Lecture 47 – More Thoughts on Wavelets: Zooming In

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Wavelet Transform

- Decomposes signal into two separate series

- Single series to represent most coarse version
- Double series to represent refined version

- Scaling Function
- Wavelet Function

Wavelet Transform: Specialty

- Scaling and Translation are indeed <u>Hallmarks</u> of Wavelet transform

- They lead us to MultiResolutioAnalysis (MRA) !!

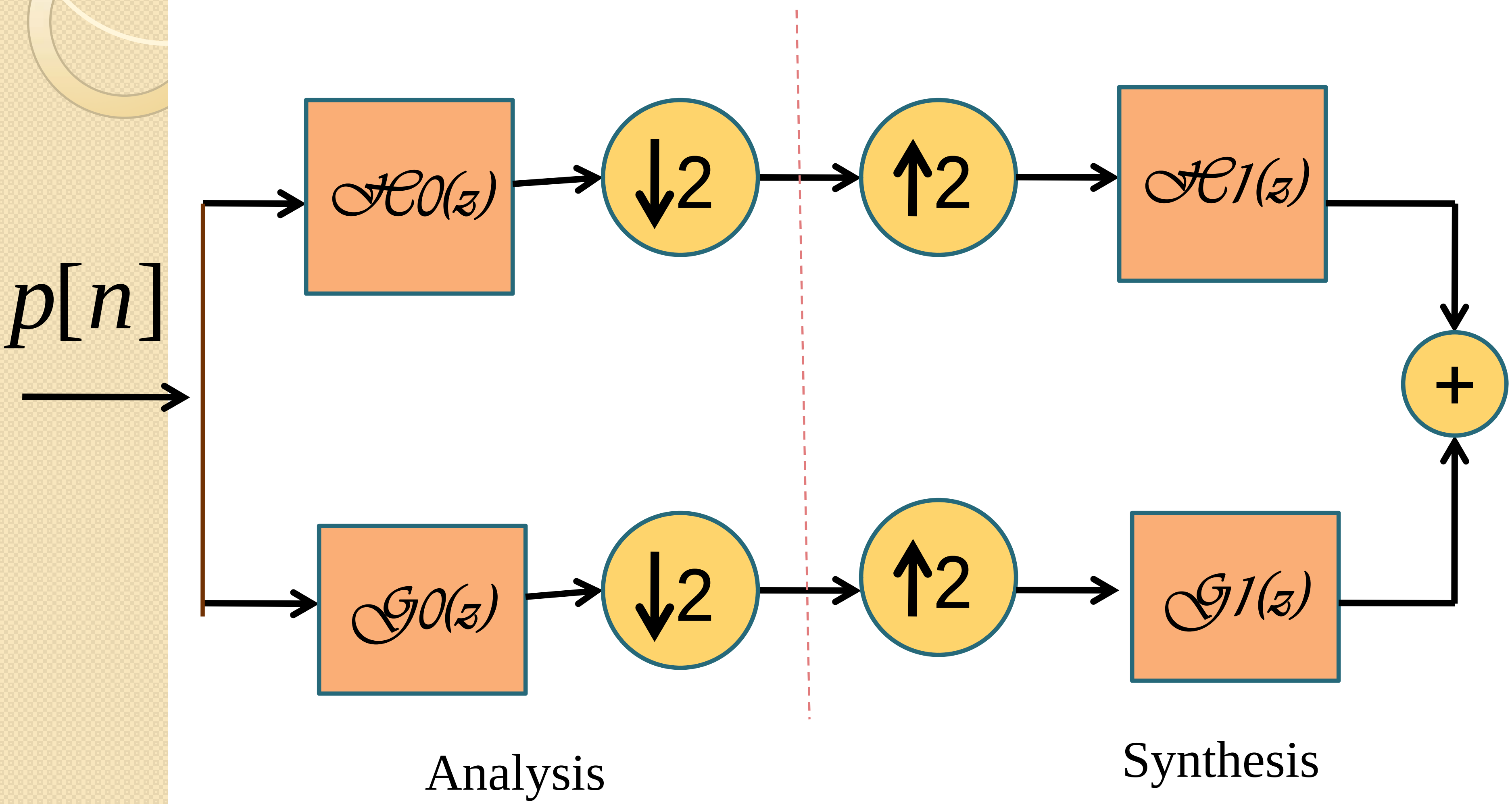
Relationship

- As the spaces and spans are clear now
- Intuitionally, we observe a relationship between these spaces!

