



A vertical decorative bar on the left side of the slide features a repeating pattern of small black dots on a light beige background. At the top, there is a stylized graphic element consisting of two overlapping circles: one large yellow circle and one smaller white circle with a black outline. A small blue circle is positioned near the bottom of the yellow circle, accompanied by a small blue dot above it.

Lecture 50 – Wavelet Applications

Dr. Aditya Abhyankar

Wavelet Transform

- Decomposes signal into two separate series
 - Single series to represent most coarse version
 - Scaling Function
 - Double series to represent refined version
 - Wavelet Function



Framework

- Gave us power to move up or down the ladder
 - We can now indeed zoom-in or zoom-out of any part of the signal
 - This makes the entire analysis ‘scalable’!!
 - Scalability stems out of multi-resolution framework !
-



Framework

$$\phi(t) = \sqrt{2} \sum_k h_k \phi(2t - k)$$

Framework

$$\phi(t) = \sqrt{2} \sum_k h_k \phi(2t - k)$$
$$\phi(1t) = \phi(2^0 t) \in V_0$$
$$\phi(2t - k) = \phi(2^1 t - k) \in V_1$$

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graph TD; C["phi(t) = sqrt(2) sum_k h_k phi(2t - k)"] --> T1["phi(1t) = phi(2^0 t) in V_0"]; C --> T2["phi(2t - k) = phi(2^1 t - k) in V_1"]
```

Framework

$$\phi(t) = \sqrt{2} \sum_k h_k \phi(2t - k)$$
$$\phi(1t) = \phi(2^0 t) \in V_0$$
