Lecture 48 – Towards selecting wavelets through vanishing moments

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Wavelet Transform

- Decomposes signal into two separate series

- Single series to represent most coarse version
  - Scaling Function

- Double series to represent refined version
  - Wavelet Function

Wavelet Transform: Specialty

- Scaling and Translation are indeed <u>Hallmarks</u> of Wavelet transform

- They lead us to MultiResolutioAnalysis (MRA) !!

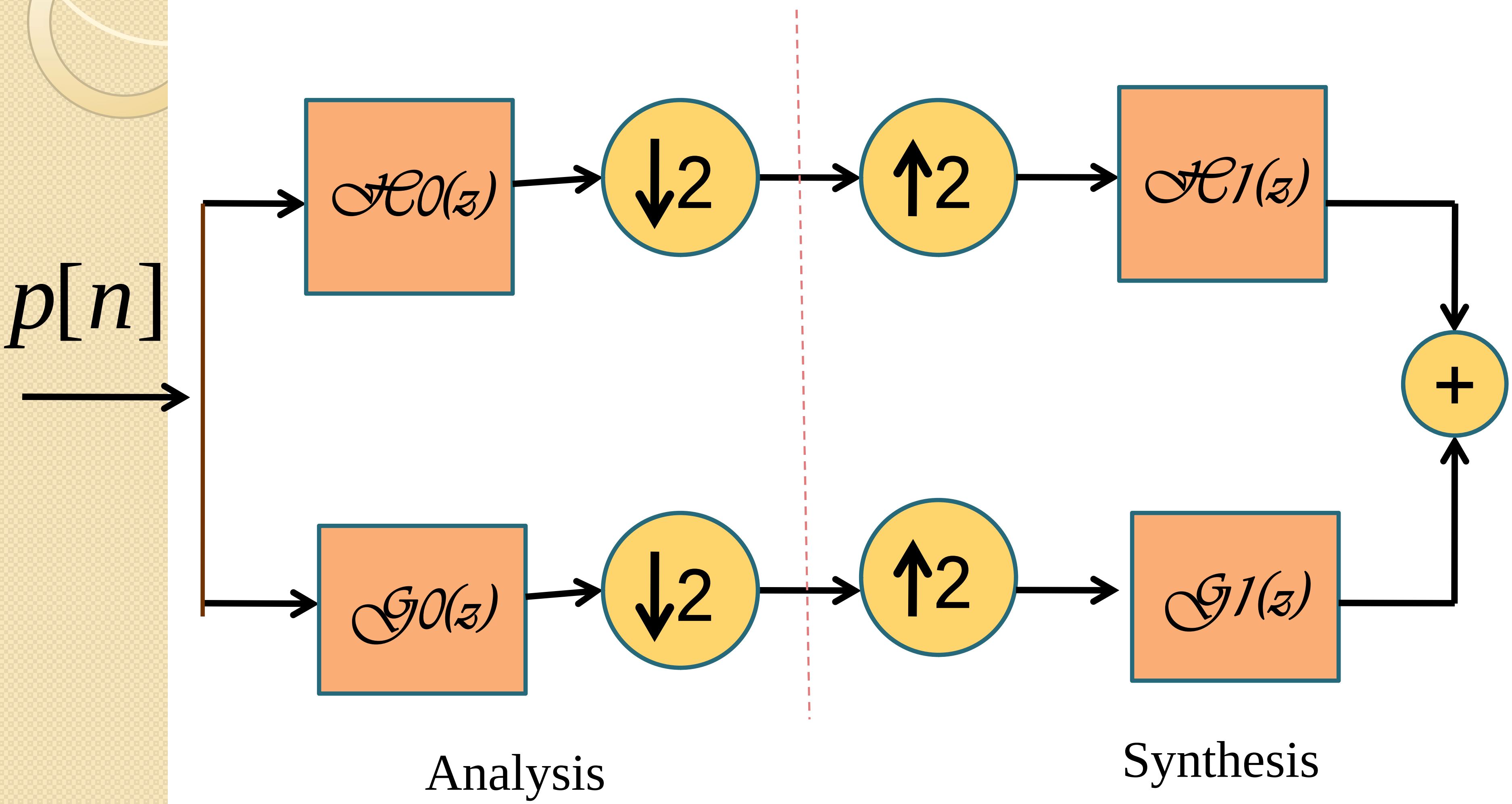
Relationship

- As the spaces and spans are clear now
- Intuitionally, we observe a relationship between these spaces!

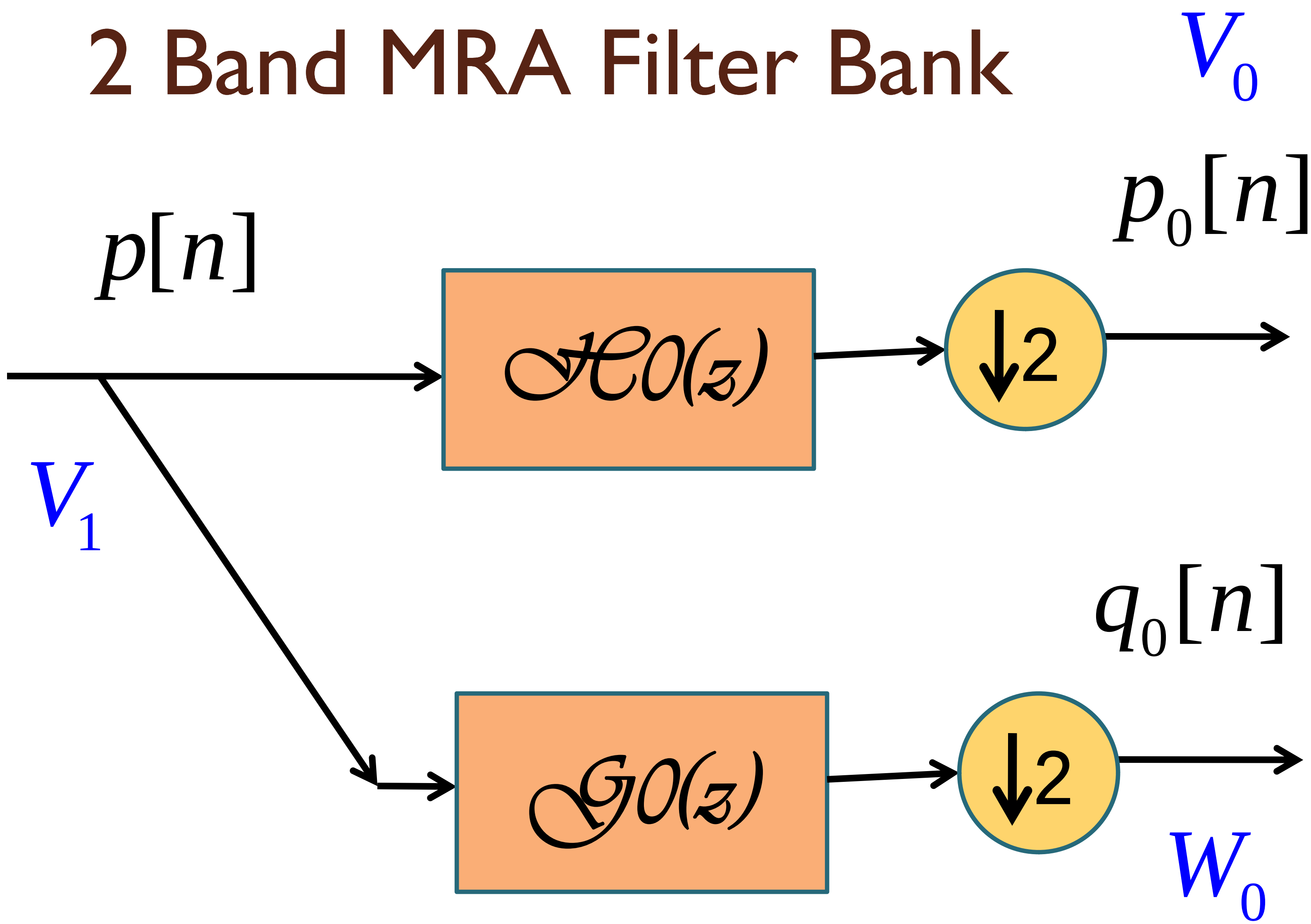
$$\ldots\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots\ldots$$

- Intuitively we can see that as we move towards right, i.e. ***up the ladder***, we are moving towards $L_2(\Re)$