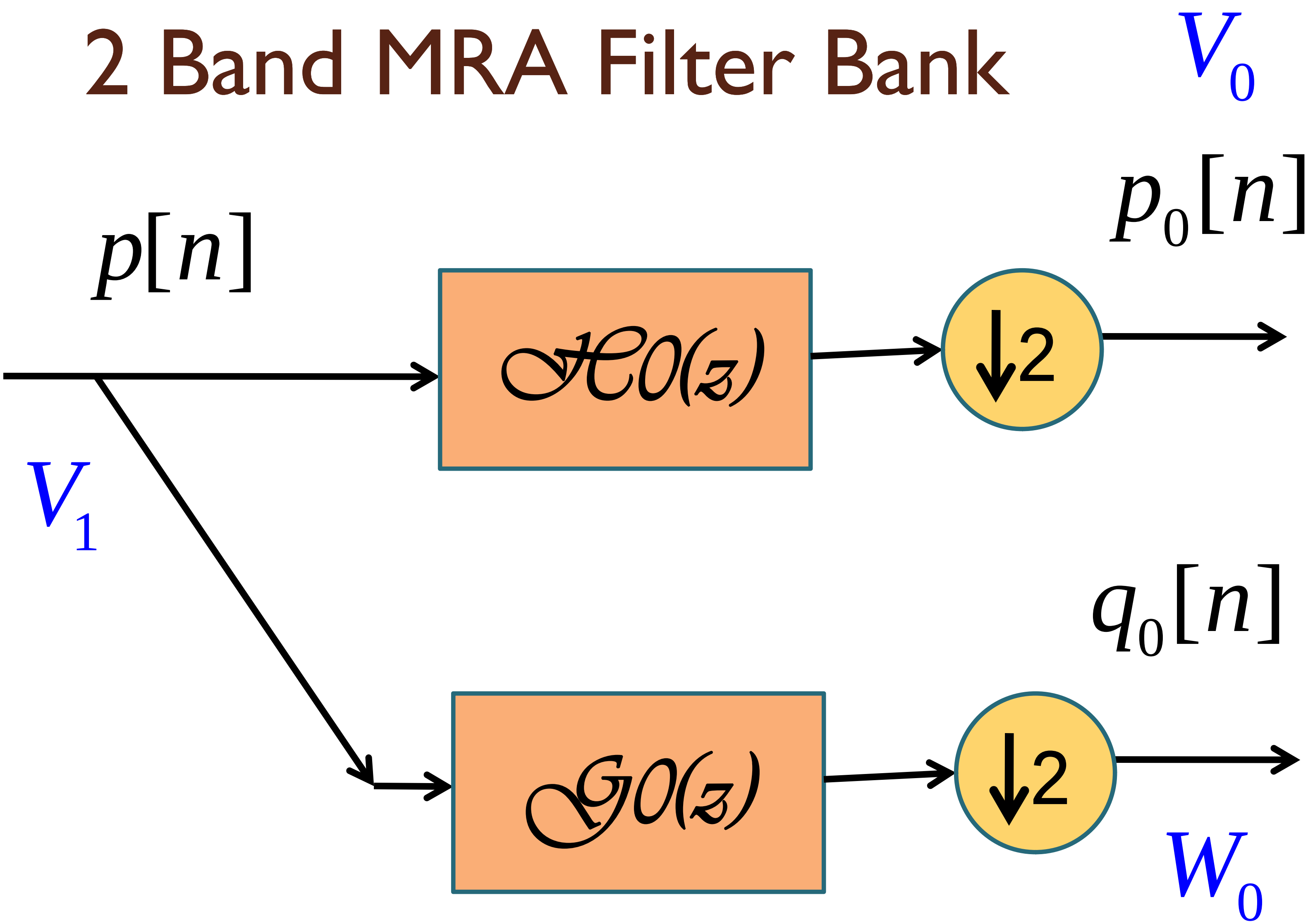
$$.......V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2.......$$

- Intuitively we can see that as we move towards right, i.e. ***up the ladder***, we are moving towards $L_2(\Re)$

2 Band MRA Filter Bank



2 Band MRA Filter Bank



Wavelet Transform

- Decomposes signal into two separate series

- Single series to represent most coarse version
  - Scaling Function

- Double series to represent refined version
  - Wavelet Function

Application

- Detecting hidden jump discontinuity
- Consider function

$$g(t) = \begin{cases} t, 0 \le t < \dfrac{1}{2} \\ t-1, \dfrac{1}{2} \le t < 1 \end{cases}$$

