function obey!

- Roof scaling function

$$\phi(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \end{cases}$$

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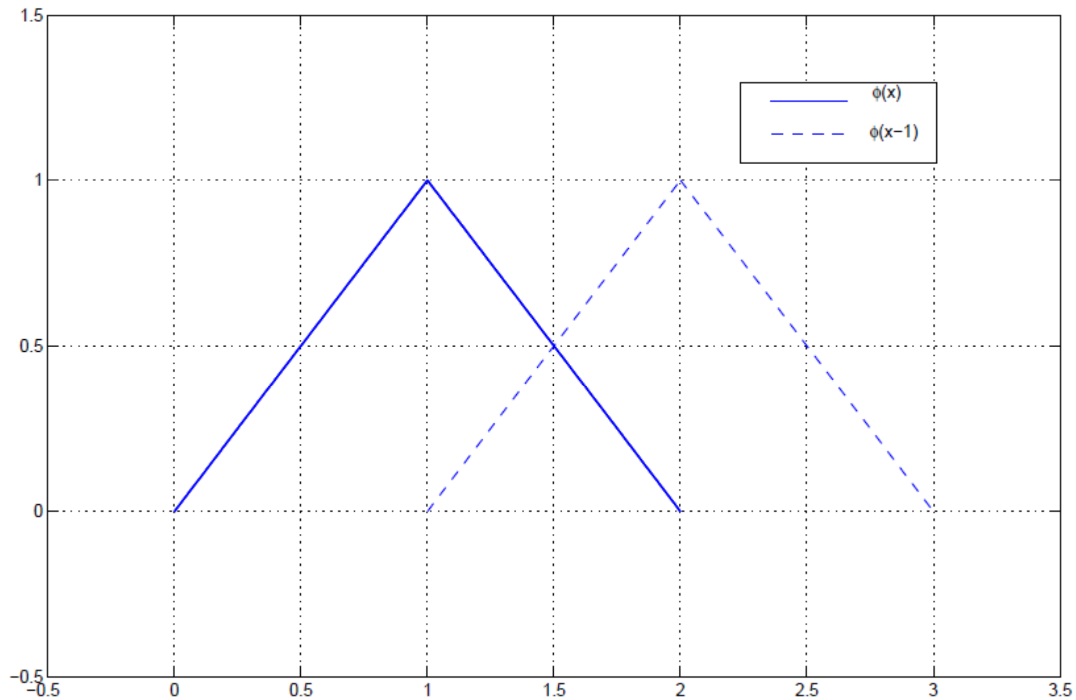


Framework

- Leads us to two questions

1) How do we go about selecting the mother wavelet and scale of analysis?
2) What is the procedure to calculate scaling and wavelet coefficients?

Applications

- Detecting (hidden) jumps/discontinuities
- Suppressing polynomials towards de-noising
- 2D applications: compression and pattern recognition

Properties of Unitary Transform

- Energy compaction: only few transform coefficients have large magnitude
  - Such property is related to the decorrelating role of unitary transform
- Energy conservation: unitary transform preserves the 2-norm of input vectors
  - Such property essentially comes from the fact that rotating coordinates does not affect Euclidean distance

Unitary Matrix and 1D Unitary Transform

<u>Definition</u>

conjugate transpose

A matrix A is called unitary if $A^{-1}=A^{*T}$

When the transform matrix A is unitary, the defined transform $\vec{y} = A\vec{x}$ is called unitary transform

<u>Example</u>

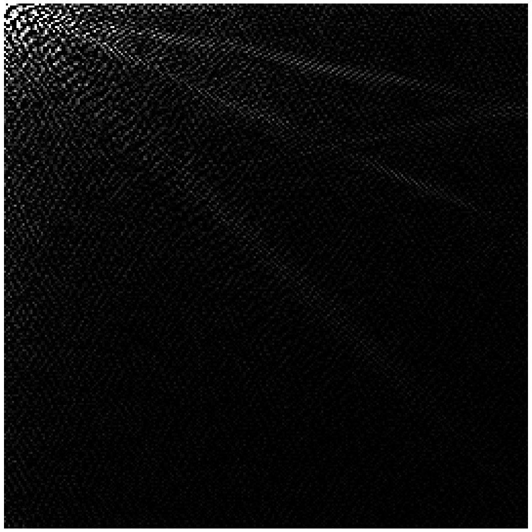
$$\mathbf{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \mathbf{A}^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \mathbf{A}^{T}$$

For a real matrix A, it is unitary if $A^{-1}=A^{T}$

2D DCT

$$\mathbf{Y} = \mathbf{C}\mathbf{X}\mathbf{C}^{T} = \mathbf{C}\mathbf{X}\mathbf{C}^{-1}$$

Original cameraman image X



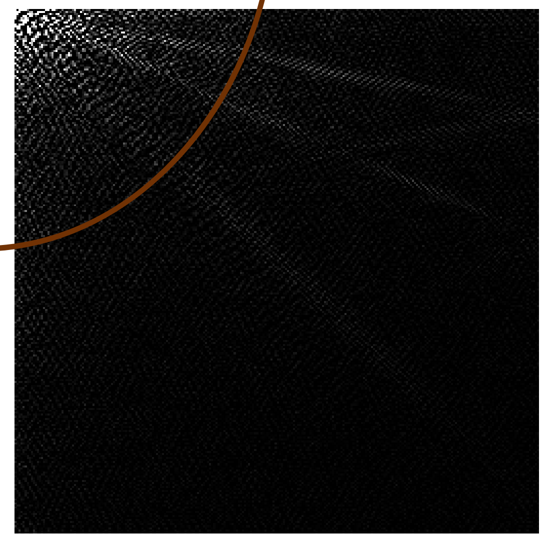
Its DCT coefficients Y
(2451 significant coefficients, *th=64*)

Notice the excellent energy compaction property of DCT

2D DCT

$$\mathbf{Y} = \mathbf{CXC}^T = \mathbf{CXC}^{-1}$$

Original cameraman image X



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Thank You!

Questions ??