2 Band MRA Filter Bank



2 Band MRA Filter Bank



Framework

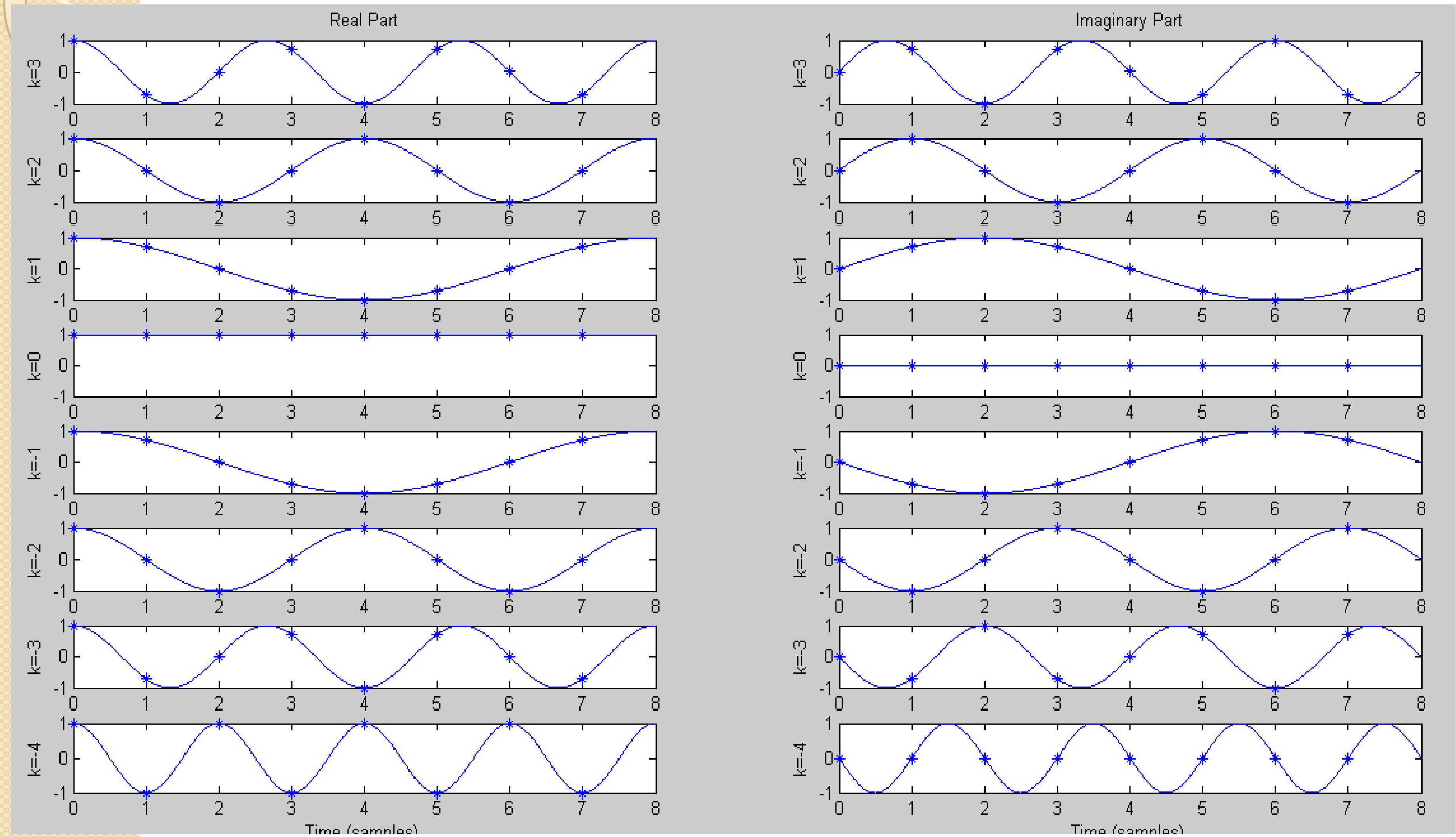
- Gave us power to move up or down the ladder
- We can now indeed zoom-in or zoom-out of any part of the signal
- This makes the entire analysis ‘scalable’!!
- Scalability stems out of multi-resolution framework !

Framework

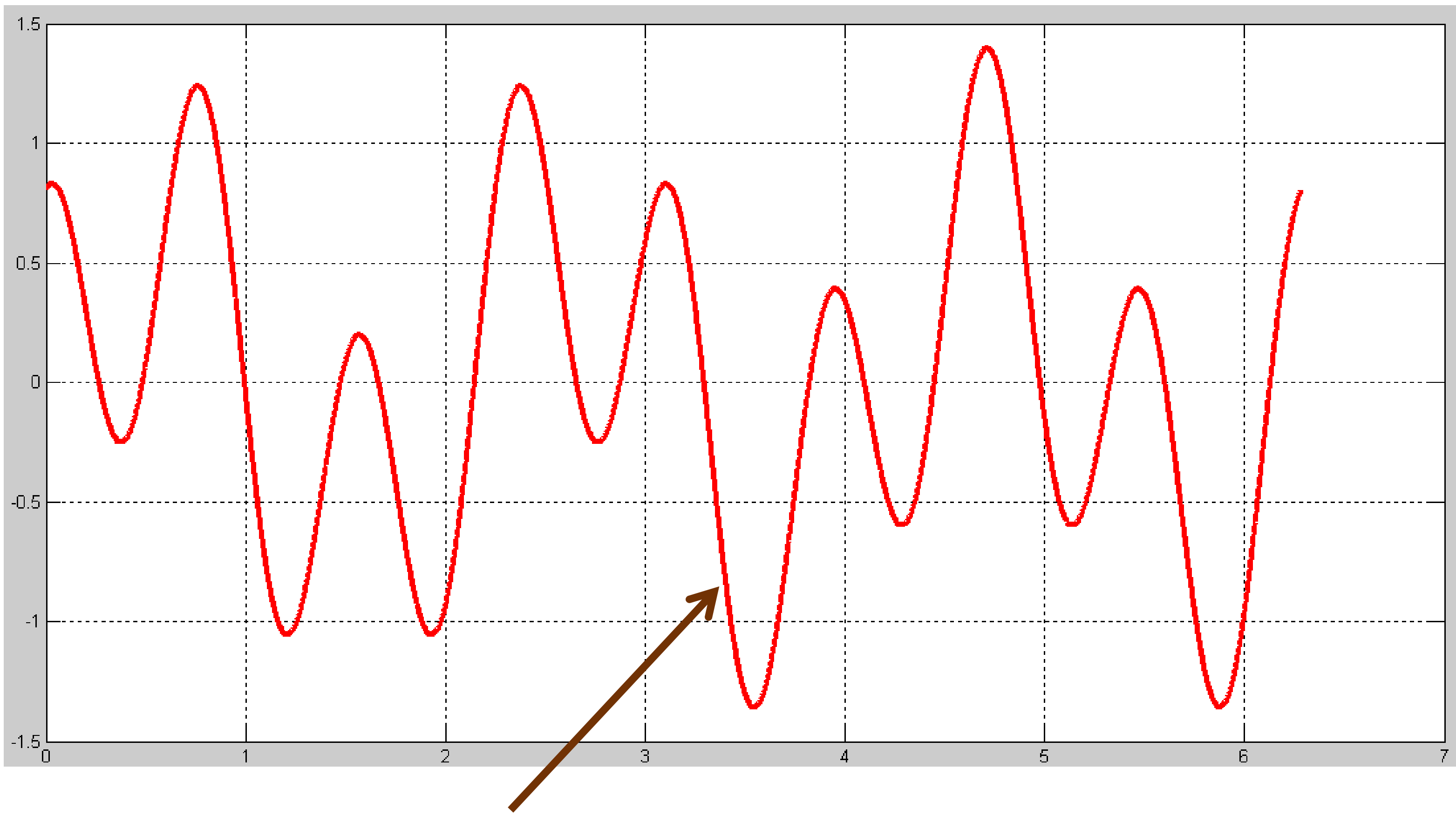
- Leads us to two questions

1) How do we go about selecting the mother wavelet and scale of analysis?
2) What is the procedure to calculate scaling and wavelet coefficients?

How Fourier Works – Basis Functions !!



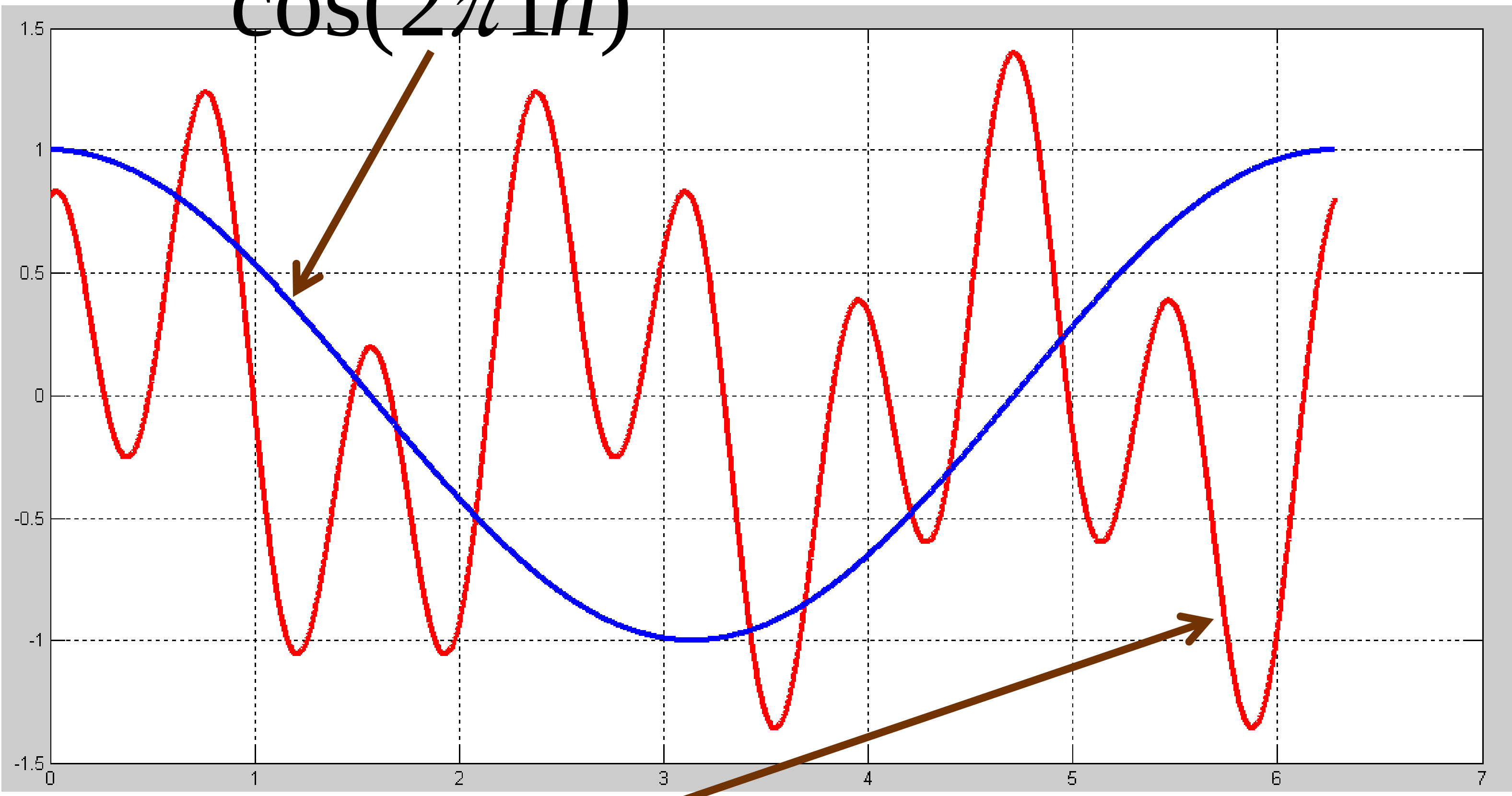
How Fourier Works!!