- Clear jump at $t=0.5$

Application

- Detecting hidden jump discontinuity
- Let's integrate

$$h(t) = \int g(t)dt = \begin{cases} \dfrac{t^2}{2}, 0 \le t < \dfrac{1}{2} \\ \dfrac{t^2}{2} - t + \dfrac{1}{2}, \dfrac{1}{2} \le t < 1 \end{cases}$$

- Cusp jump at $t=0.5$

Application

- Detecting hidden jump discontinuity
- Let's integrate again

$$f(t) = \int h(t)dt = \begin{cases} \dfrac{t^3}{6}, 0 \le t < \dfrac{1}{2} \\ \dfrac{t^3}{6} - \dfrac{t^2}{2} + \dfrac{t}{2} - \dfrac{1}{8}, \dfrac{1}{2} \le t < 1 \end{cases}$$

- Appears smooth to eye

Wavelet Packet Analysis

$$W^{[2n]}(t) = \sqrt{2}\sum_k h[k].W^{[n]}(2t-k)$$

$$W^{[2n+1]}(t) = \sqrt{2}\sum_k g[k].W^{[n]}(2t-k)$$

Wavelet Packet Analysis

$$W^{[2n]}(t) = \sqrt{2}\sum_{k} h[k].W^{[n]}(2t-k)$$

$$h[k] = \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

$$W^{[2n+1]}(t) = \sqrt{2}\sum_{k} g[k].W^{[n]}(2t-k)$$

$$g[k] = \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$$

Wavelet Packet Analysis

$$n = 0$$

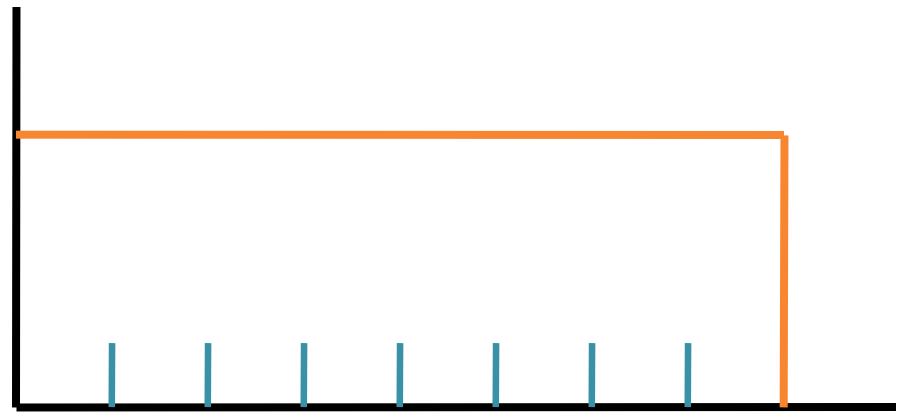
$$W_{(j,k)}^{[0]}(t) = \phi(2^j t - k)$$