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$V_0 \subset V_1$

Framework

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$$\varphi(t) = \sqrt{2} \sum_k g_k \phi(2t - k)$$

Framework

$$V_0 \subset V_1$$

$$\phi(t) = \sqrt{2} \sum_k h_k \phi(2t - k)$$

$$\phi(1t) = \phi(2^0 t) \in V_0$$

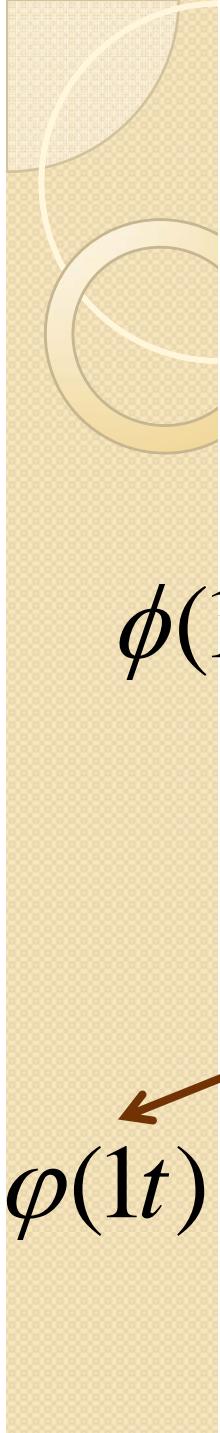
$$\phi(2t - k) = \phi(2^1 t - k) \in V_1$$

$$\varphi(t) = \sqrt{2} \sum_k g_k \phi(2t - k)$$

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Framework



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The diagram illustrates a framework for signal processing or analysis. It shows two parallel paths. The top path starts with the equation $\phi(t) = \sqrt{2} \sum_k h_k \phi(2t - k)$, which is enclosed in a light brown box. This equation is connected by arrows to $\phi(1t) = \phi(2^0 t) \in V_0$ on the left and $\phi(2t - k) = \phi(2^1 t - k) \in V_1$ on the right. The sets V_0 and V_1 are also shown at the top right. The bottom path starts with the equation $\varphi(t) = \sqrt{2} \sum_k g_k \phi(2t - k)$, which is enclosed in a light brown box. This equation is connected by arrows to $\varphi(1t) = \varphi(2^0 t) \in W_0$ on the left and $\phi(2t - k) = \phi(2^1 t - k) \in V_1$ on the right. The sets V_1 and W_0 are also shown at the bottom right.

Framework

$$V_1 = V_0 \oplus W_0$$

$$V_0 \subset V_1$$

$$\phi(t) = \sqrt{2} \sum_k h_k \phi(2t - k)$$