$$y[n] = 0.6\sin(2\pi 3n) + 0.8\cos(2\pi 8n)$$

How Fourier Works!!

$$\cos(2\pi 1n)$$



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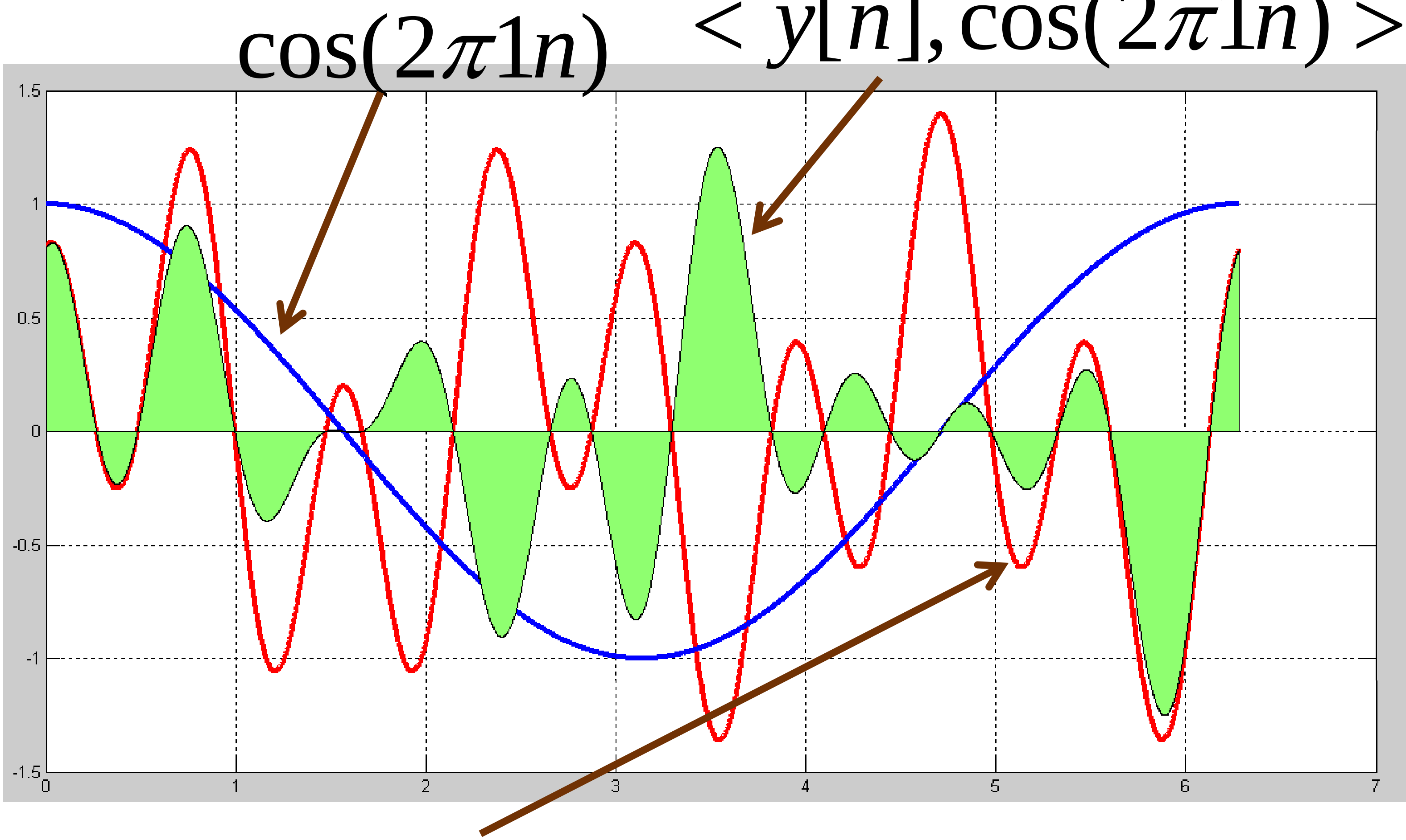

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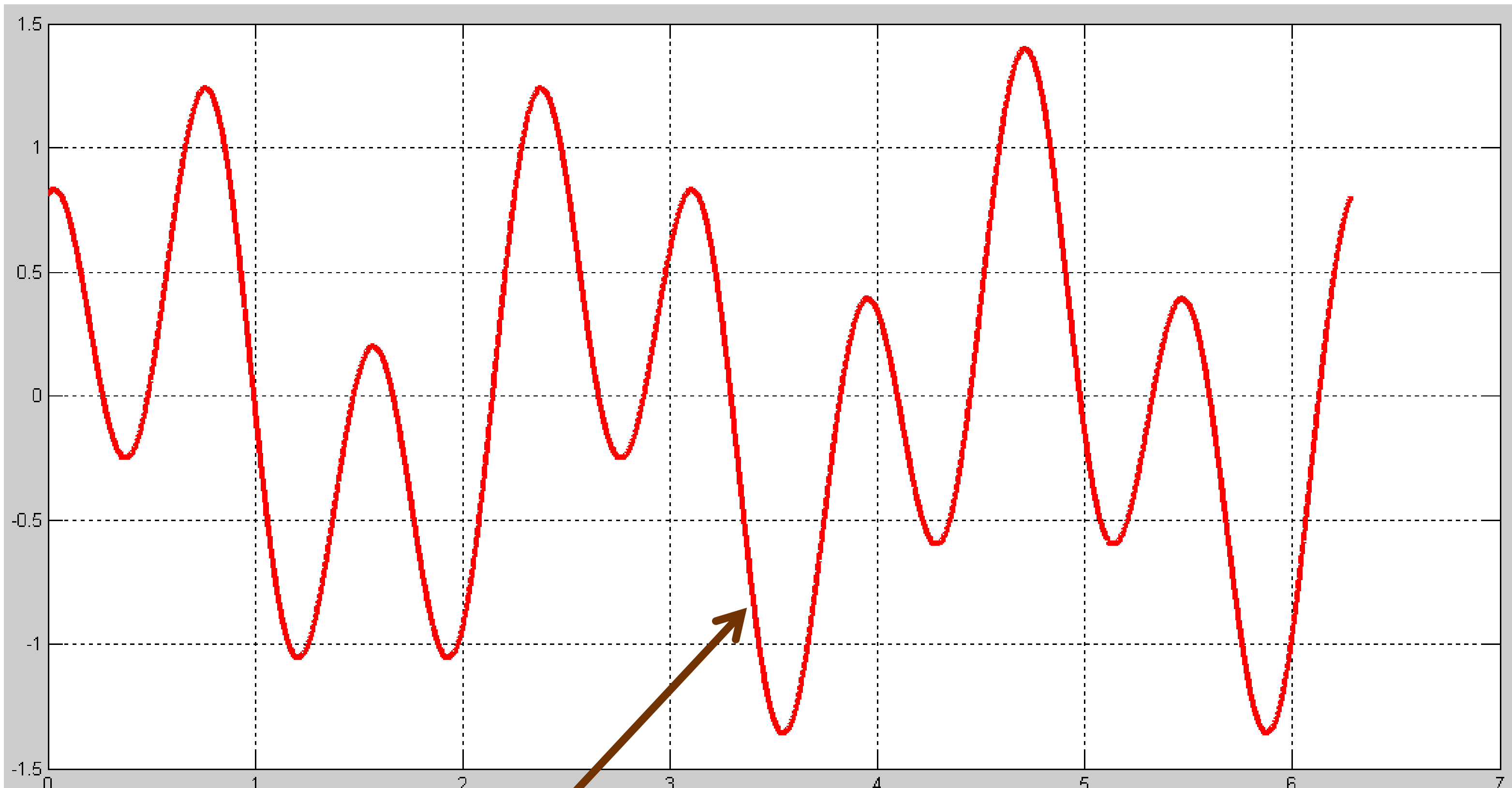