$$W_{(j,k)}^{[1]}(t) = \psi(2^j t - k)$$

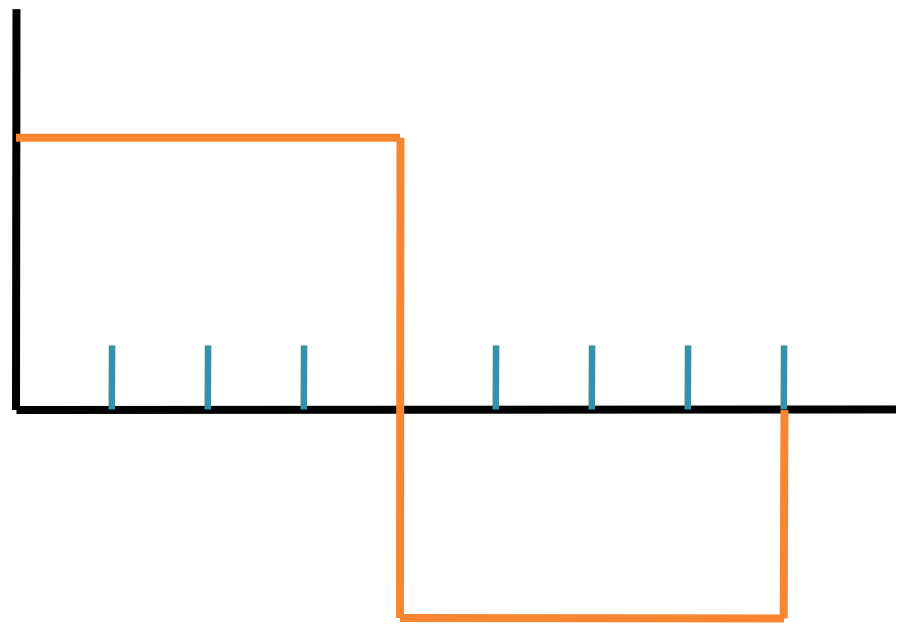
Wavelet Packet Analysis

$$n = 0$$

$$W_{(j,k)}^{[0]}(t) = \phi(2^j t - k)$$



$$W_{(j,k)}^{[1]}(t) = \psi(2^j t - k)$$



Wavelet Packet Analysis

$$n = 1$$

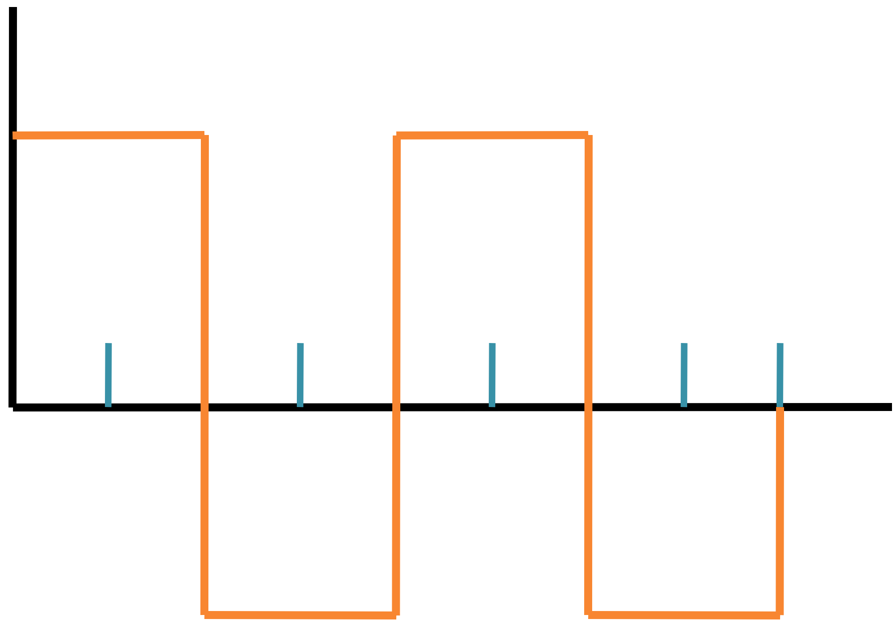
$$W^{[2]}(t) = \sqrt{2}\sum_{k} h[k].W^{[1]}(2t - k)$$

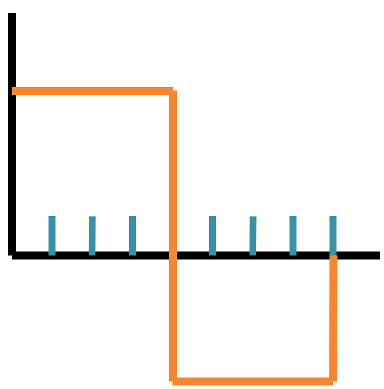
$$W^{[3]}(t) = \sqrt{2}\sum_{k} g[k].W^{[1]}(2t - k)$$