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Framework

$$\phi\left(\frac{t}{2}\right) = \phi(2^{-1}t) \in V_{-1}$$

$$\phi\left(\frac{t}{2}\right) = \sqrt{2} \sum_k h_k \phi(t - k)$$

$$\phi(t - k) = \phi(2^0 t - k) \in V_0$$

$$\varphi\left(\frac{t}{2}\right) = \varphi(2^{-1}t) \in W_{-1}$$

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$$V_{-1} \subset V_0$$

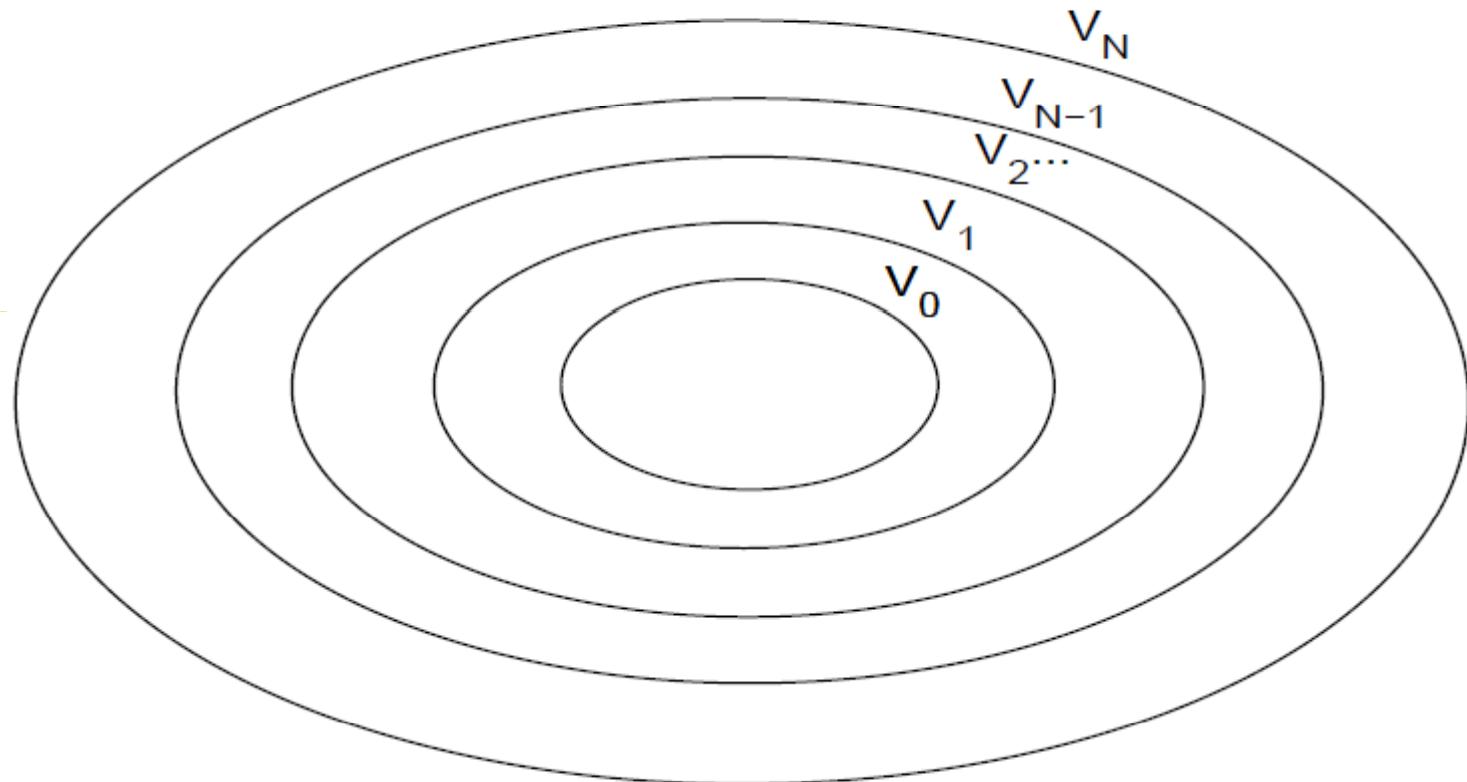
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$$V_0 = V_{-1} \oplus W_{-1}$$

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Framework



..... $V_{-3} \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 \dots$

Framework

$f_j(x) \in V_j$, Scale $\frac{1}{2^j}$

$$\{2^{j/2} \phi(2^j x - k)\}_k$$

$$f_j(x) = \sum_k \alpha_{j,k} 2^{j/2} \phi(2^j x - k)$$

$$\alpha_{j,k} = \int_{-\infty}^{\infty} f_j(x) 2^{j/2} \phi(2^j x - k) dx$$

Framework

$f_j(x) \in V_j$, Scale $\frac{1}{2^j}$

Normalization

$$\{2^{j/2}\phi(2^j x - k)\}_k$$

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Framework

Normalization

$$f_j(x) \in V_j, \text{ Scale } \frac{1}{2^j} \rightarrow \text{Orthonormal basis}$$

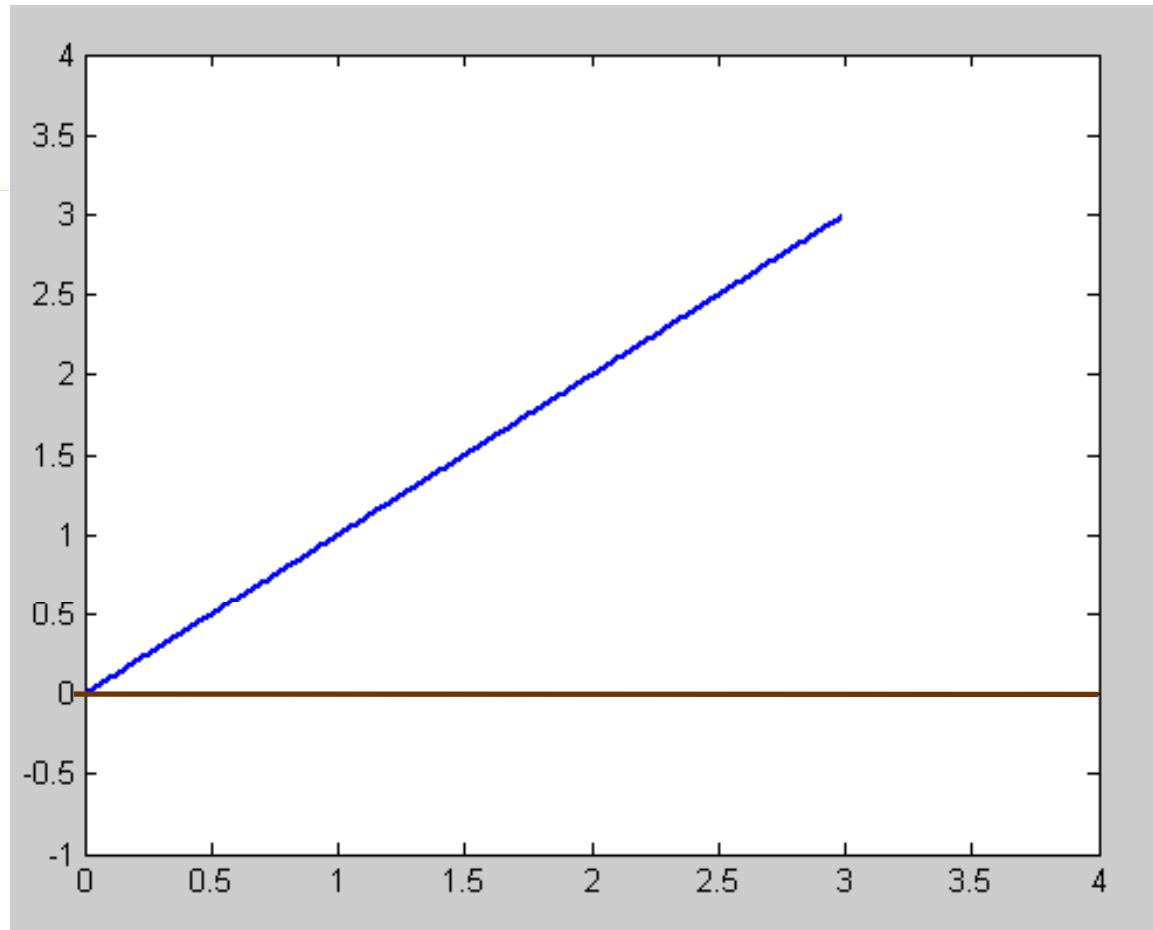
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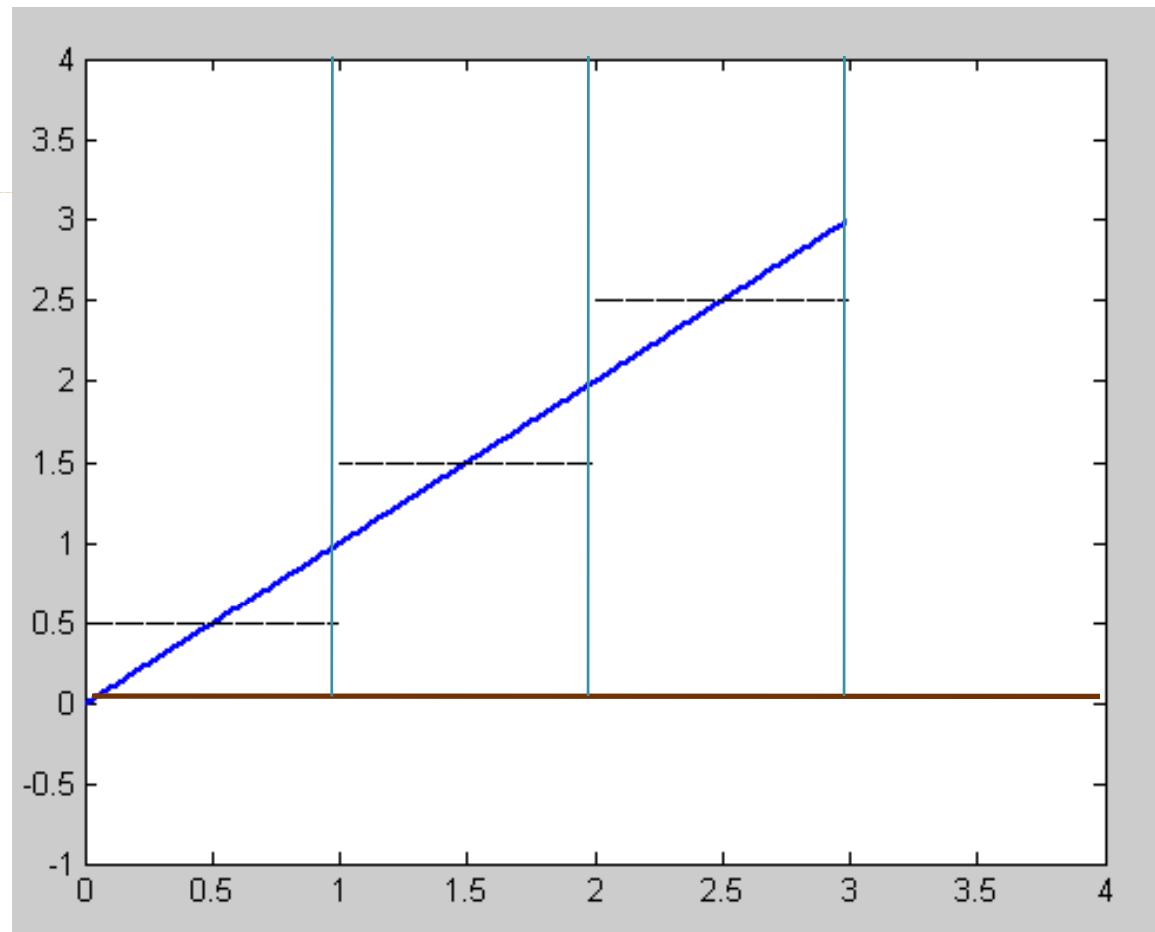
Test Signal

$$f(x) = \begin{cases} x, & 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$



Test Signal