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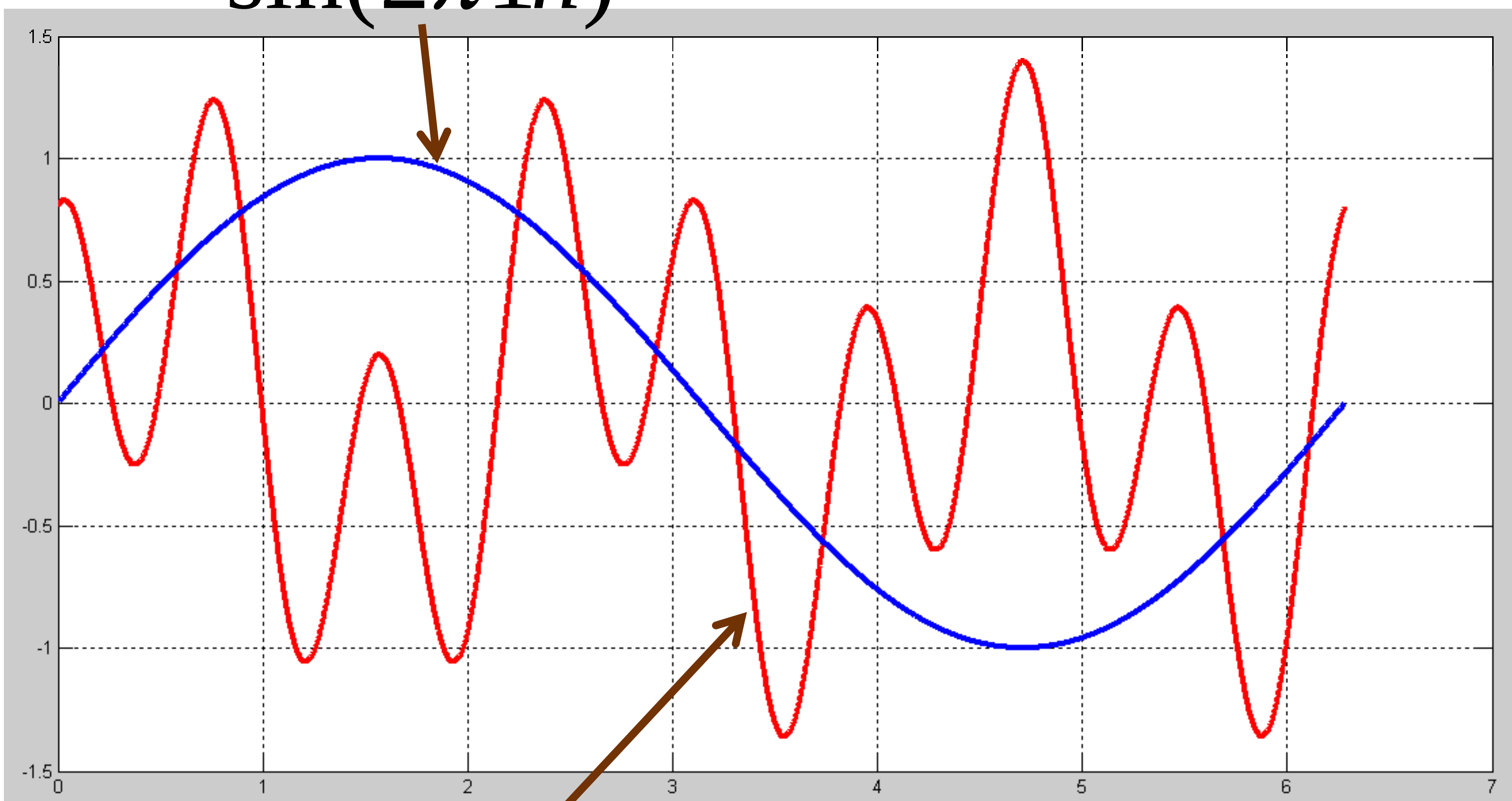
How Fourier Works!!

$\sin(2\pi 1 n)$



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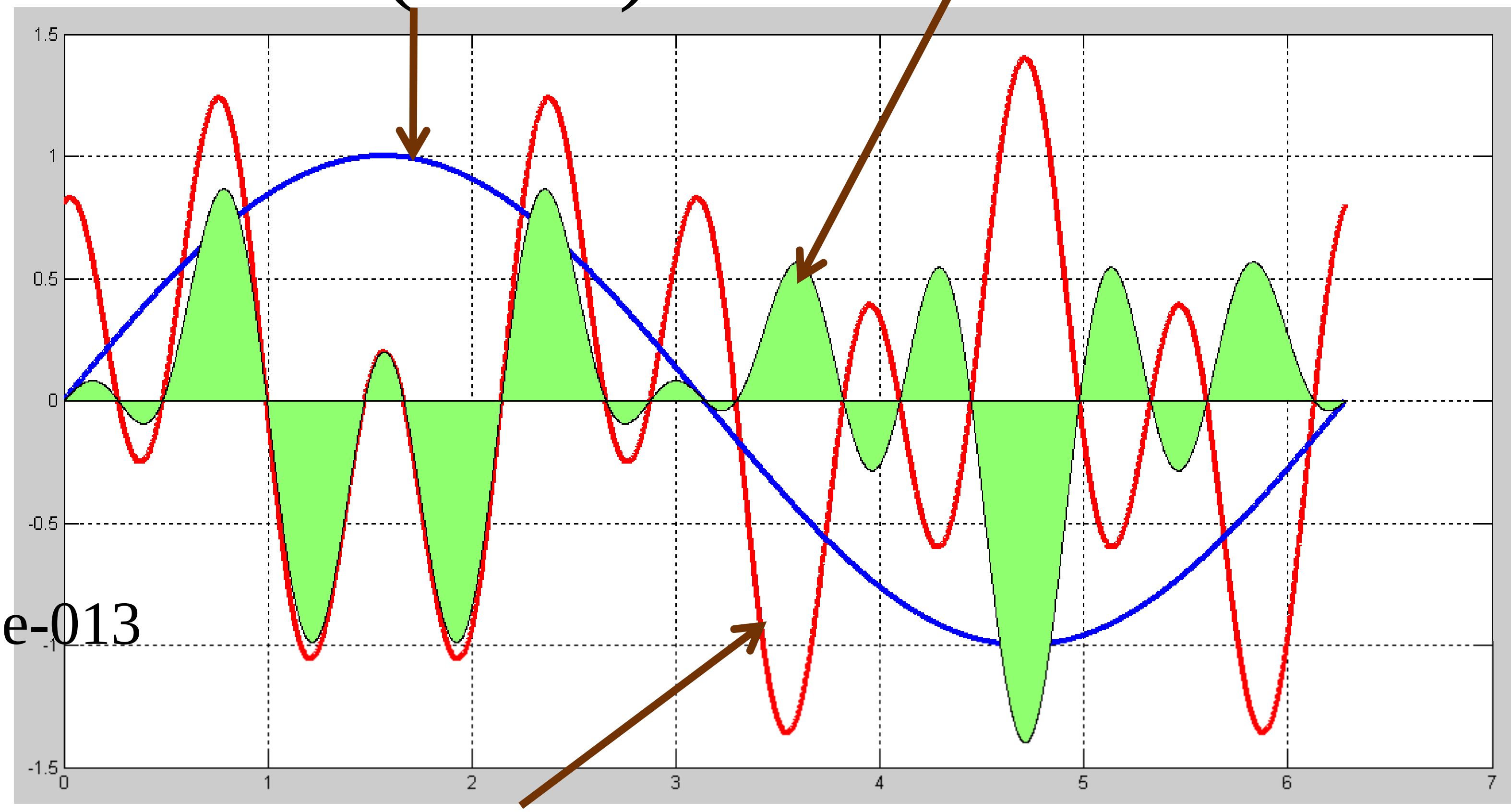
How Fourier Works!!