Wavelet Packet Analysis

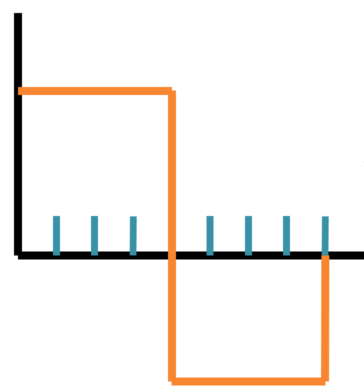
$$n = 1$$

$$W^{[2]}_{(j,k)}(t)$$





$$+$$



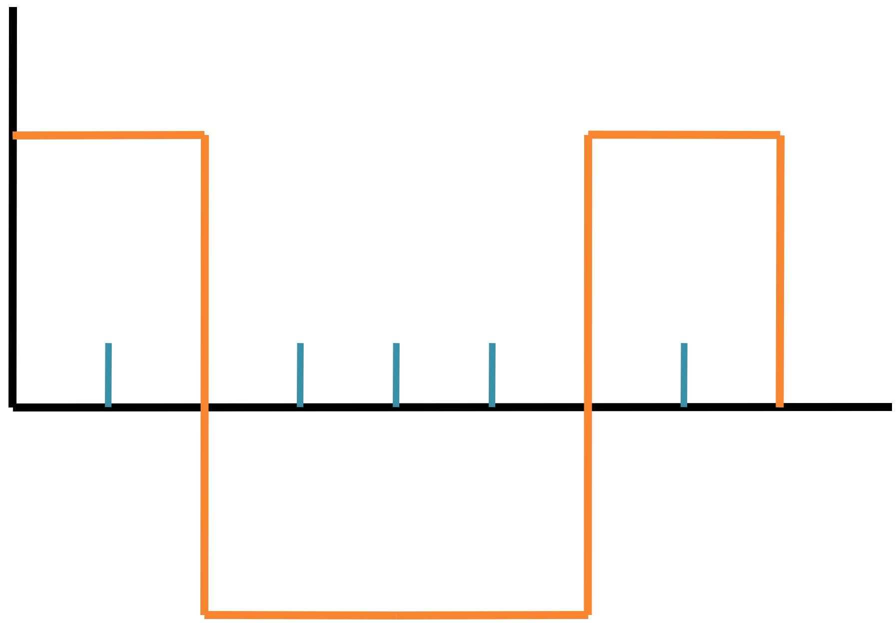
$$W^{[1]}(2t-1)$$

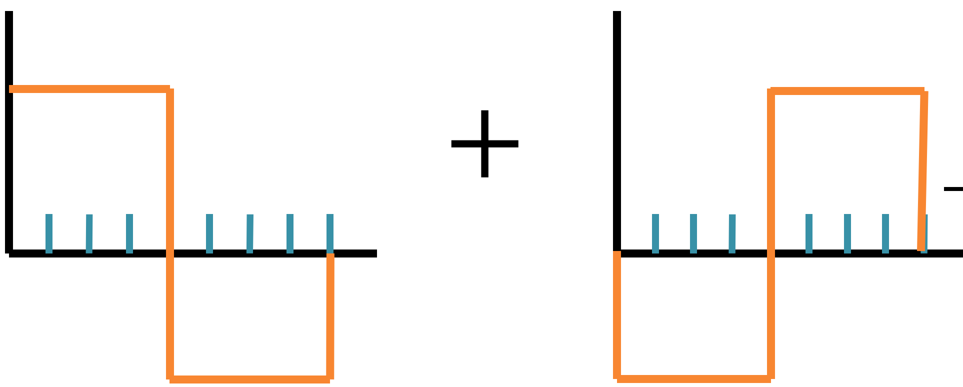
$$W^{[1]}(2t)$$