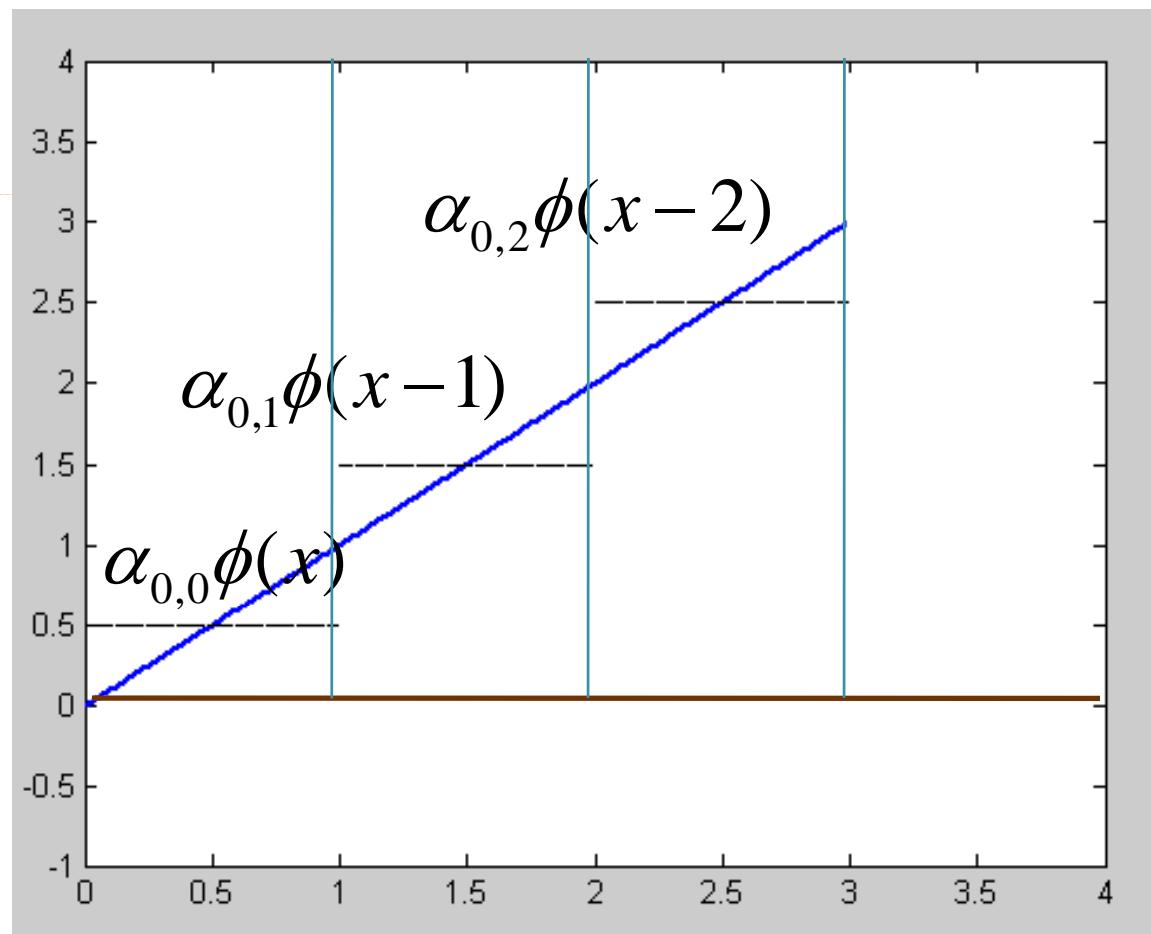
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$$f_0(x) \in V_0$$

Test Signal

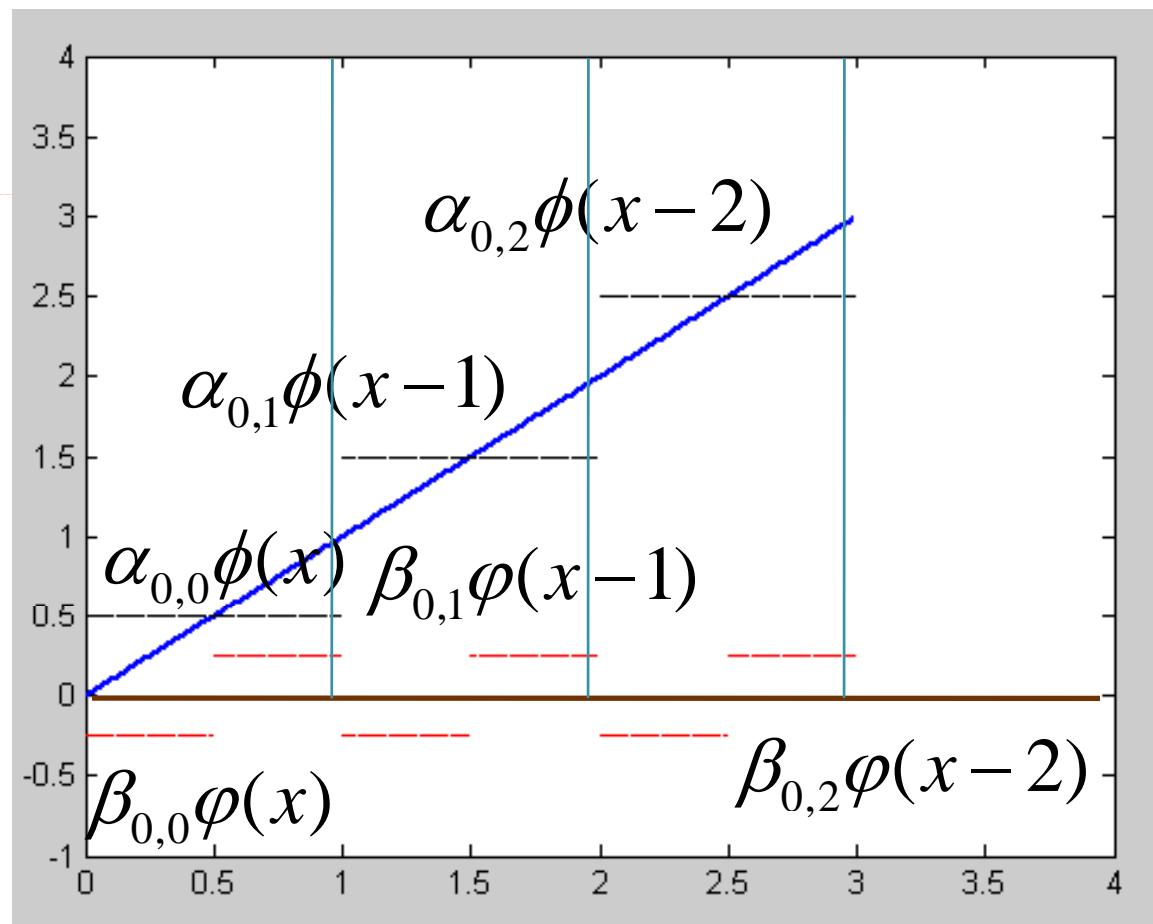
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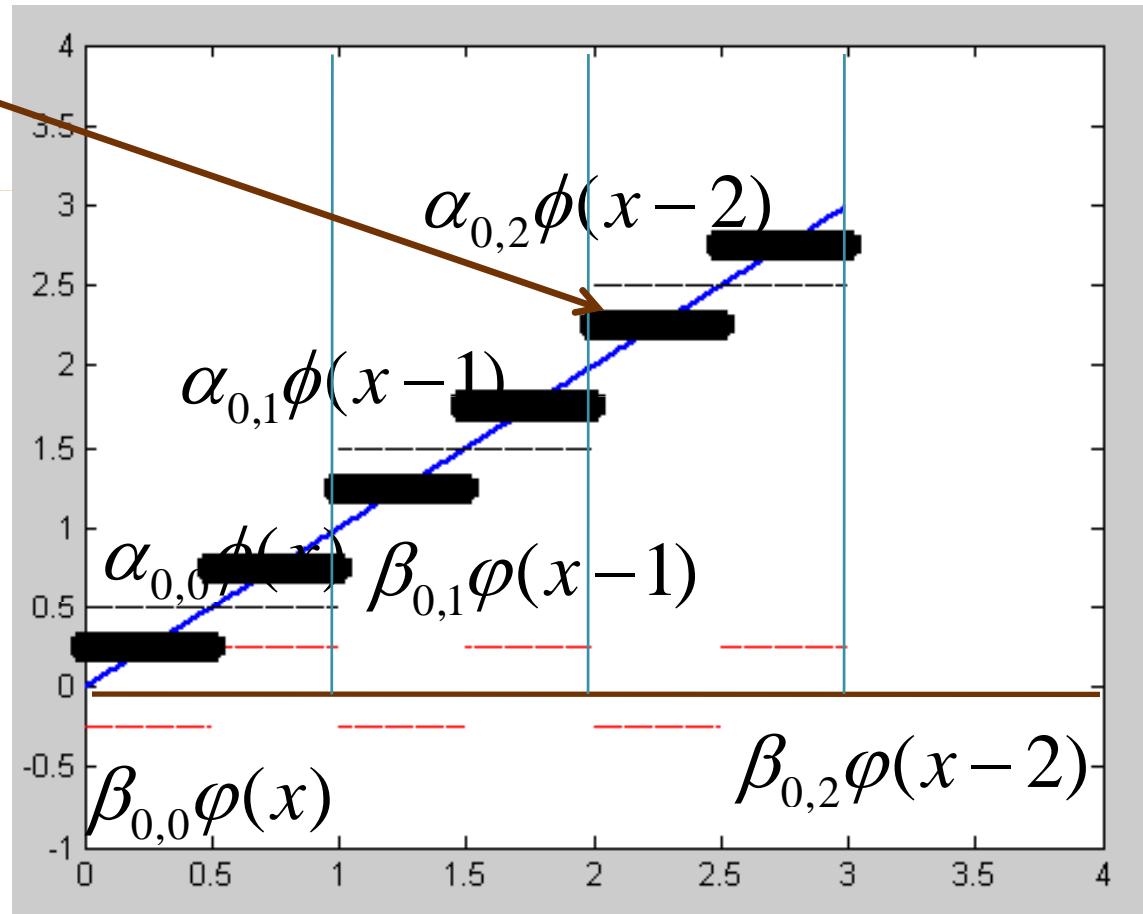
$$f_0(x) \in V_0$$

$$g_0(x) \in W_0$$

Test Signal

$$f(x) = \begin{cases} x, & 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

$f_1(x) \in V_1$



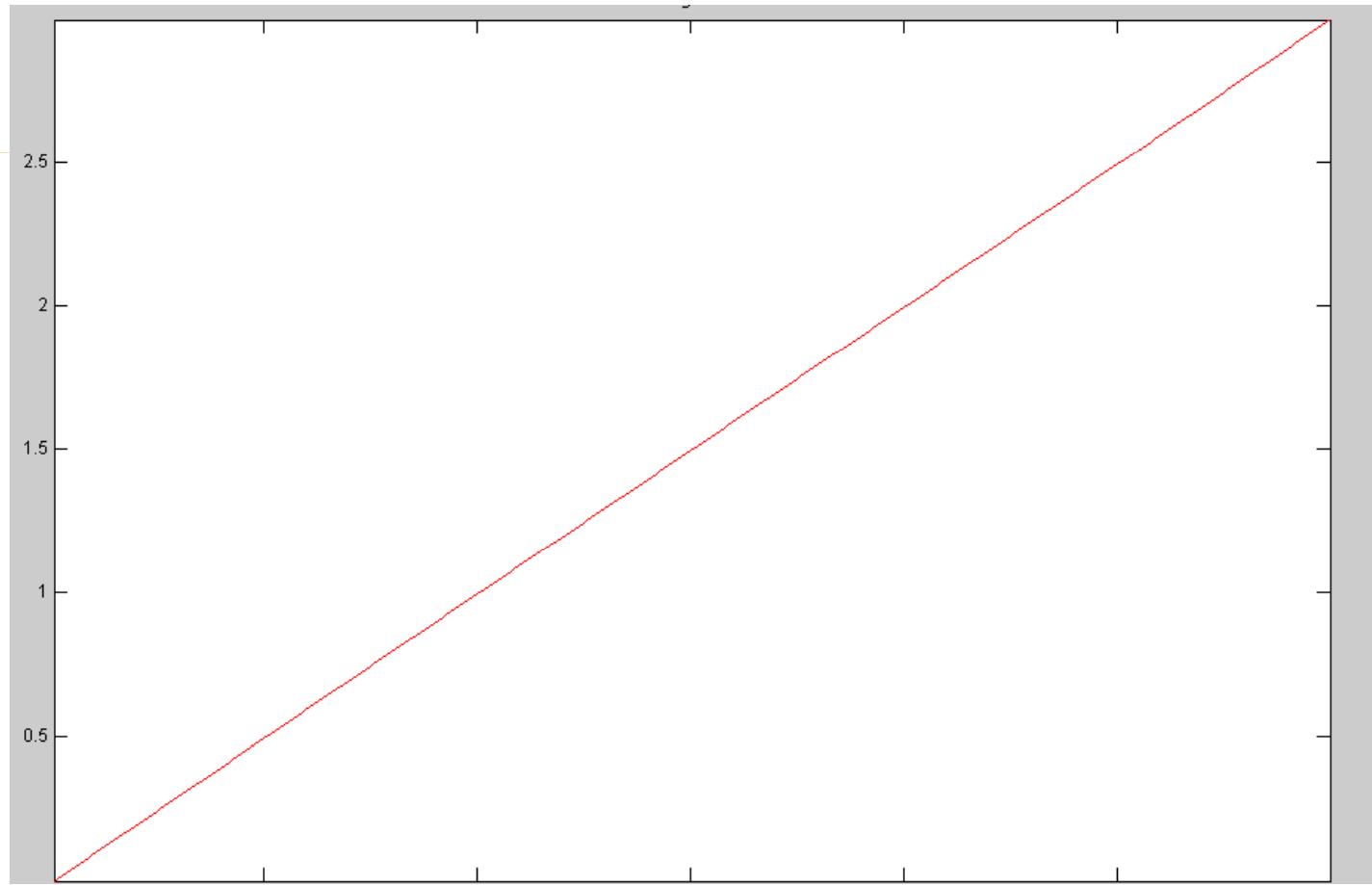
$f_0(x) \in V_0$

\oplus

$g_0(x) \in W_0$

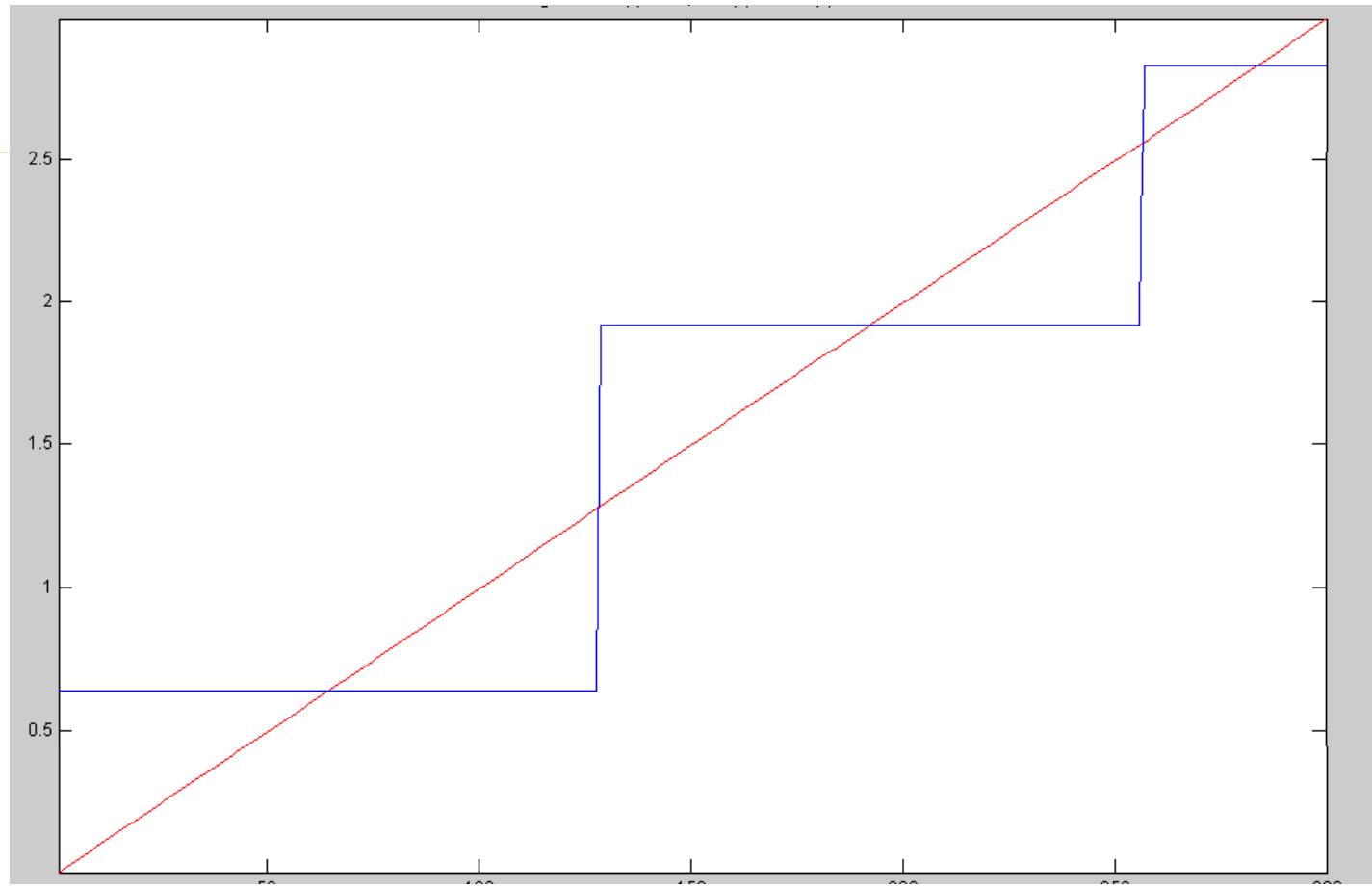
Test Signal

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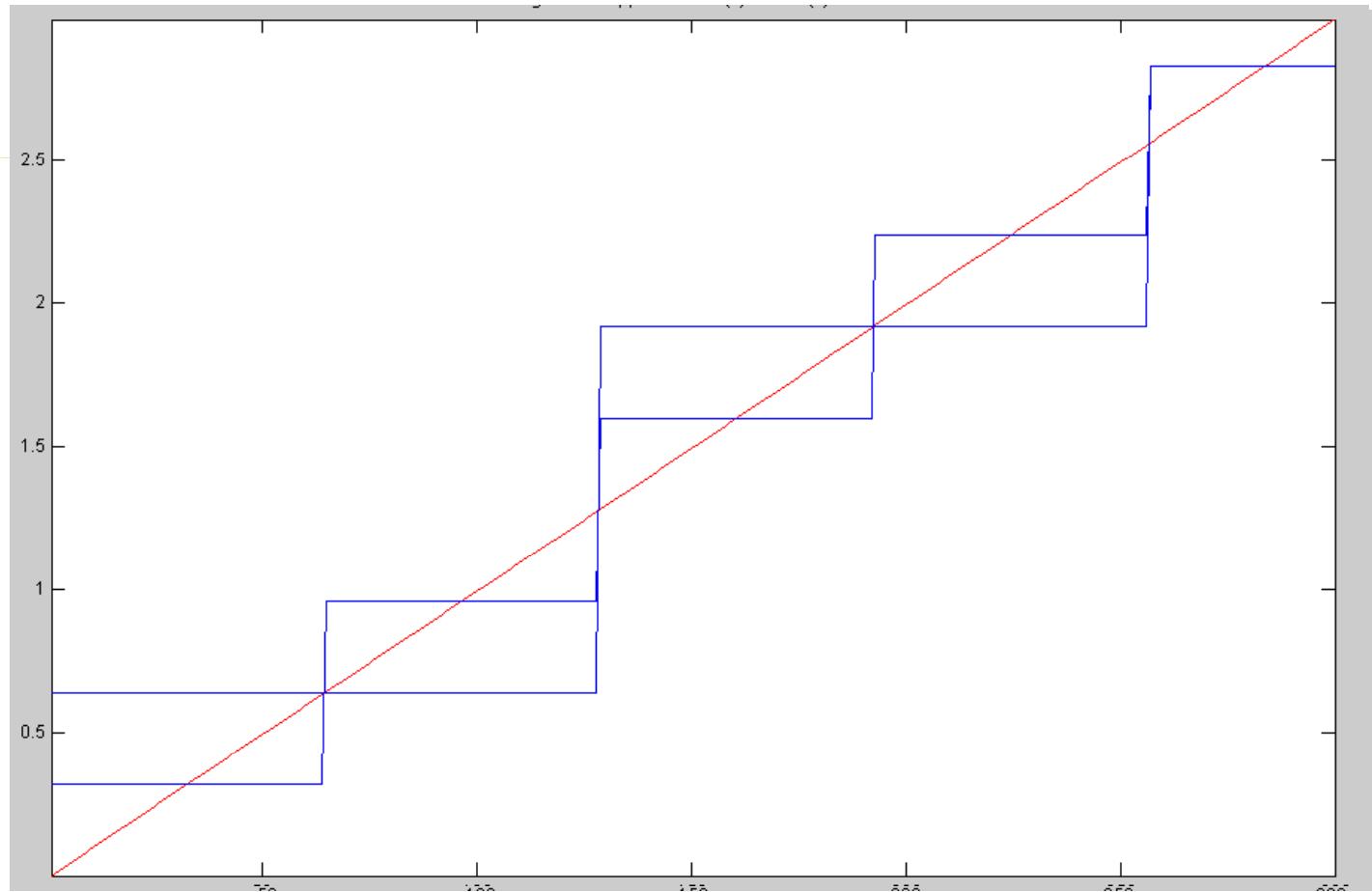
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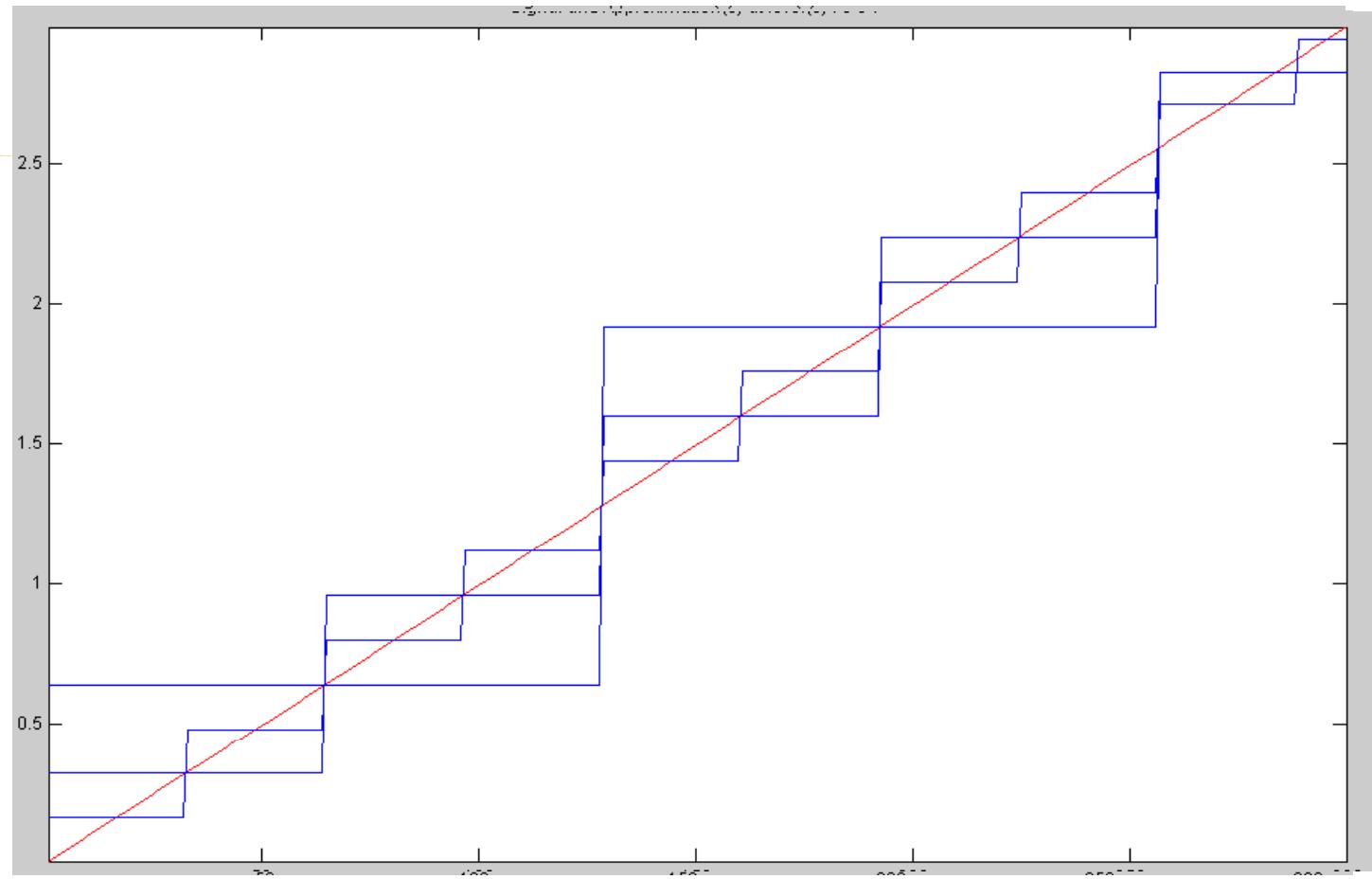
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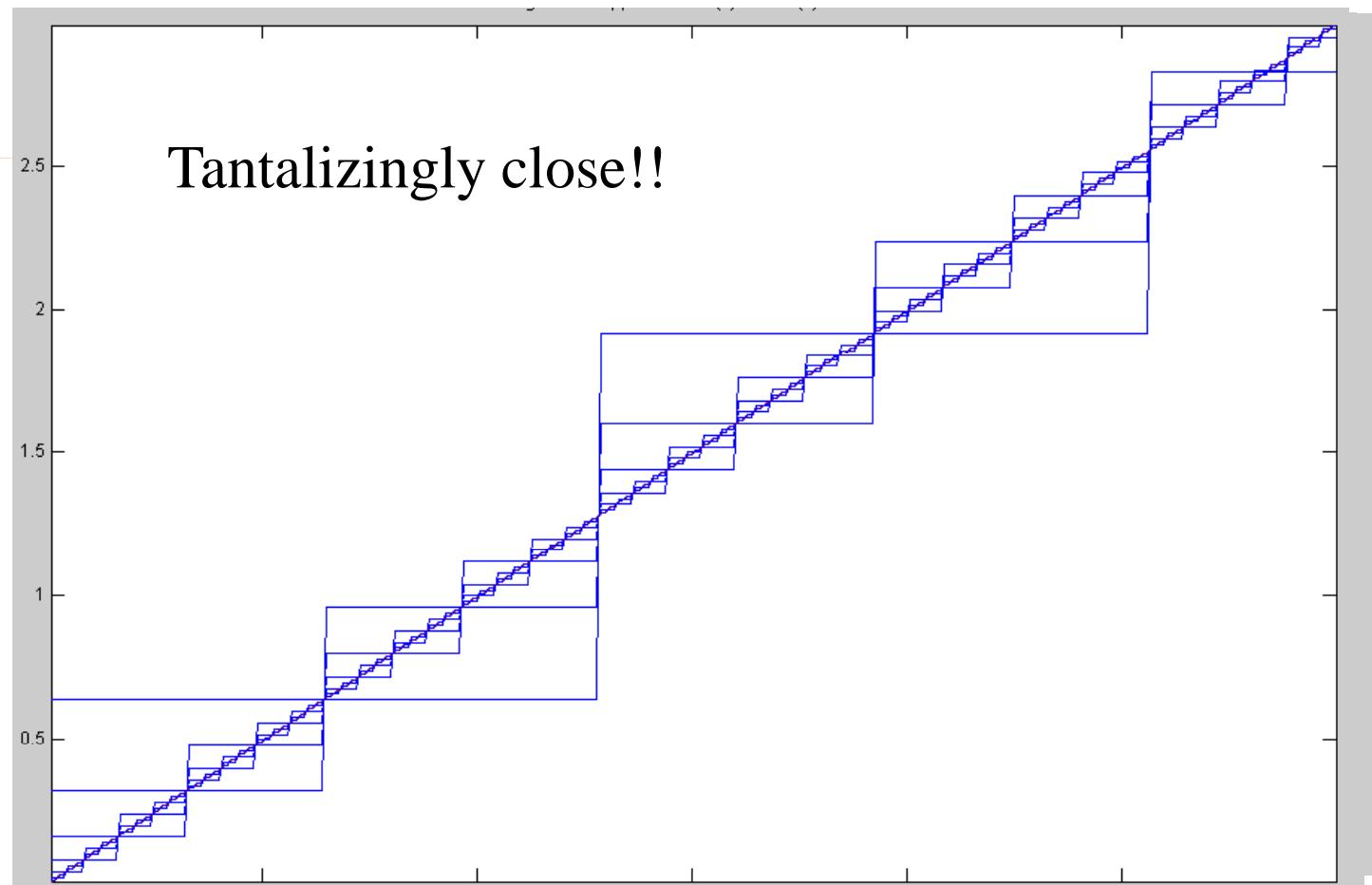
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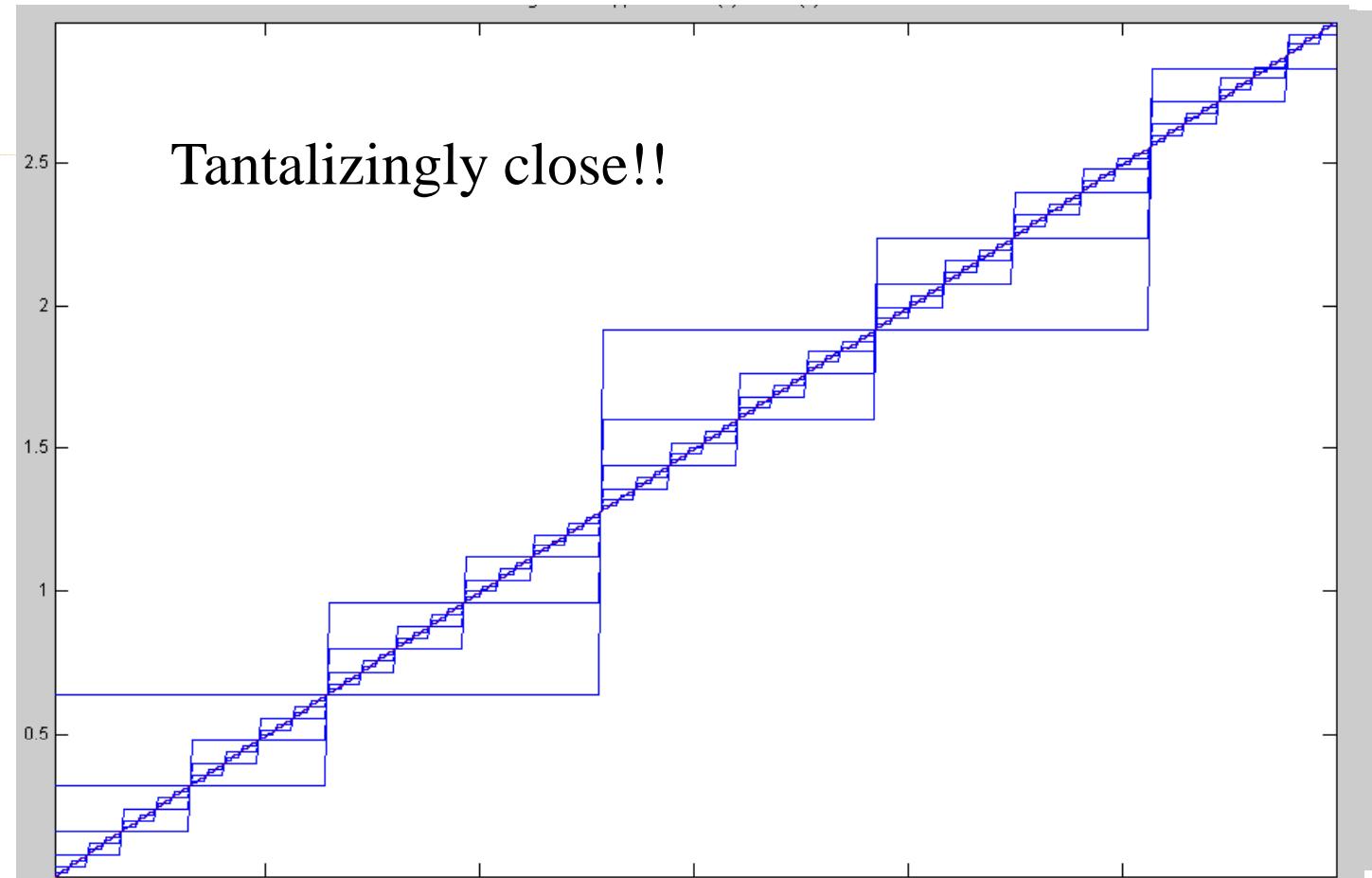
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Framework

- Leads us to two questions
 - 1) How do we go about selecting the mother wavelet and scale of analysis?
 - 2) What is the procedure to calculate scaling and wavelet coefficients?



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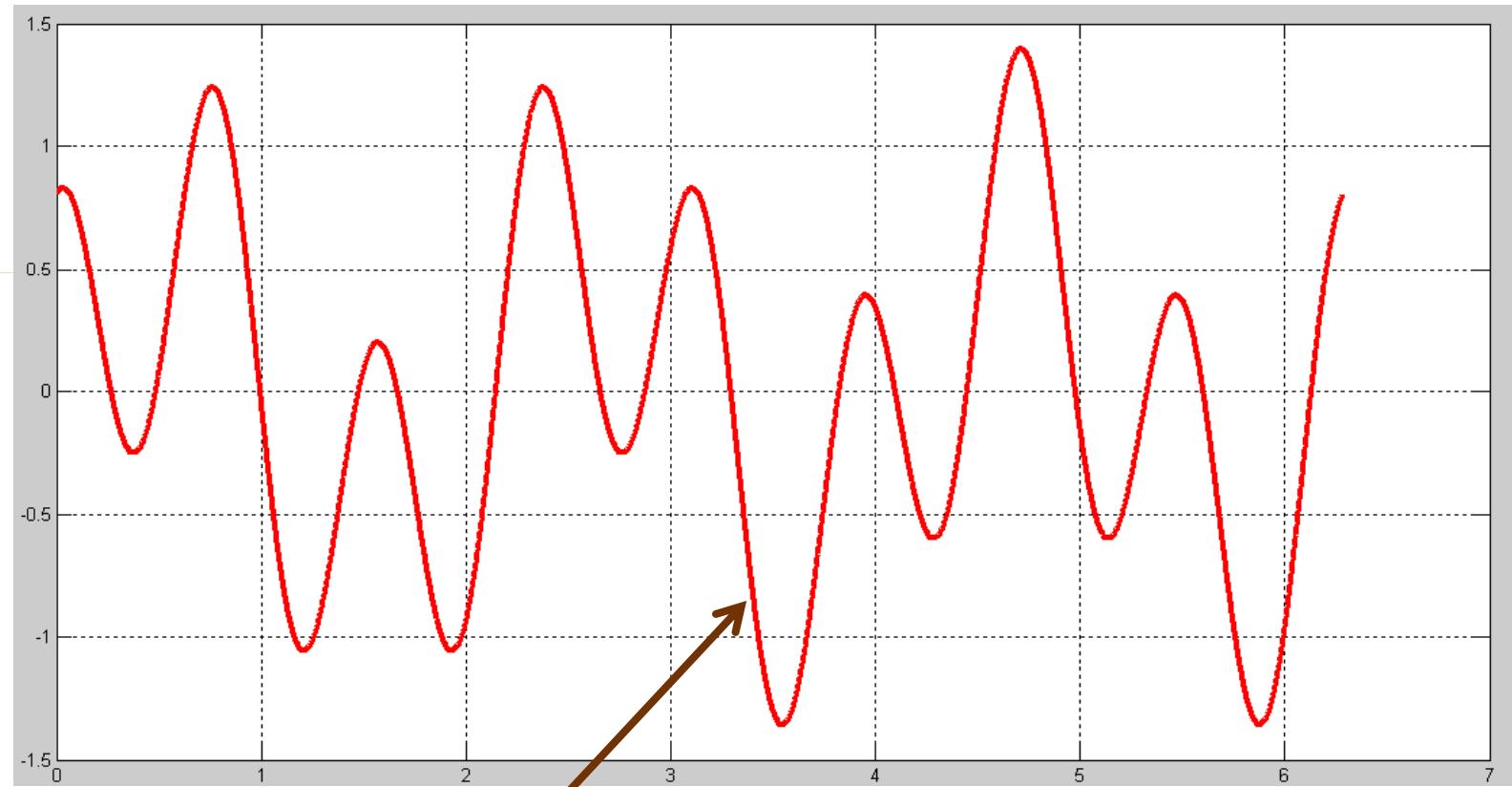
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Vanishing moments

Correlation

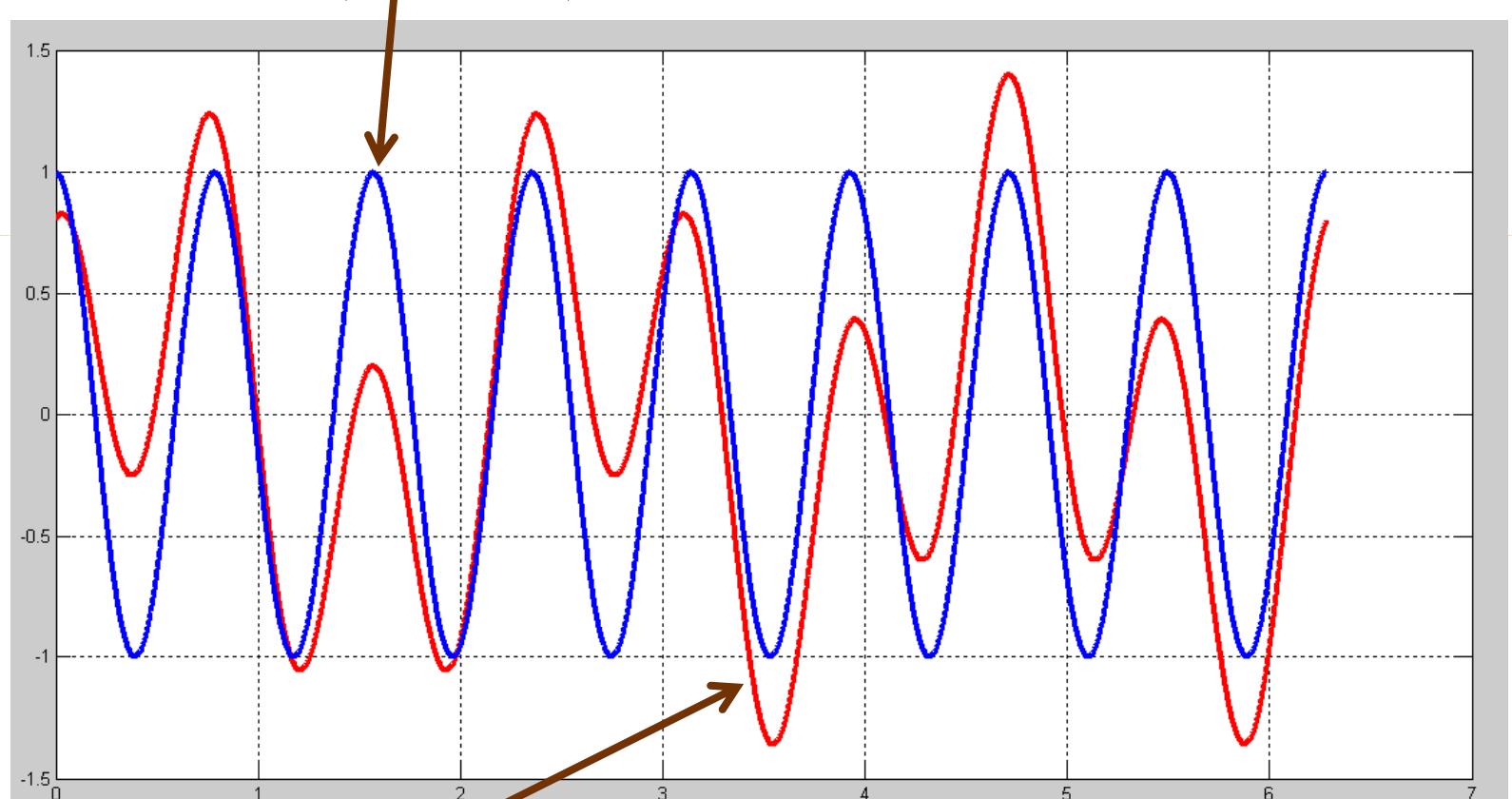
How Fourier Works!!