$\sin(2\pi 1n)$ $< y[n], \sin(2\pi 1n) >$



5.4573e-013

$$y[n] = 0.6\sin(2\pi 3n) + 0.8\cos(2\pi 8n)$$

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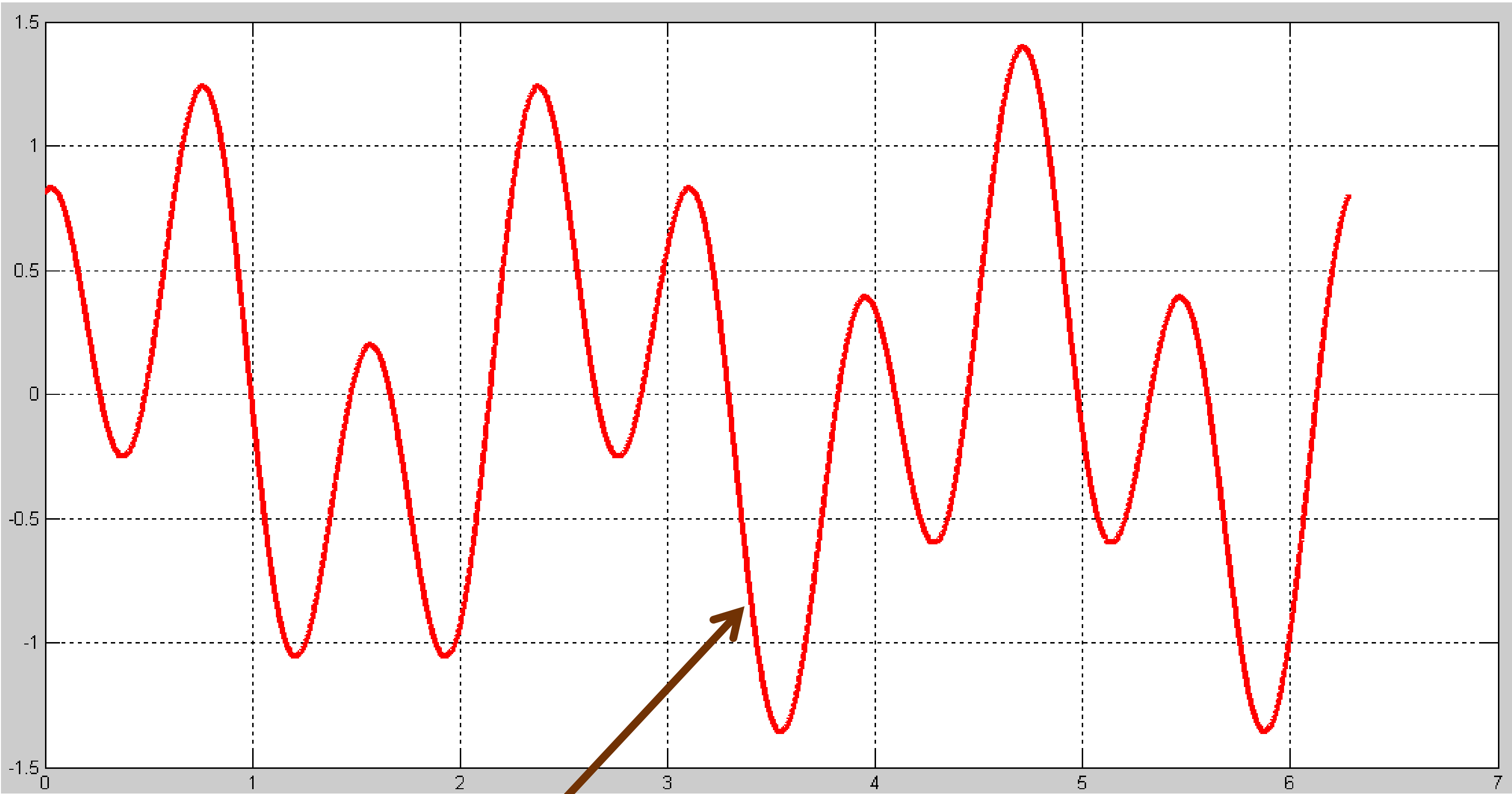

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How Fourier Works!!

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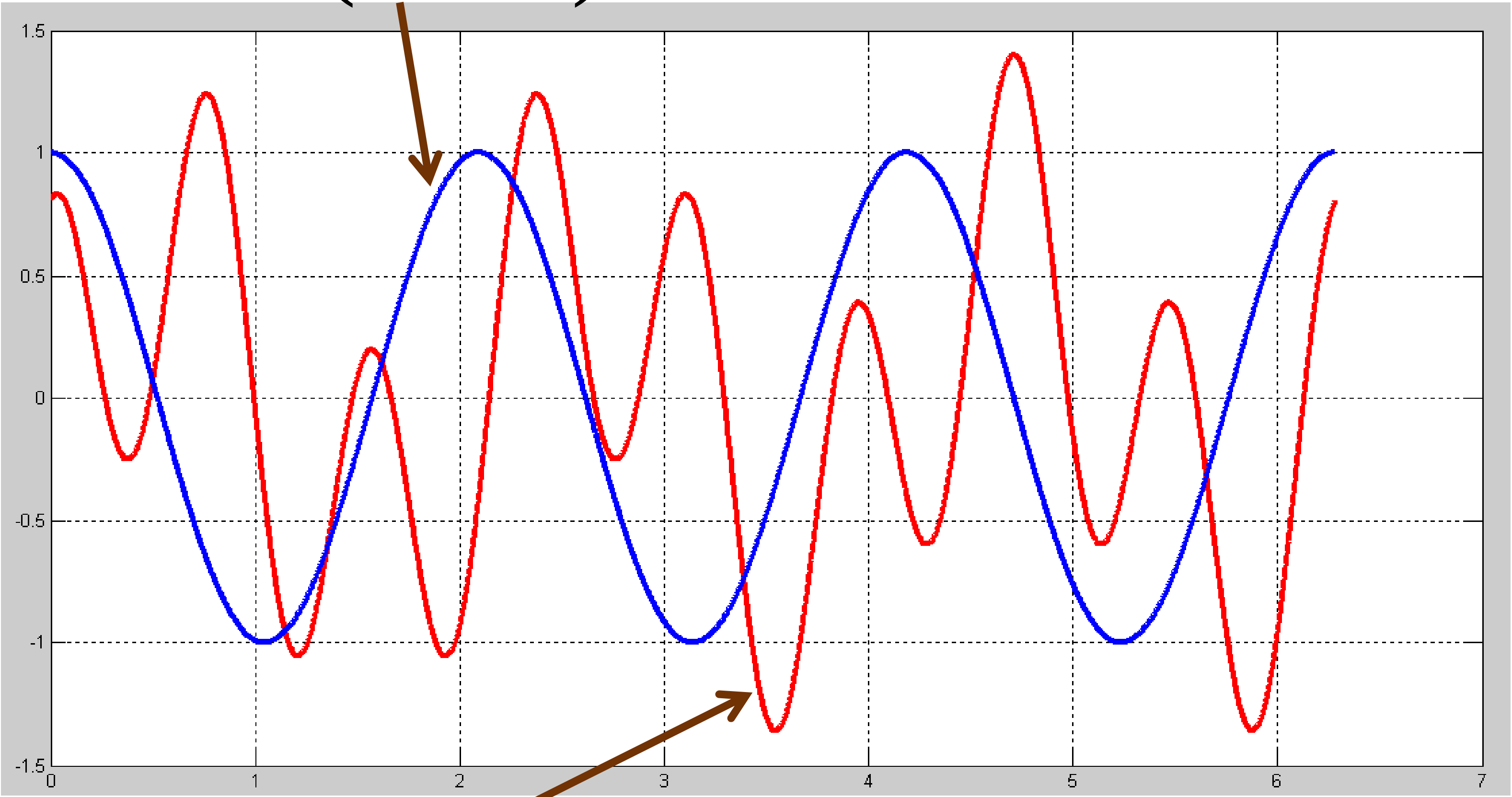

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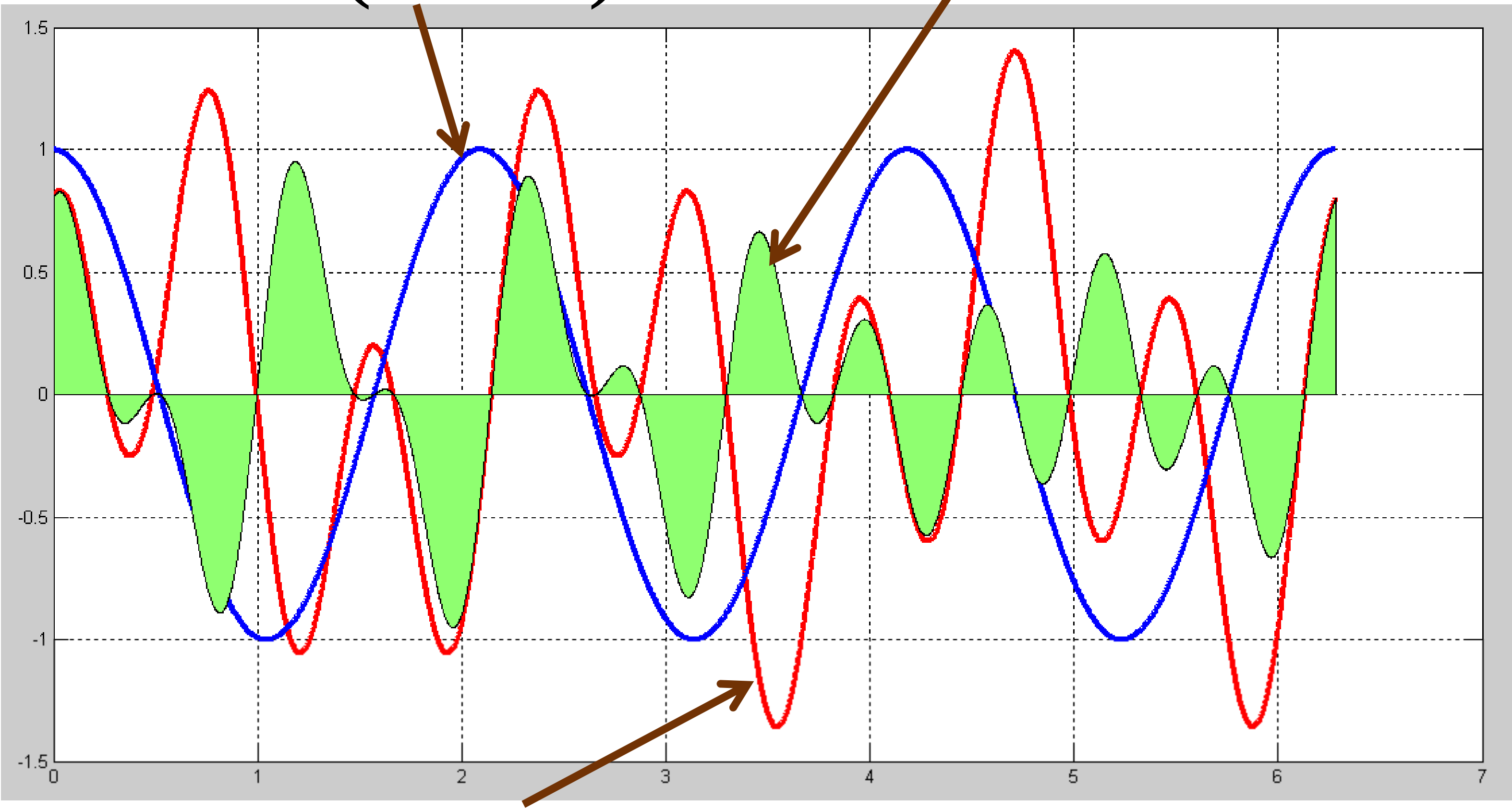
How Fourier Works!!