Wavelet Packet Analysis

$$n = 1$$

$$W_{(j,k)}^{[3]}(t)$$

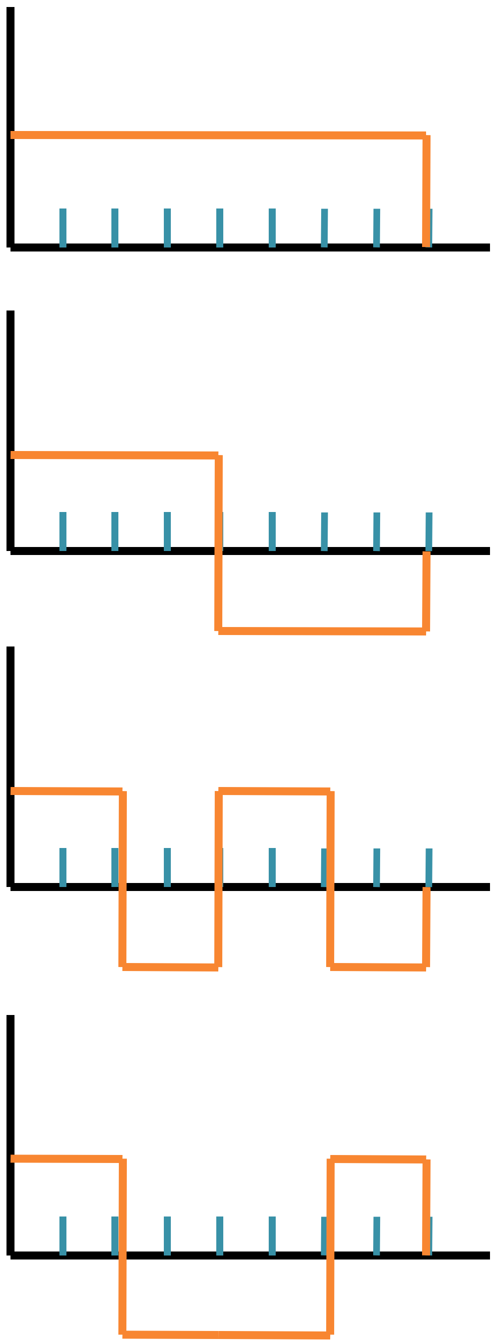




$$-W^{[1]}(2t-1)$$

$$W^{[1]}(2t)$$

Bases



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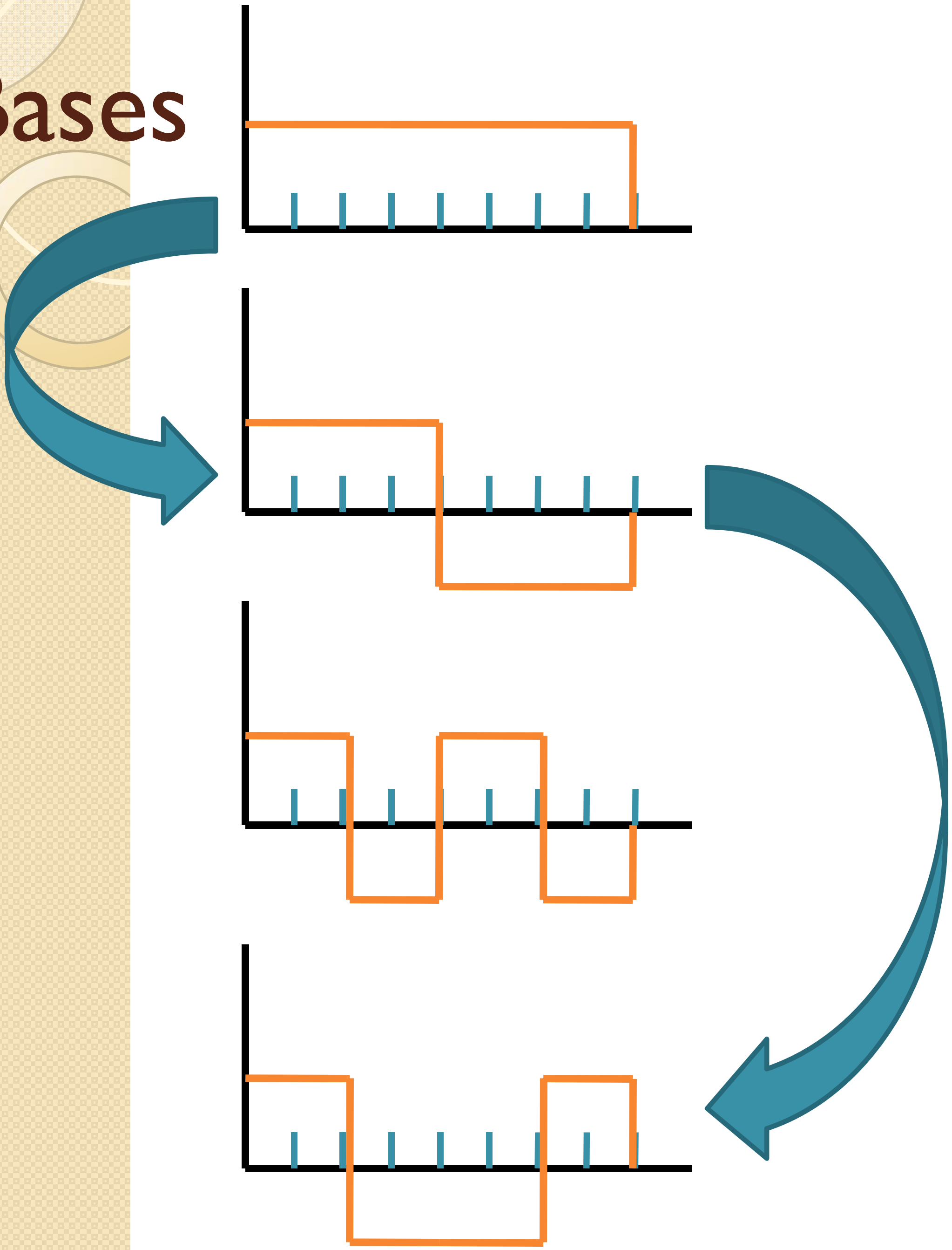