$$y[n] = 0.6\sin(2\pi 3n) + 0.8\cos(2\pi 8n)$$

How Fourier Works!!

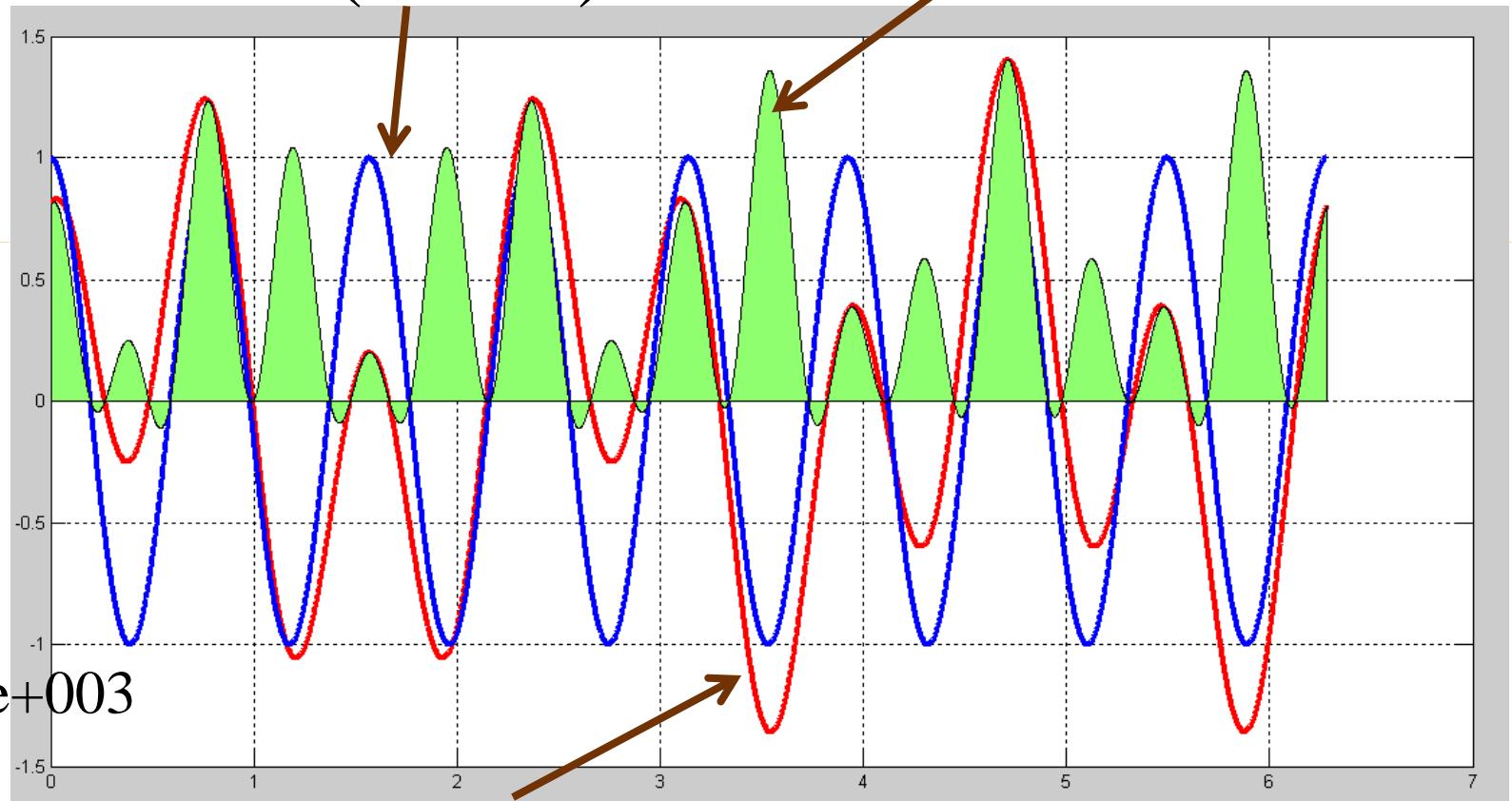
$$\cos(2\pi 8n)$$



$$y[n] = 0.6\sin(2\pi 3n) + 0.8\cos(2\pi 8n)$$

How Fourier Works!!

$$\cos(2\pi 8n) \quad < y[n], \cos(2\pi 8n) >$$



Moments

- The moment of order m , of function $f(x)$ on (a,b) can be given as
-

$$M_m = \int_a^b x^m f(x) dx$$

Application

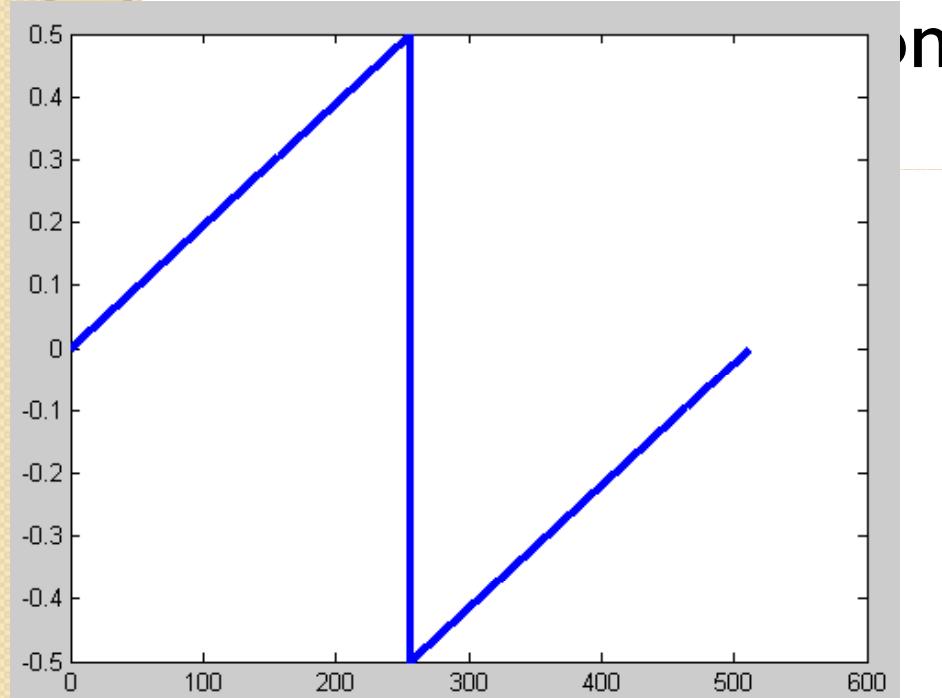
- Detecting hidden jump discontinuity
 - Consider function
-

$$g(t) = \begin{cases} t, & 0 \leq t < \frac{1}{2} \\ t - 1, & \frac{1}{2} \leq t < 1 \end{cases}$$

- Clear jump at $t=0.5$

Application

- Detecting hidden jump discontinuity



on

$$g(t) = \begin{cases} t, & 0 \leq t < \frac{1}{2} \\ t - 1, & \frac{1}{2} \leq t < 1 \end{cases}$$

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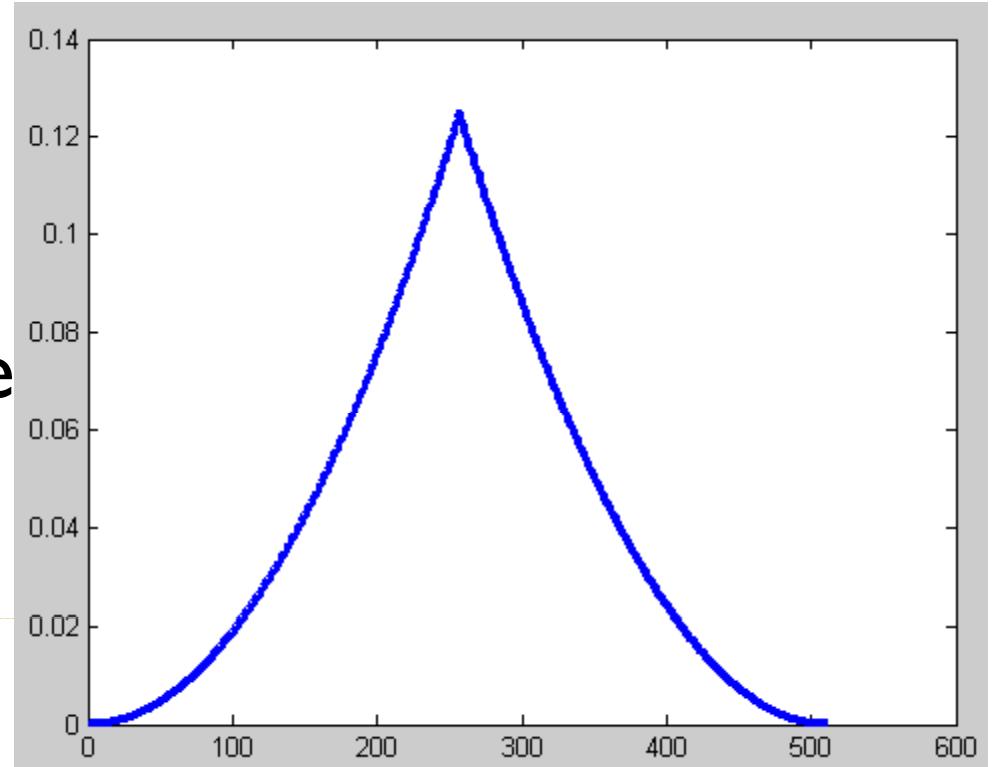
- Detecting hidden jump discontinuity
- Let's integrate

$$h(t) = \int g(t)dt = \begin{cases} \frac{t^2}{2}, & 0 \leq t < \frac{1}{2} \\ \frac{t^2}{2} - t + \frac{1}{2}, & \frac{1}{2} \leq t < 1 \end{cases}$$

- Cusp jump at $t=0.5$

Application

- Detecting hidden
- Let's integrate