$\cos(2\pi 3n)$ $\quad < y[n], \cos(2\pi 3n) >$



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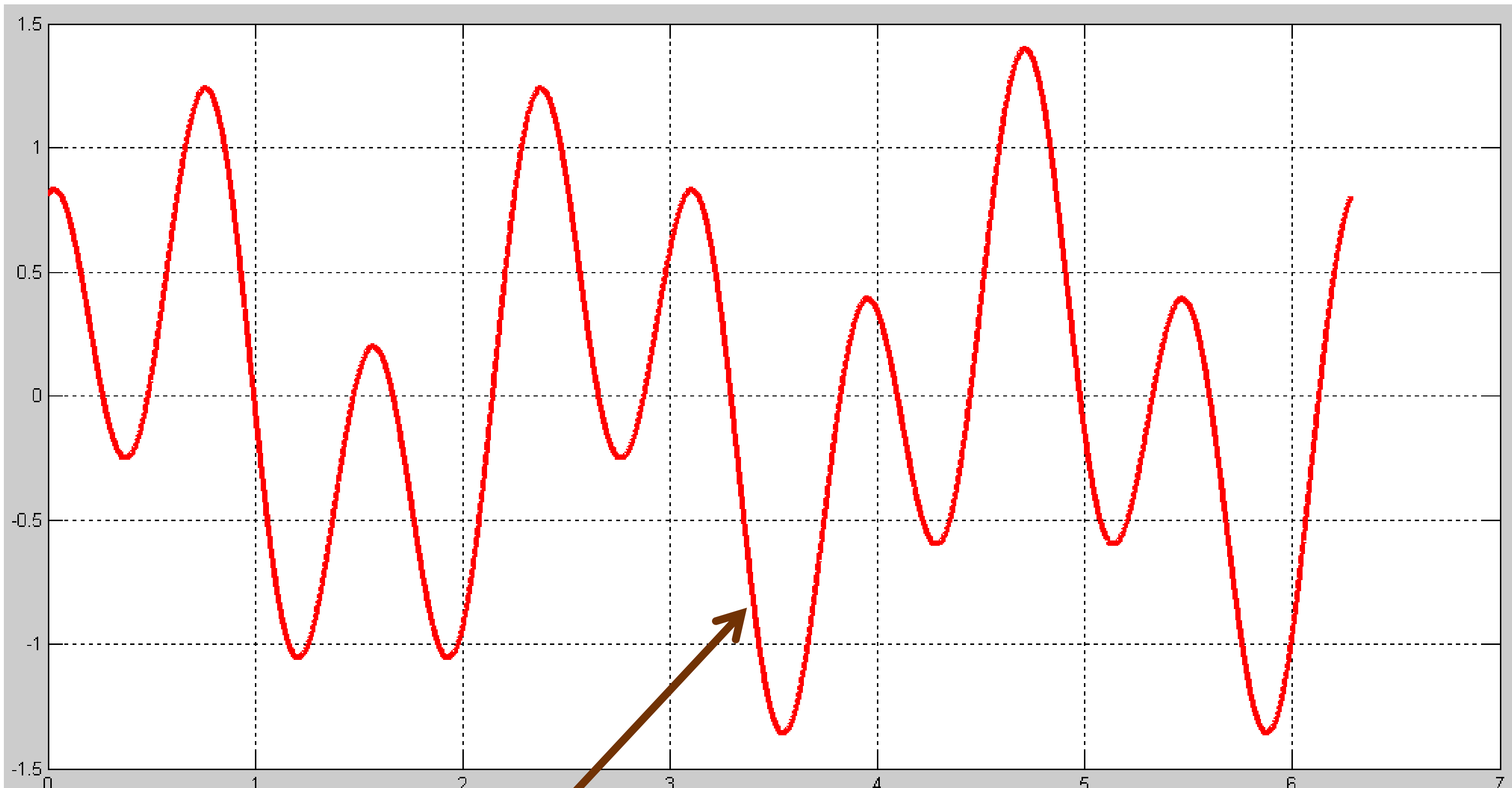

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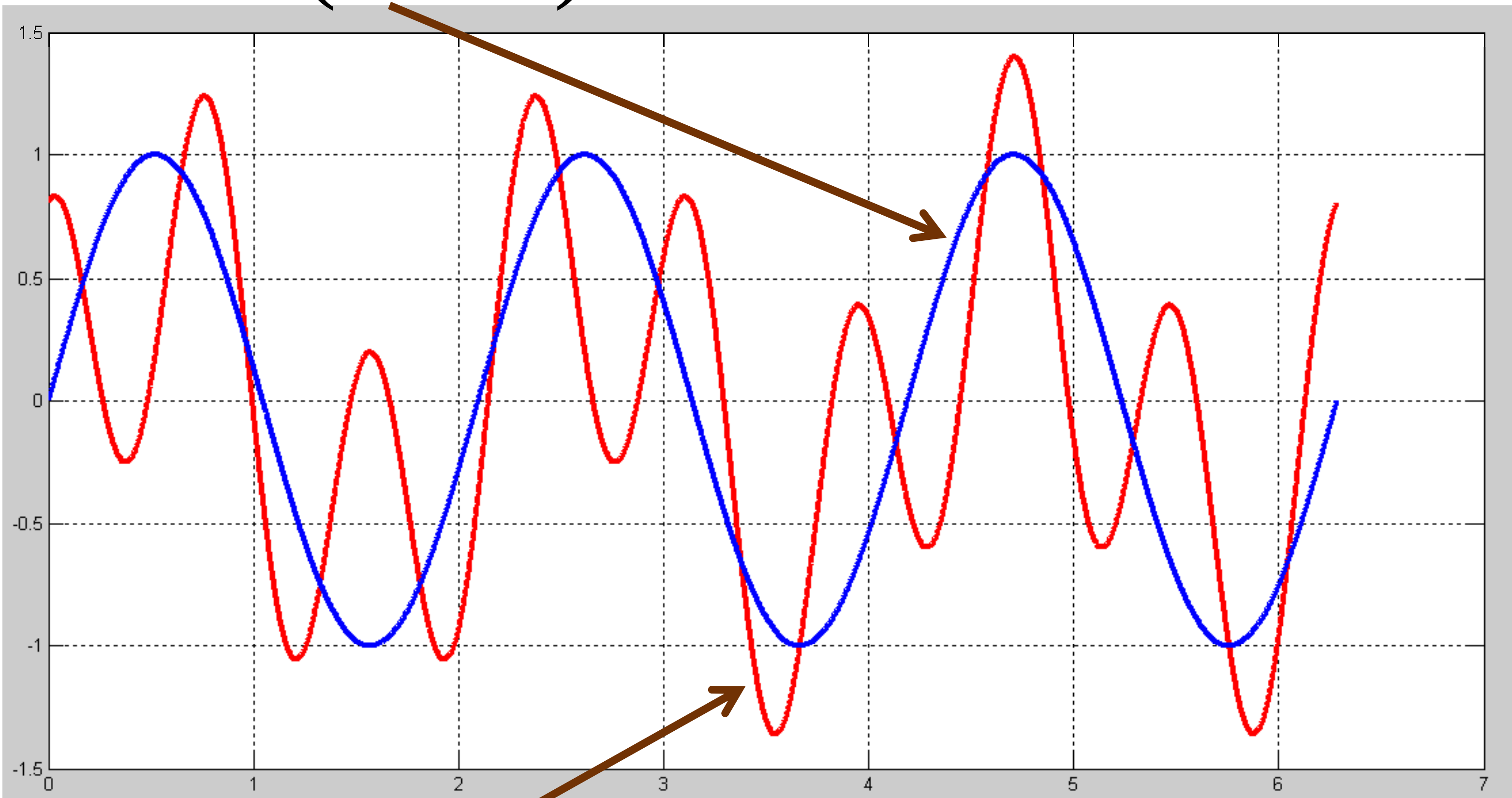