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Application

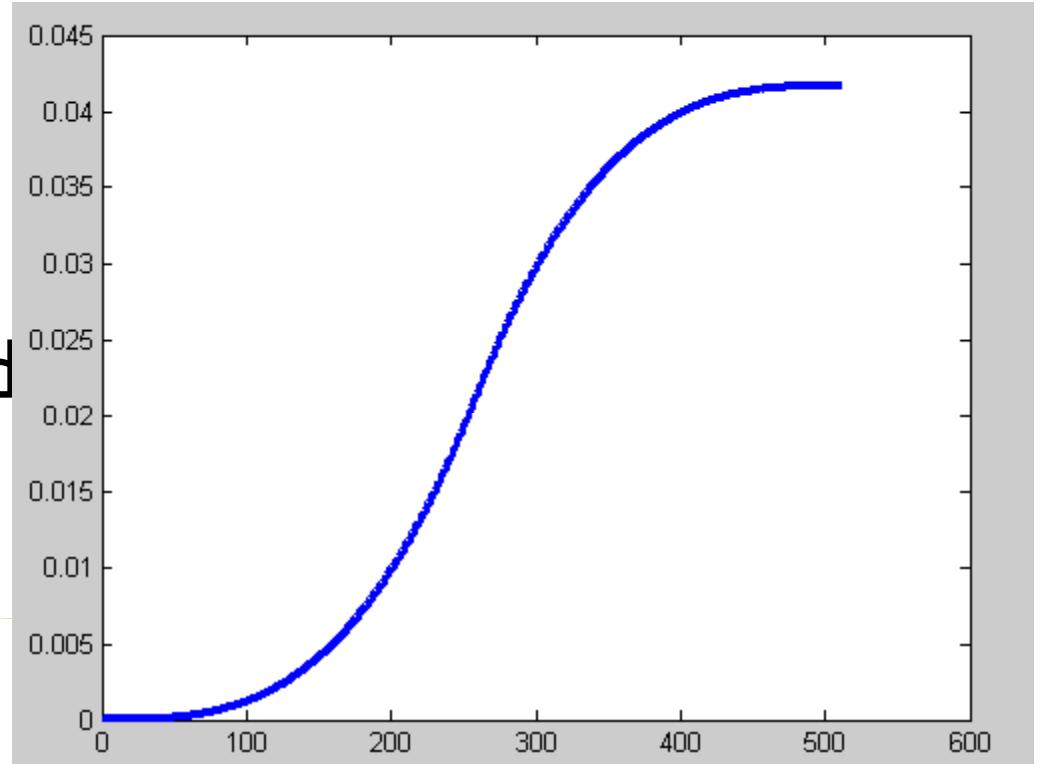
- Detecting hidden jump discontinuity
- Let's integrate again

$$f(t) = \int h(t)dt = \begin{cases} \frac{t^3}{6}, & 0 \leq t < \frac{1}{2} \\ \frac{t^3}{6} - \frac{t^2}{2} + \frac{t}{2} - \frac{1}{8}, & \frac{1}{2} \leq t < 1 \end{cases}$$

- Appears smooth to eye

Application

- Detecting hidden
- Let's integrate

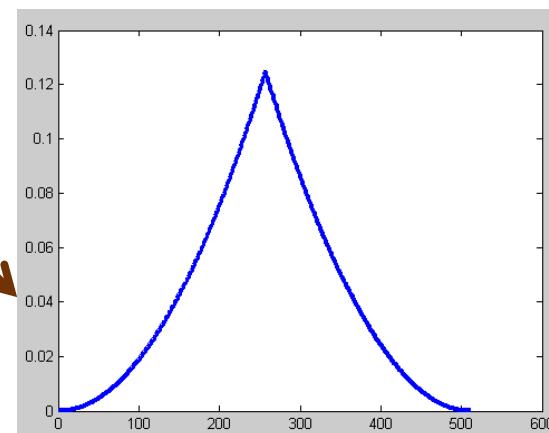
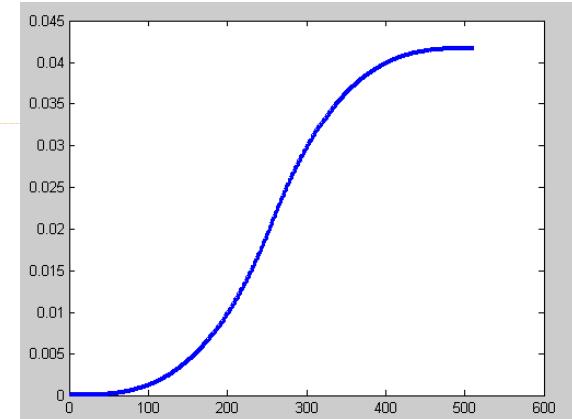
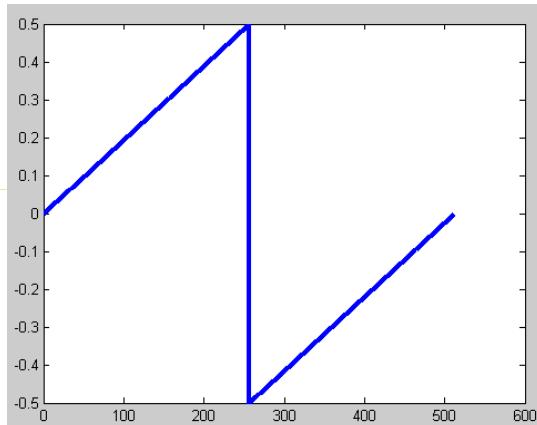


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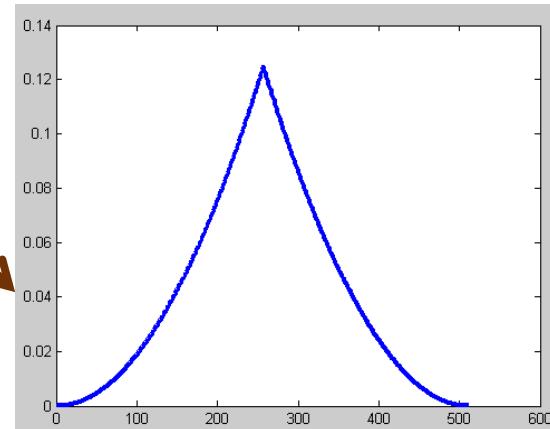
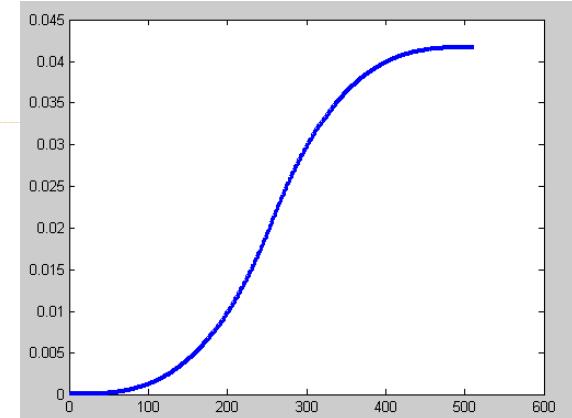
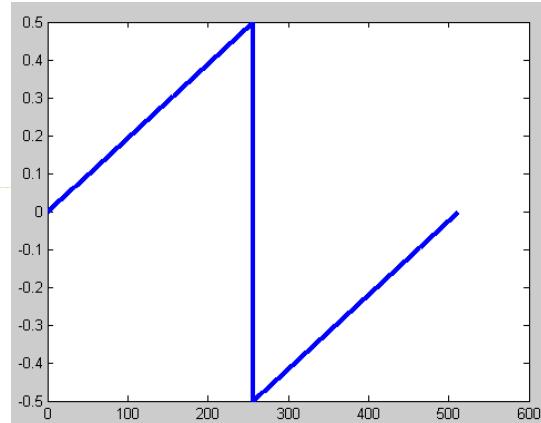
Application

- Detecting hidden jump discontinuity



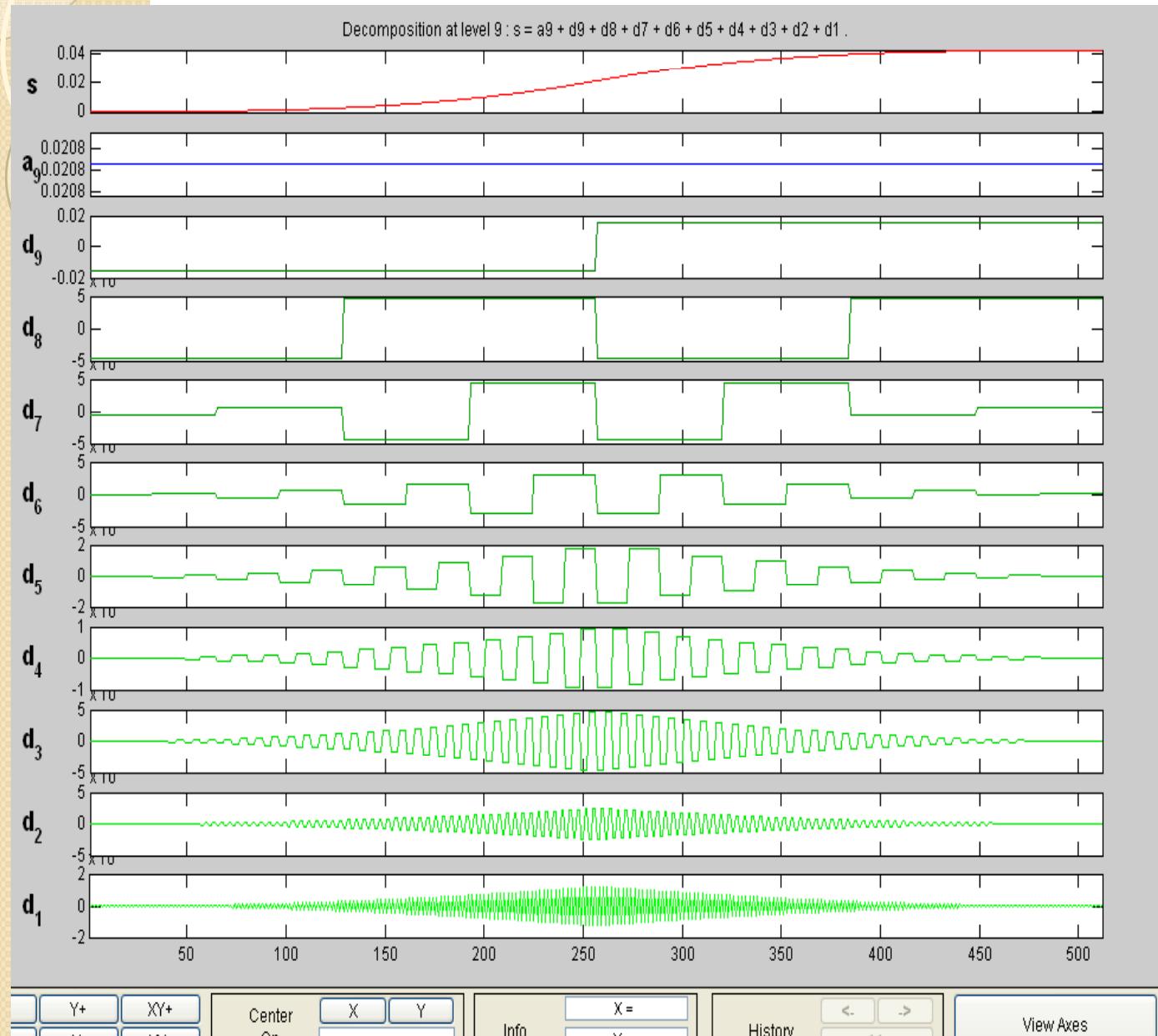
Application

- Detecting hidden jump discontinuity



While selecting
mother wavelet
At least $M_0=M_1=0$

Application



Data (Size)	f (512)
Wavelet	haar
Level	9

Analyze

Statistics Compress

Histograms De-noise

Display mode :

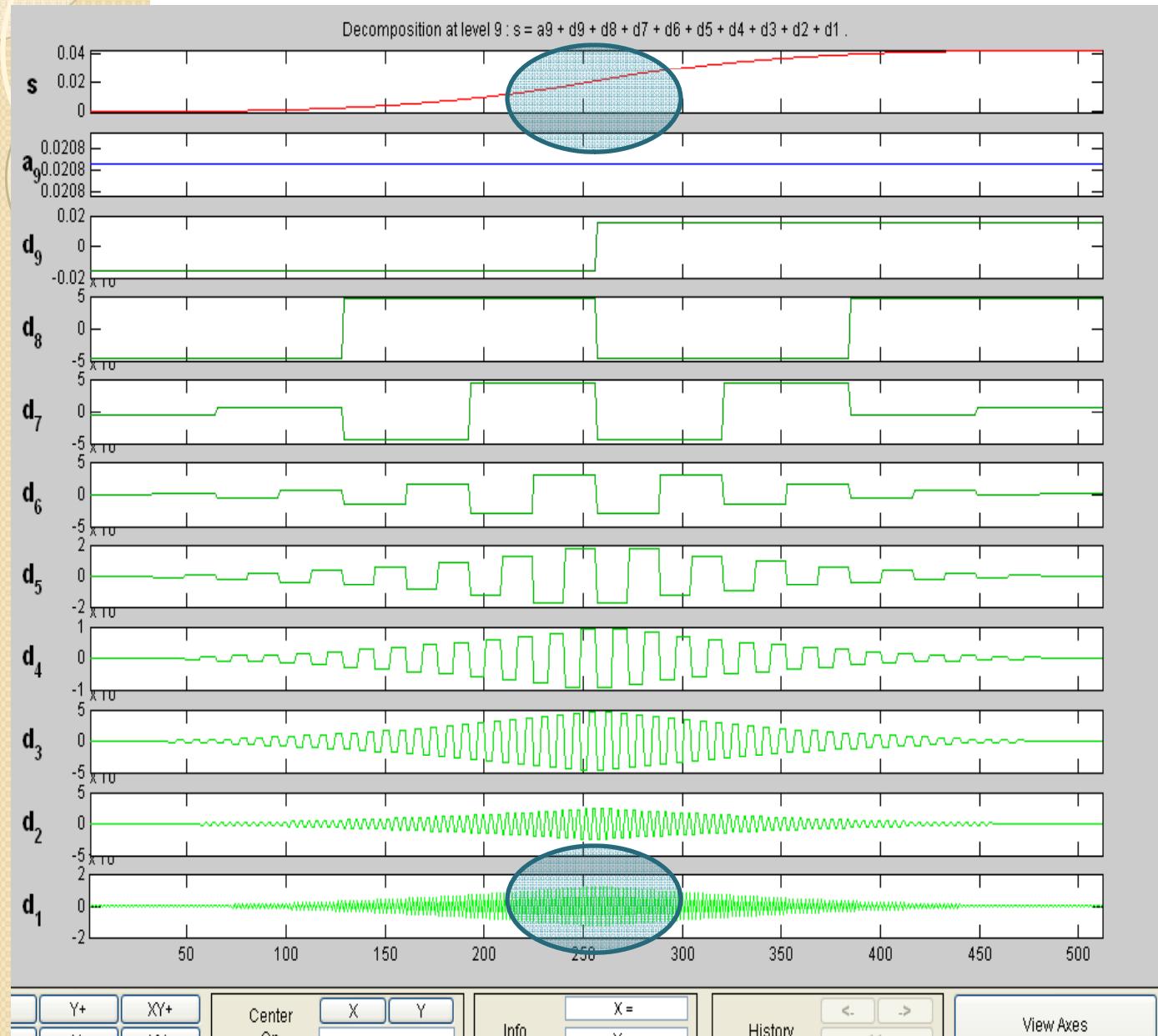
Full Decomposition

at level 9

Show Synthesized Sig.