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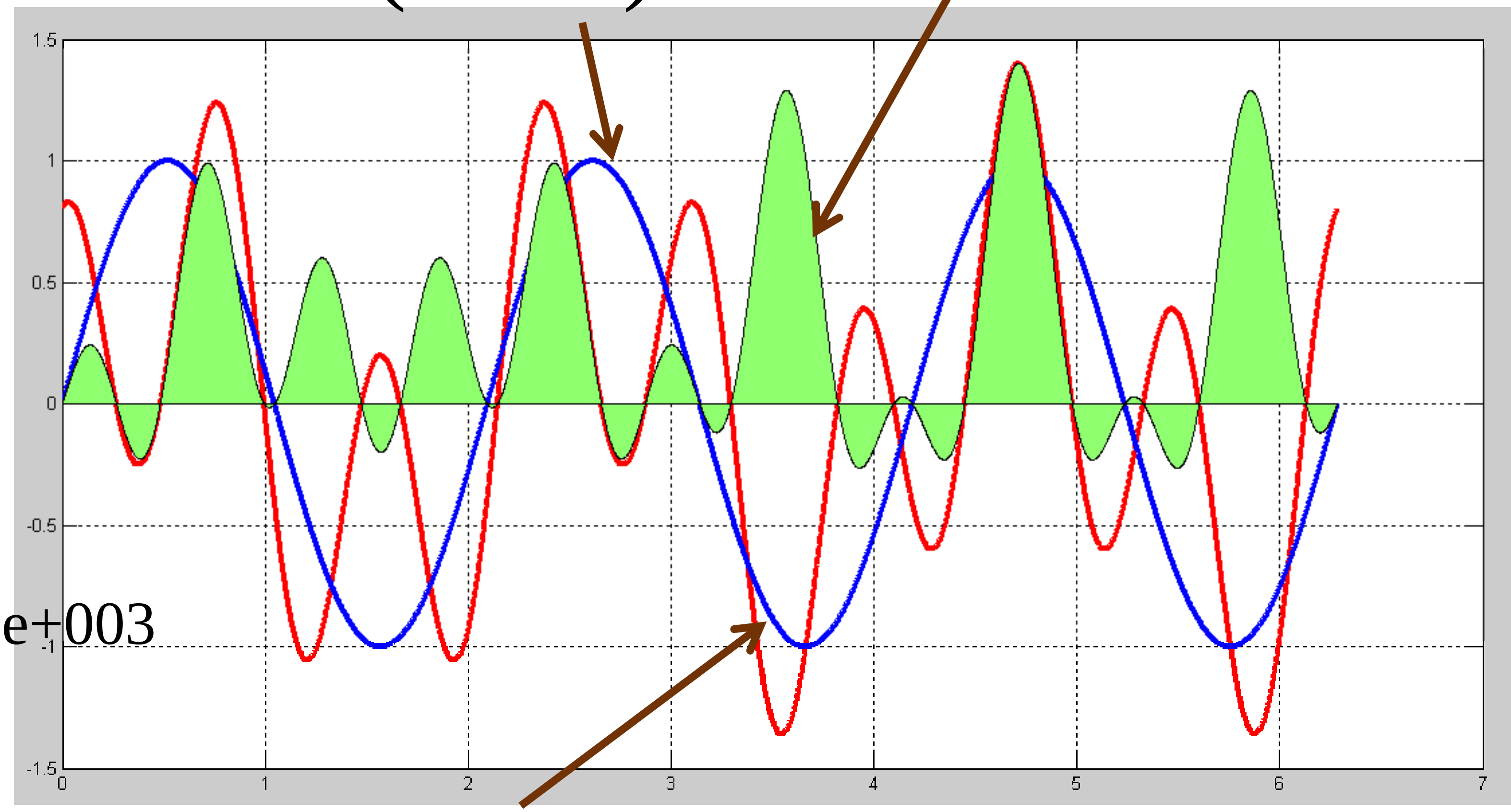
How Fourier Works!!

$$\sin(2\pi 3n) \qquad < y[n], \sin(2\pi 3n) >$$



1.4997e+003

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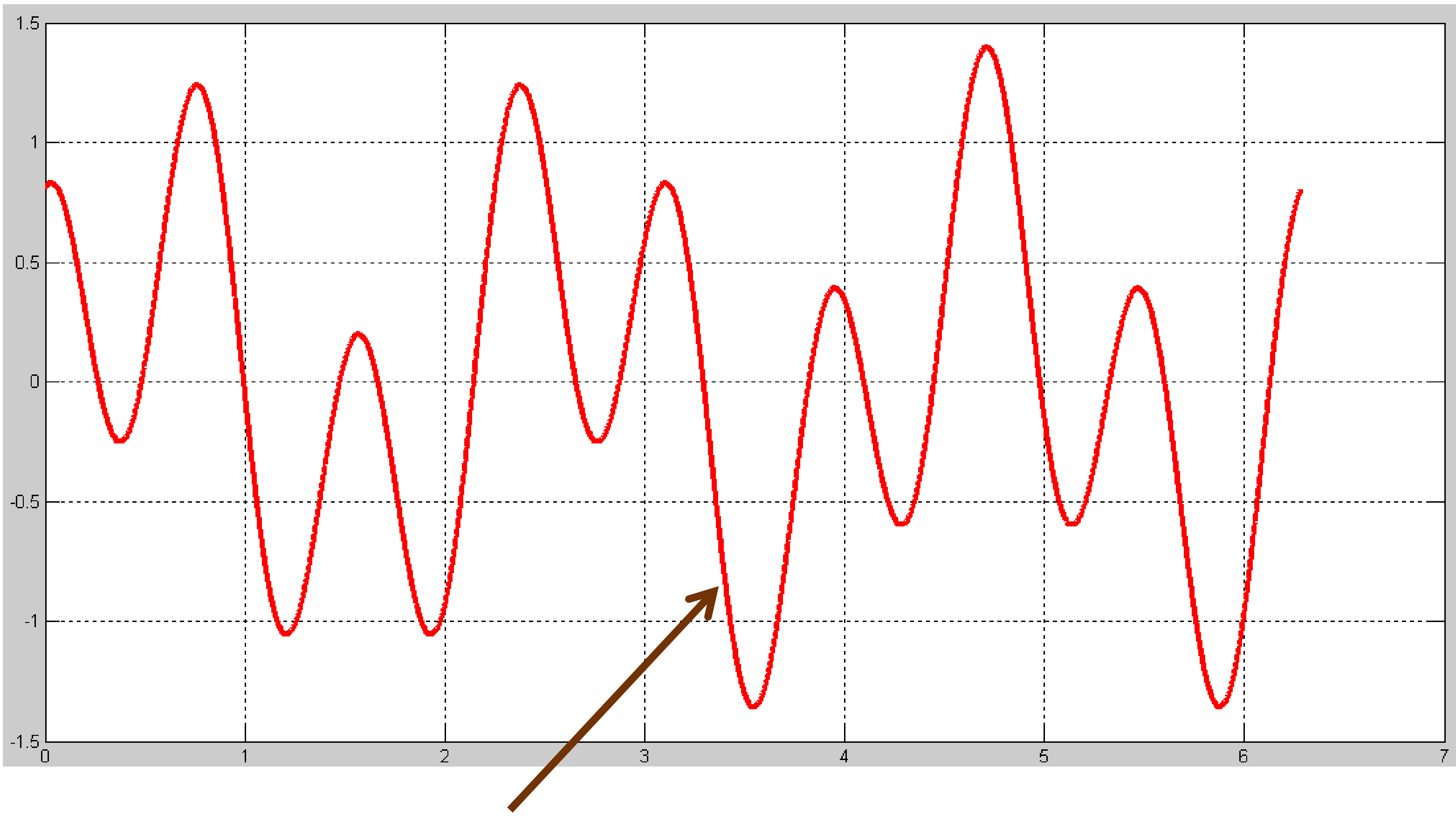