Close

Application



Data (Size)	f (512)
Wavelet	haar
Level	9

Analyze

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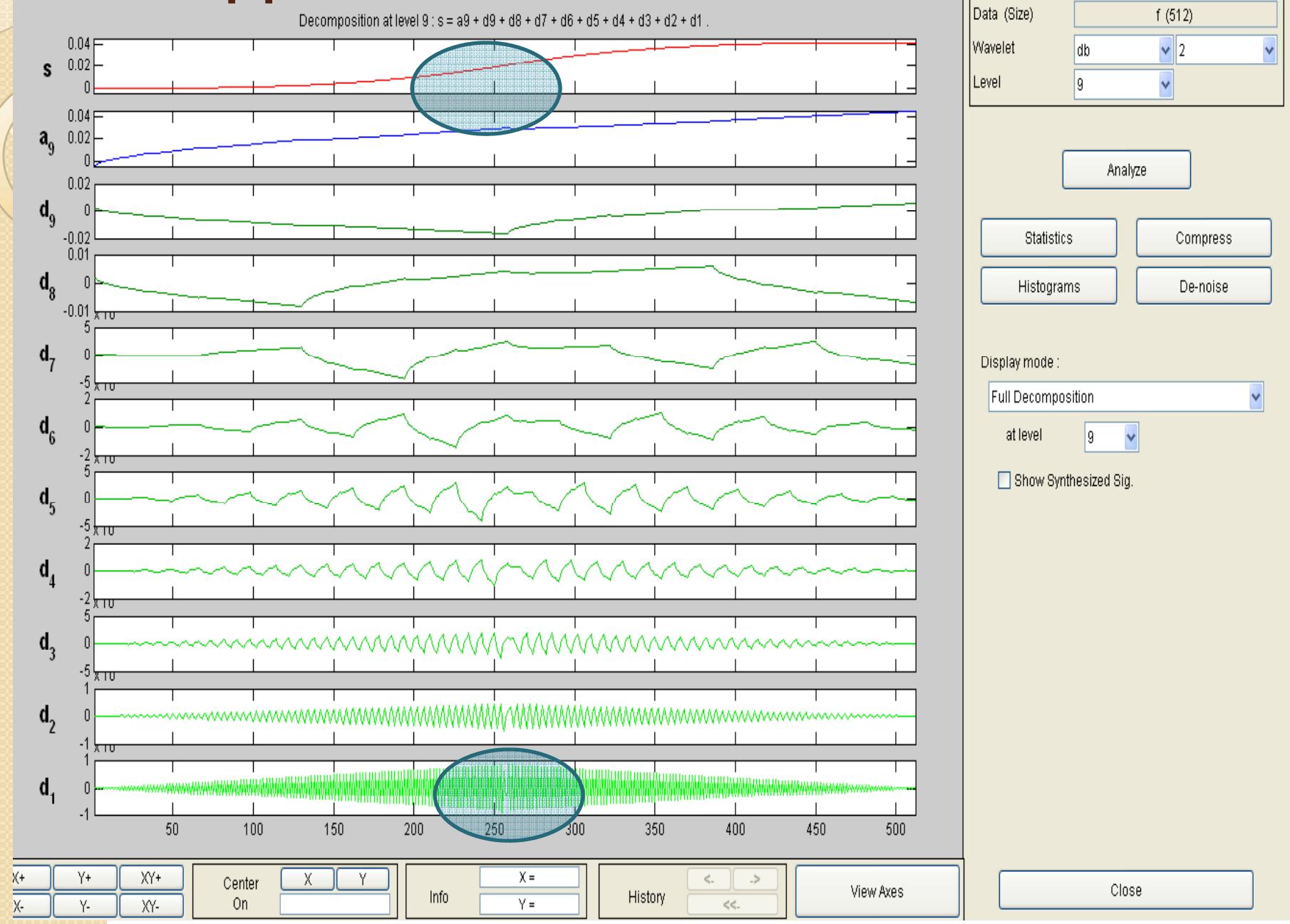
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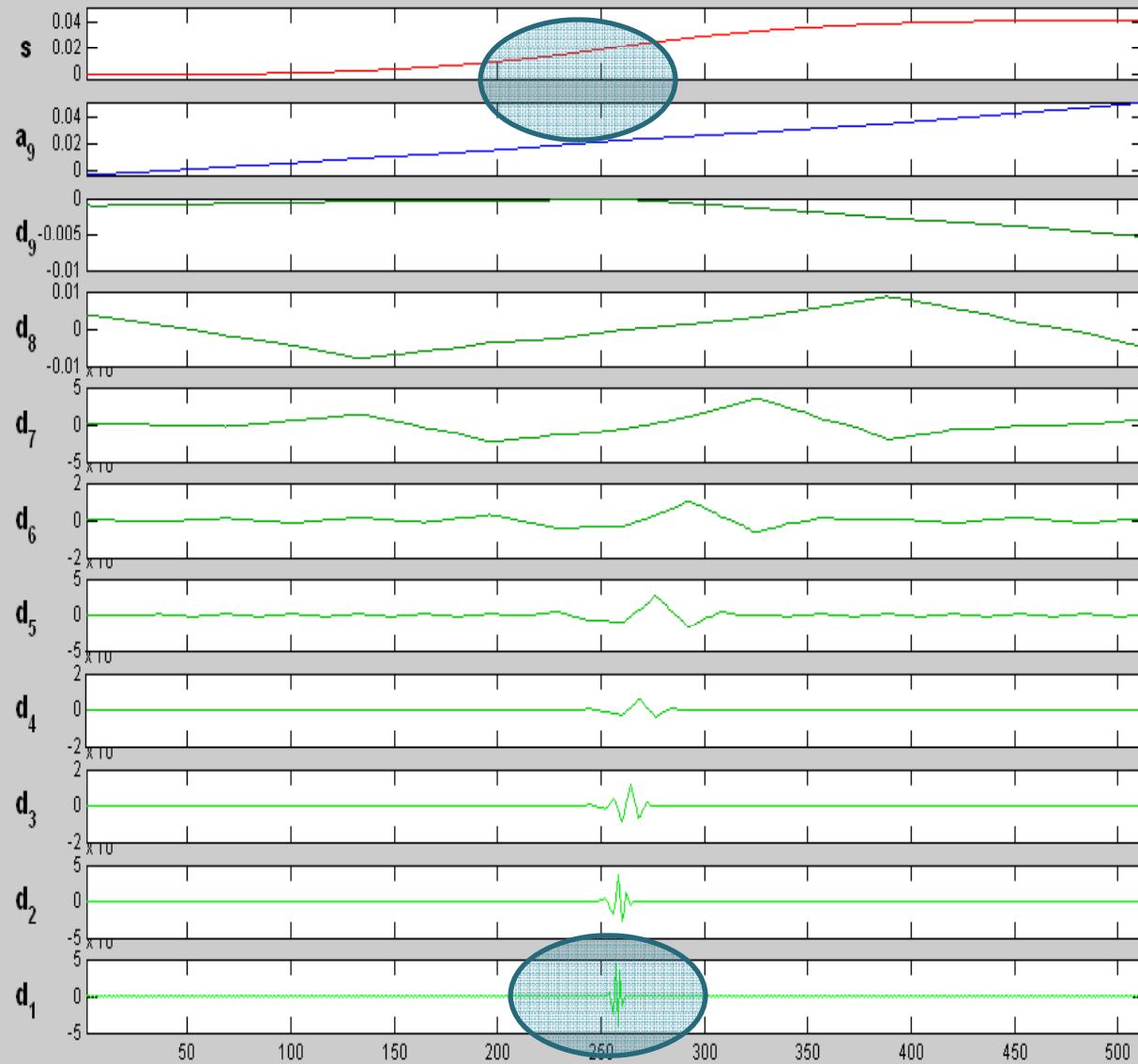
Close

Application



Application

Decomposition at level 9 : $s = a_9 + d_9 + d_8 + d_7 + d_6 + d_5 + d_4 + d_3 + d_2 + d_1$.



Data (Size)	f (512)
Wavelet	db 3
Level	9

Analyze

Statistics

Compress

Histograms

De-noise

Display mode :

Full Decomposition

at level

9

Show Synthesized Sig.

X+ Y+ XY+
X- Y- XY-

Center
On

X Y

Info

X =
Y =

History

<- >
<<

View Axes

Close

Framework

- Leads us to two questions
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Framework

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Framework

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Coefficients

- Who gives us coefficients of scaling equation?
- Haar

$$\psi(t) = \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ -1, & \frac{1}{2} \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\phi(t) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Framework

$$V_1 = V_0 \oplus W_0$$

$$V_0 \subset V_1$$

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In search of coefficients

- In search of scaling equation coefficients!!!

$$\phi(t) = \sqrt{2} \sum_k h_k \phi(2t - k)$$

$$\varphi(t) = \sqrt{2} \sum_k (-1)^k h_{1-k} \phi(2t - k)$$



In search of coefficients

- In search of scaling equation coefficients!!!
- We can thinks of using three guiding theorems !



In search of coefficients

- We can thinks of using three guiding theorems !
 - Theorem I:
-

For the scaling equation $\phi(x) = \sum_k h_k \sqrt{2} \phi(2x - k)$, with non-vanishing coefficients $\{h_k\}_{k=N}^M$ only for $N \leq k \leq M$, its $\phi(x)$ is with a compact support contained in interval $[N, M]$



In search of coefficients

- We can thinks of using three guiding theorems !
 - Theorem 2:
-

If the scaling function $\phi(x)$ has compact support on $0 \leq x \leq N - 1$ and if, $\{\phi(x - k)\}$ are linearly independent, then $h_n = h(n) = 0$, for $n < 0$ and $n > N - 1$.

Hence N is the length of the sequence.

In search of coefficients

- We can think of using three guiding theorems !

- Theorem 3:

If the scaling coefficients $\{h_k\}$ satisfy the condition for existence and orthogonality of $\phi(x)$, then

$$\varphi(x) = \sum_k g_k \sqrt{2} \phi(2x - k)$$

where, $g_k = \pm(-1)^k h_{N-k}$

and, $\int_{-\infty}^{\infty} \varphi(x - l) \phi(x - k) dx = \delta_{l,k} = 0, l \neq k$

Properties of scaling coefficients

$$1. \quad \sum h_k = \sqrt{2}$$

$$2. \quad \sum h_{2k} = \frac{1}{\sqrt{2}}$$

$$3. \quad \sum h_{2k+1} = \frac{1}{\sqrt{2}}$$



Properties of scaling coefficients

$$4. \quad \sum |h_k|^2 = 1$$

$$5. \quad \sum h_{k-2l} h_k = \delta_{l,0}$$

$$6. \quad \sum 2h_{k-2l} h_{k-2j} = \delta_{l,j}$$

Lets derive PI 1. $\sum h_k = \sqrt{2}$

$$\phi(t) = \sum_k h_k \sqrt{2} \phi(2t - k)$$

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(t) dt &= \int_{-\infty}^{\infty} [\sum_k h_k \sqrt{2} \phi(2t - k)] dt \\ &= \sum_k h_k \sqrt{2} \int_{-\infty}^{\infty} \phi(2t - k) dt\end{aligned}$$

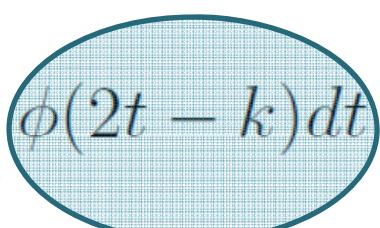
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$$\int_{-\infty}^{\infty} \phi(2t - k) dt = \frac{1}{2} \int_{-\infty}^{\infty} \phi(x) dx$$

Lets derive PI 1. $\sum h_k = \sqrt{2}$

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Lets derive P5 5. $\sum h_{k-2l}h_k = \delta_{l,0}$

$$\phi(t) = \sum_k h_k \sqrt{2} \phi(2t - k)$$

change of the variable t in $\phi(t)$ to $x = 2^{j-1}t - l$,

$$\phi(2^{j-1}t - l) = \sum_{k=-\infty}^{\infty} h_k \sqrt{2} \phi(2(2^{j-1}t - l) - k)$$

change in the index k to $m = k + 2l$,

$$\begin{aligned}\phi(2^{j-1}t - l) &= \sum_{m=-\infty}^{\infty} h_{m-2l} \sqrt{2} \phi(2^j t - 2l - m + 2l) \\ &= \sum_{m=-\infty}^{\infty} h_{m-2l} \sqrt{2} \phi(2^j t - m)\end{aligned}$$

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do this we multiply both sides of the above equation by $\phi(t)$, and integrate from $-\infty$ to ∞ , allowing the exchange of the integration

$$\int_{-\infty}^{\infty} \phi(t) \phi(2^{j-1}t - l) dt = \sum_{k=-\infty}^{\infty} h_{k-2l} \int_{-\infty}^{\infty} \sqrt{2} \phi(t) \phi(2^j t - k) dt$$

Lets derive P5 5. $\sum h_{k-2l}h_k = \delta_{l,0}$

$$\int_{-\infty}^{\infty} \phi(t)\phi(t-l)dt = \delta_{0,l} = \sum_{k=-\infty}^{\infty} h_{k-2}h_k$$

$$\sum_{k=-\infty}^{\infty} h_{k-2l}h_k = \delta_{l,0}$$

Lets derive P4 4. $\sum |h_k|^2 = 1$

special case of $l = 0$ gives the property 4

$$\sum_{k=-\infty}^{\infty} h_{k-2l} h_k = \delta_{l,0}$$

$$\sum_{k=-\infty}^{\infty} h_k \overline{h_k} = \sum_{k=-\infty}^{\infty} |h_k|^2 = \delta_{0,0} 1$$

Lets derive P3

$$3. \quad \sum h_{2k+1} = \frac{1}{\sqrt{2}}$$

$$\delta_{l,0} = \sum_{k=-\infty}^{\infty} 2h_{k-2l}\overline{h_k} = 2 \sum_{k=-\infty}^{\infty} h_{k+2l}\overline{h_k}$$