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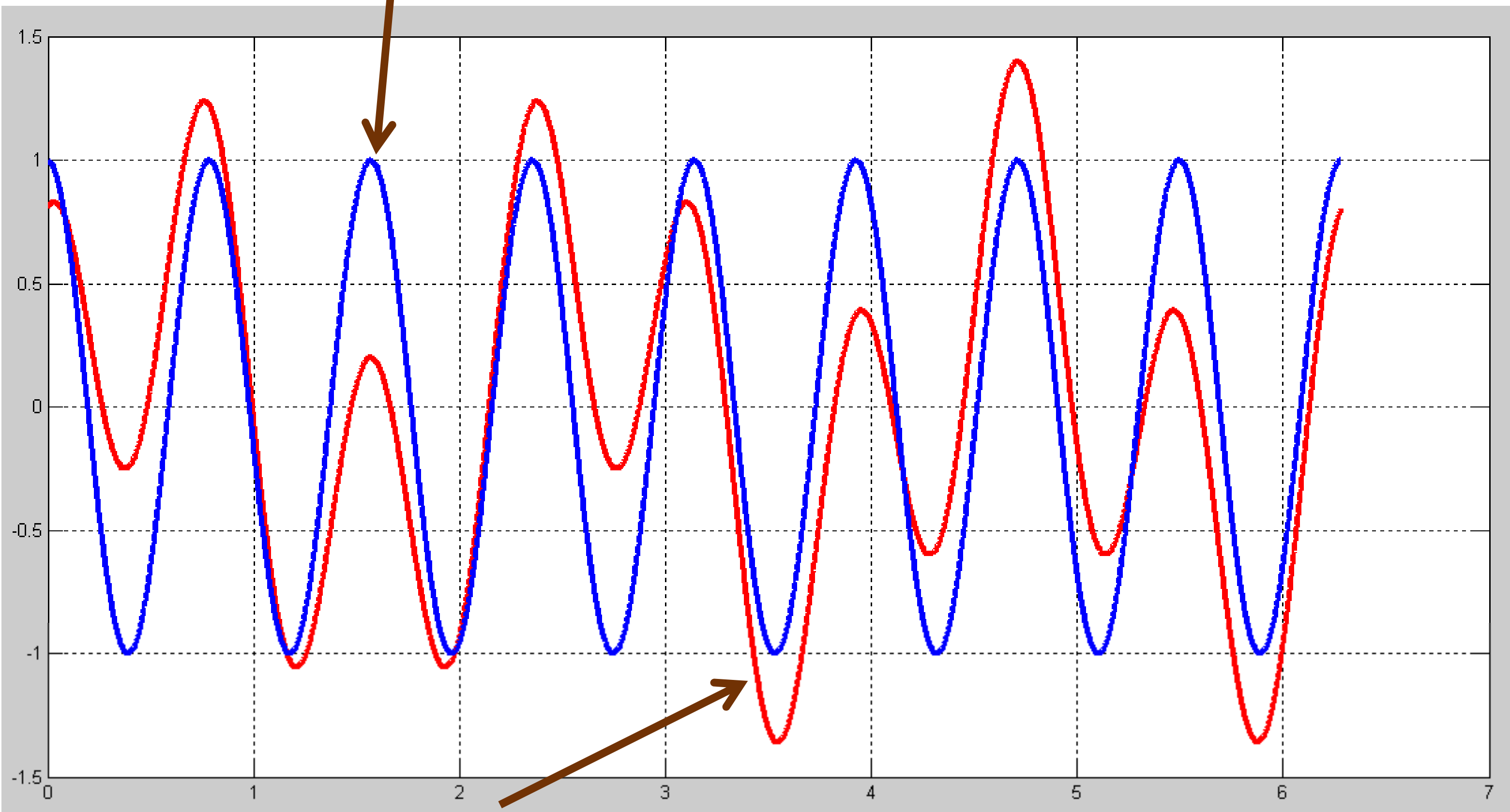

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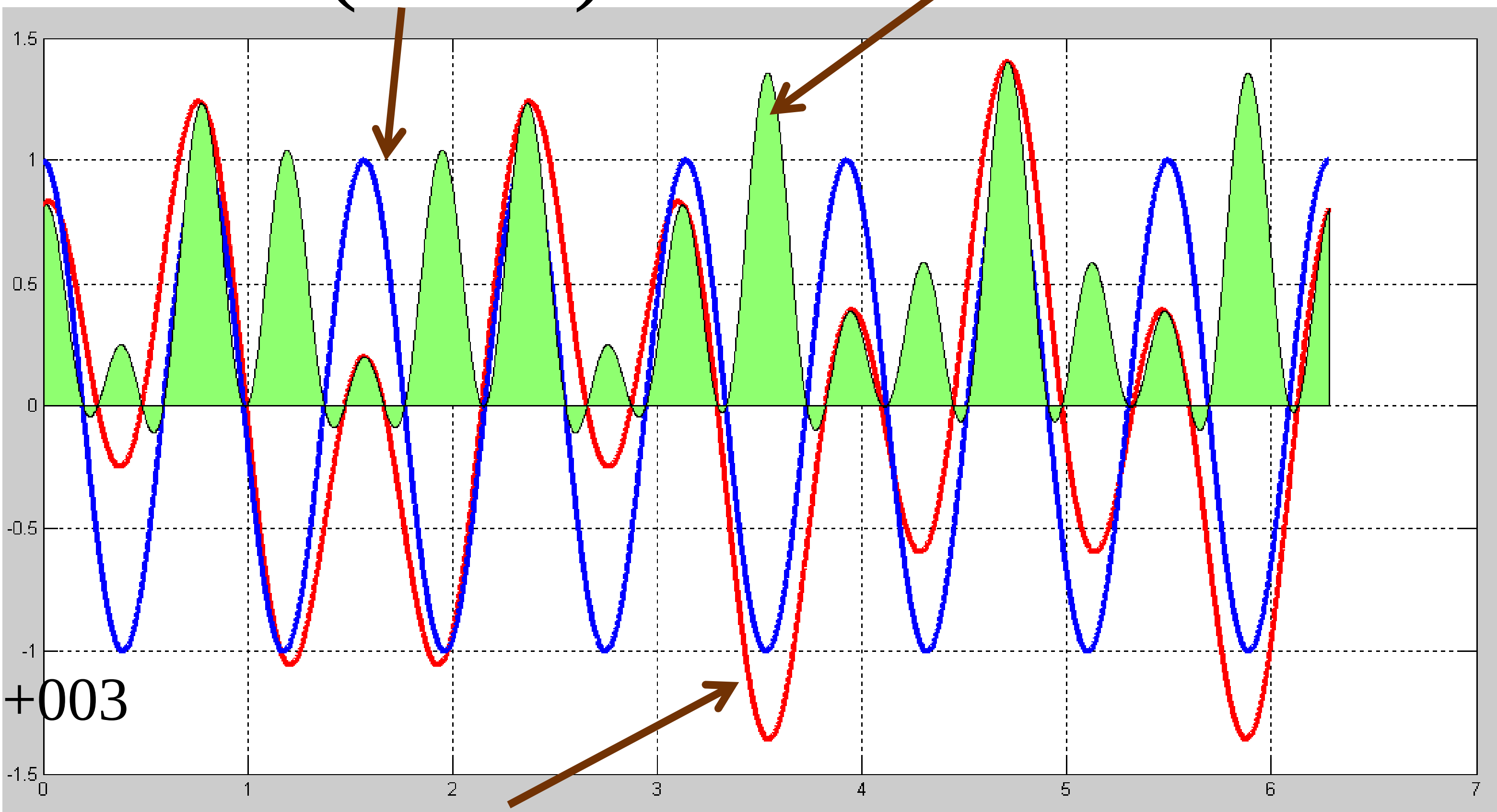
How Fourier Works!!

$\cos(2\pi 8n)$ $\quad < y[n], \cos(2\pi 8n) >$



2.0004e+003

$$y[n] = 0.6\sin(2\pi 3n) + 0.8\cos(2\pi 8n)$$

Application

- Detecting hidden jump discontinuity
- Consider function

$$g(t) = \begin{cases} t, 0 \le t < \dfrac{1}{2} \\ t-1, \dfrac{1}{2} \le t < 1 \end{cases}$$

- Clear jump at $t=0.5$

Application

- Detecting hidden jump discontinuity
- Let's integrate

$$h(t) = \int g(t)dt = \begin{cases} \frac{t^2}{2}, 0 \le t < \frac{1}{2} \\ \frac{t^2}{2} - t + \frac{1}{2}, \frac{1}{2} \le t < 1 \end{cases}$$

- Cusp jump at $t=0.5$

Application

- Detecting hidden jump discontinuity
- Let's integrate again

$$f(t) = \int h(t)dt = \begin{cases} \dfrac{t^3}{6}, 0 \le t < \dfrac{1}{2} \\ \dfrac{t^3}{6} - \dfrac{t^2}{2} + \dfrac{t}{2} - \dfrac{1}{8}, \dfrac{1}{2} \le t < 1 \end{cases}$$

- Appears smooth to eye

Coefficients

- Who gives us coefficients of scaling equation?
- Haar

$$\psi(t) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\phi(t) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

In search of coefficients

- We can thinks of using three guiding theorems !

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- We can thinks of using three guiding theorems !
- Theorem 1:

For the scaling equation $\phi(x) = \sum_k h_k \sqrt{2}\phi(2x-k)$, with

non-vanishing coefficients $\{h_k\}_{k=N}^{M}$ only for $N \le k \le M$,

its $\phi(x)$ is with a compact support contained in interval $[N, M]$

In search of coefficients

- We can thinks of using three guiding theorems !
- Theorem 2:

If the scaling function $\phi(x)$ has compact support on $0 \le x \le N-1$ and if, $\{\phi(x-k)\}$ are linearly independent, then $h_n = h(n) = 0$, for n<0 and $n > N-1$.

Hence N is the length of the sequence.

In search of coefficients

- We can thinks of using three guiding theorems !
- Theorem 3:

If the scaling coefficieents $\{h_k\}$

satisfy the condition for existence and

orthogonality of $\phi(x)$, then

$$\varphi(x) = \sum_k g_k \sqrt{2}\phi(2x - k)$$

where, $g_k = \pm(-1)^k h_{N-k}$

and, $\int_{-\infty}^{\infty} \varphi(x - l)\phi(x - k)dx = \delta_{l,k} = 0, l \neq k$

Properties of scaling coefficients

1. $\sum h_k = \sqrt{2}$

2. $\sum h_{2k} = \frac{1}{\sqrt{2}}$

3. $\sum h_{2k+1} = \frac{1}{\sqrt{2}}$

Properties of scaling coefficients

4. $\sum |h_k|^2 = 1$

5. $\sum h_{k-2l} h_k = \delta_{l,0}$

6. $\sum 2h_{k-2l} h_{k-2j} = \delta_{l,j}$

Thank You!

Questions ??