$$\delta_{l,0} = 2 \sum_{k=-\infty}^{\infty} h_{2k+2l}\overline{h_{2k}} + 2 \sum_{k=-\infty}^{\infty} h_{2k+1+2l}\overline{h_{2k+1}}$$

$$\sum_{l=-\infty}^{\infty} \delta_{l,0} = 1 = 2 \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} [h_{2k+2l}h_{2k} + h_{2k+1+2l}h_{2k+1}]$$

Lets derive P3

$$3. \quad \sum h_{2k+1} = \frac{1}{\sqrt{2}}$$

$$1 = \sum_{k=-\infty}^{\infty} \overline{h_{2k}} \left[\sum_{l=-\infty}^{\infty} h_{2k+2l} \right] + \sum_{k=-\infty}^{\infty} \overline{h_{2k+1}} \left[\sum_{l=-\infty}^{\infty} h_{2k+1+2l} \right]$$

$$\sum_{l=-\infty}^{\infty} h_{2k+2l} = \sum_{l=-\infty}^{\infty} h_{2l} = \sum_{l=-\infty}^{\infty} h_{2k} \equiv A$$

$$\sum_{l=-\infty}^{\infty} h_{2k+2l+1} = \sum_{l=-\infty}^{\infty} h_{2l+1} = \sum_{l=-\infty}^{\infty} h_{2k+1} \equiv B$$



Lets derive P3

$$3. \quad \sum h_{2k+1} = \frac{1}{\sqrt{2}}$$

$$1 = \sum_{k=-\infty}^{\infty} \overline{h_{2k}} \sum_{k=-\infty}^{\infty} h_{2k} + \sum_{k=-\infty}^{\infty} \overline{h_{2k+1}} \sum_{k=-\infty}^{\infty} h_{2k+1}$$

$$1 = \overline{A}A + \overline{B}B = |A|^2 + |B|^2$$

Lets derive P3

$$3. \quad \sum h_{2k+1} = \frac{1}{\sqrt{2}}$$

$$\sum_k h_k = \sqrt{2}$$

$$\sqrt{2} = \sum_{k=-\infty}^{\infty} h_k = \sum_{k=-\infty}^{\infty} h_{2k} + \sum_{k=-\infty}^{\infty} h_{2k+1} = A + B$$

$$A + B = \sqrt{2}$$

Lets derive P3

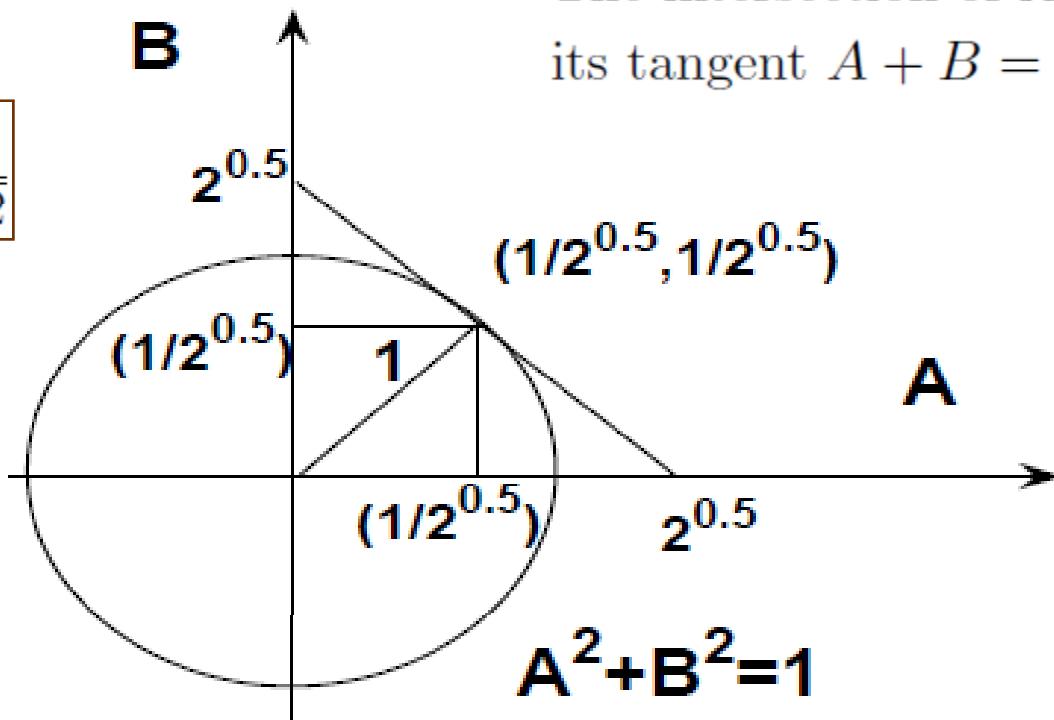
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$$A + B = \sqrt{2}$$

The intersection of $A^2 + B^2 = 1$ and its tangent $A + B = \sqrt{2}$ at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

$$A = B = \frac{1}{\sqrt{2}}$$



Lets derive P3

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Lets derive P2 2. $\sum h_{2k} = \frac{1}{\sqrt{2}}$

$$A = B = \frac{1}{\sqrt{2}}$$

$$\sum_{l=-\infty}^{\infty} h_{2k+2l} = \sum_{l=-\infty}^{\infty} h_{2l} = \sum_{l=-\infty}^{\infty} h_{2k} \equiv A$$

$$2. \quad \sum h_{2k} = \frac{1}{\sqrt{2}}$$

Verification for Haar

$$h_0 = h_1 = \frac{1}{\sqrt{2}}, h_k = 0 \text{ for } k \neq 0, 1,$$

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$$2. \quad \sum_{k=0} h_{2k} = h_0 = \frac{1}{\sqrt{2}}$$

$$3. \quad \sum_{k=1} h_{2k+1} = h_1 = \frac{1}{\sqrt{2}}$$

Verification for Haar

$$h_0 = h_1 = \frac{1}{\sqrt{2}}, \quad h_k = 0 \text{ for } k \neq 0, 1,$$

$$4. \quad \sum_{k=0,1} |h_k|^2 = |h_0|^2 + |h_1|^2 = \frac{1}{2} + \frac{1}{2} = 1$$

Verification for Haar

$$h_0 = h_1 = \frac{1}{\sqrt{2}}, h_k = 0 \text{ for } k \neq 0, 1,$$

$$5. \quad \sum_k 2h_{k-2l}h_k$$

$$\begin{aligned} \sum_{k=0,1;l=0} 2h_{k-2l}\overline{h_k} &= 2[h_0\overline{h_0} + h_1\overline{h_1}] \\ &= 2\left[\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right] = 2\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] = 1 \end{aligned}$$

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**Discrete
Orthonormality**

$$\sum_k 2h_{k-2l}\overline{h_k} = \delta_{l,0}$$

Not all function obey!

- Roof scaling function

$$\phi(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \end{cases}$$

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- Roof scaling function

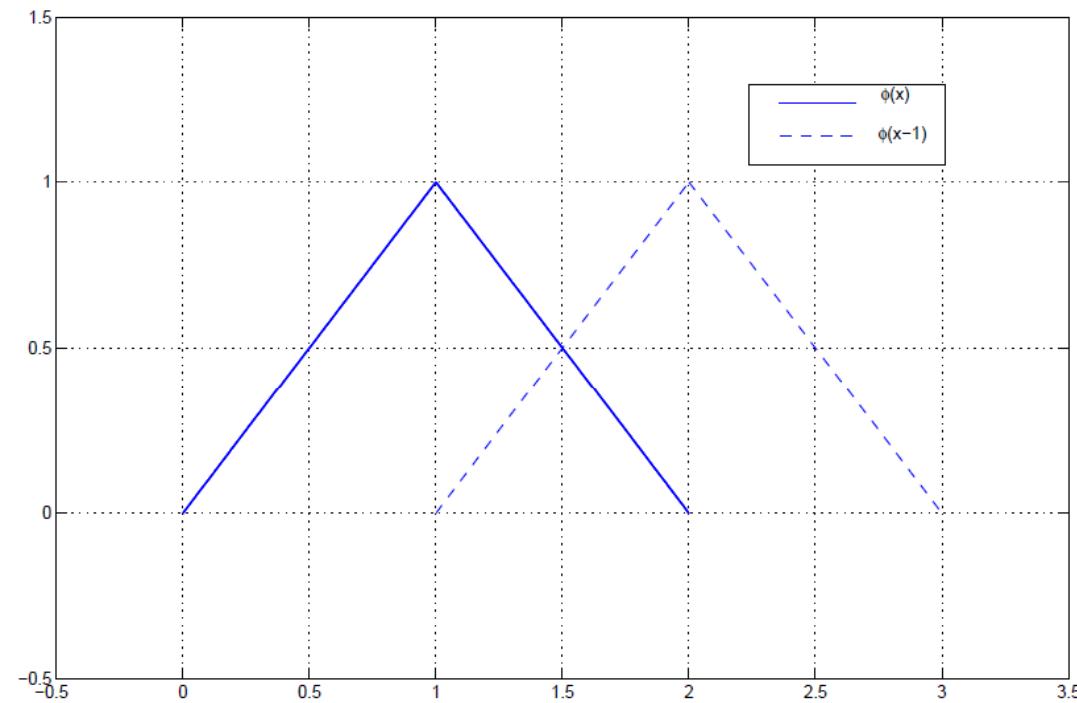
$$\phi(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \end{cases}$$

$$\int_{-\infty}^{\infty} \phi(x)\phi(x-1)dx = \int_1^2 \phi(x)\phi(x-1)dx \neq 0$$

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Framework

- Leads us to two questions
 - 1) How do we go about selecting the **mother wavelet** and **scale of analysis?**
 - 2) What is the procedure to calculate **scaling and wavelet coefficients?**



Applications

- Detecting (hidden) jumps/discontinuities
- Suppressing polynomials towards denoising
- 2D applications: compression and pattern recognition



Properties of Unitary Transform

- Energy **compaction**: only few transform coefficients have large magnitude
 - Such property is related to the decorrelating role of unitary transform
- Energy **conservation**: unitary transform preserves the 2-norm of input vectors
 - Such property essentially comes from the fact that rotating coordinates does not affect Euclidean distance

Unitary Matrix and 1D Unitary Transform

Definition

A matrix A is called **unitary** if $A^{-1} = A^*{}^T$

conjugate
transpose

When the transform matrix A is unitary, the defined transform $\vec{y} = A\vec{x}$ is called **unitary transform**

Example

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, A^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = A^T$$

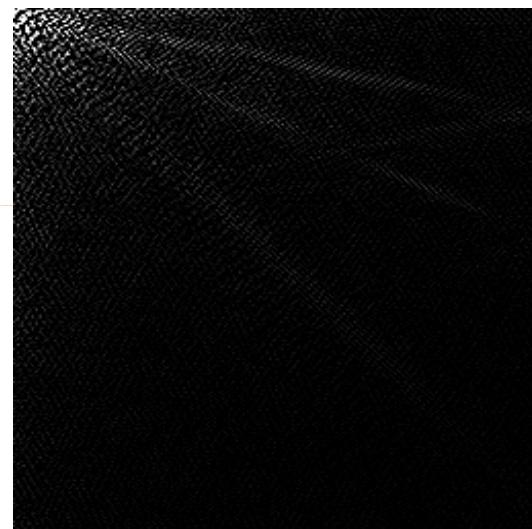
For a real matrix A , it is unitary if $A^{-1} = A^T$

2D DCT

$$\mathbf{Y} = \mathbf{C}\mathbf{X}\mathbf{C}^T = \mathbf{C}\mathbf{X}\mathbf{C}^{-1}$$



Original cameraman image \mathbf{X}



Its DCT coefficients \mathbf{Y}
(**2451** significant coefficients, $th=64$)

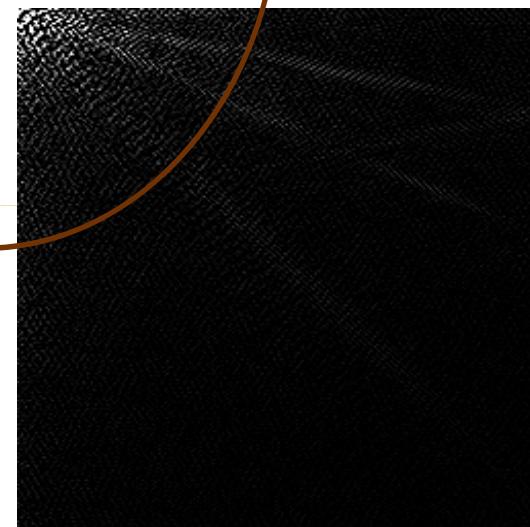
Notice the excellent energy compaction property of DCT

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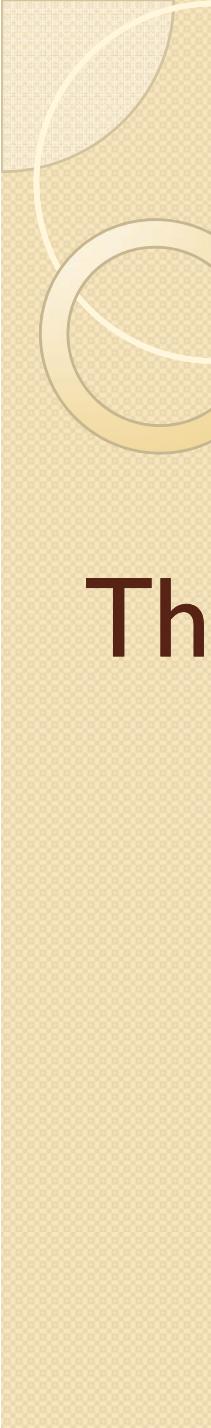


Original cameraman image \mathbf{X}



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Thank You!

Questions ??