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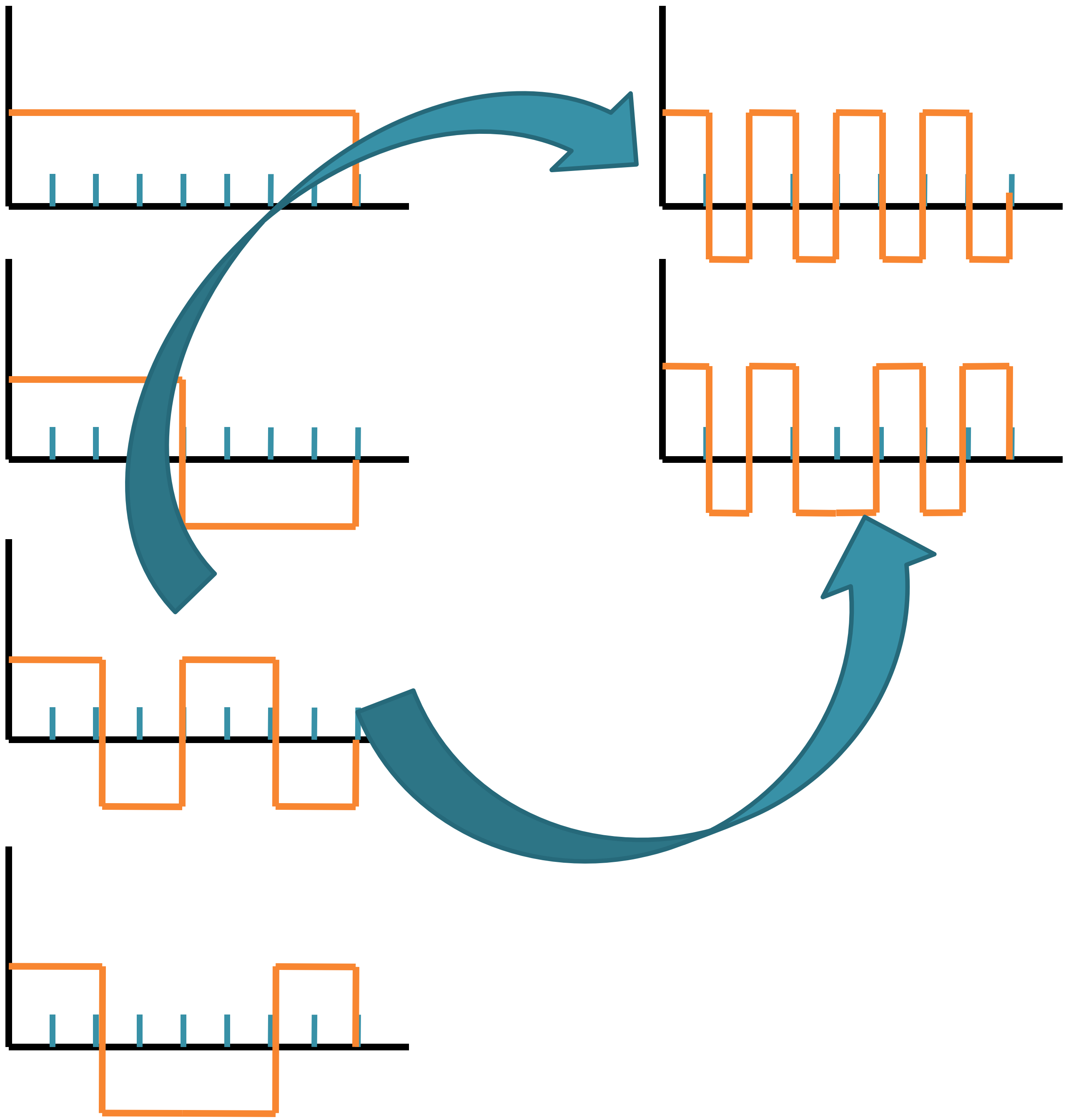

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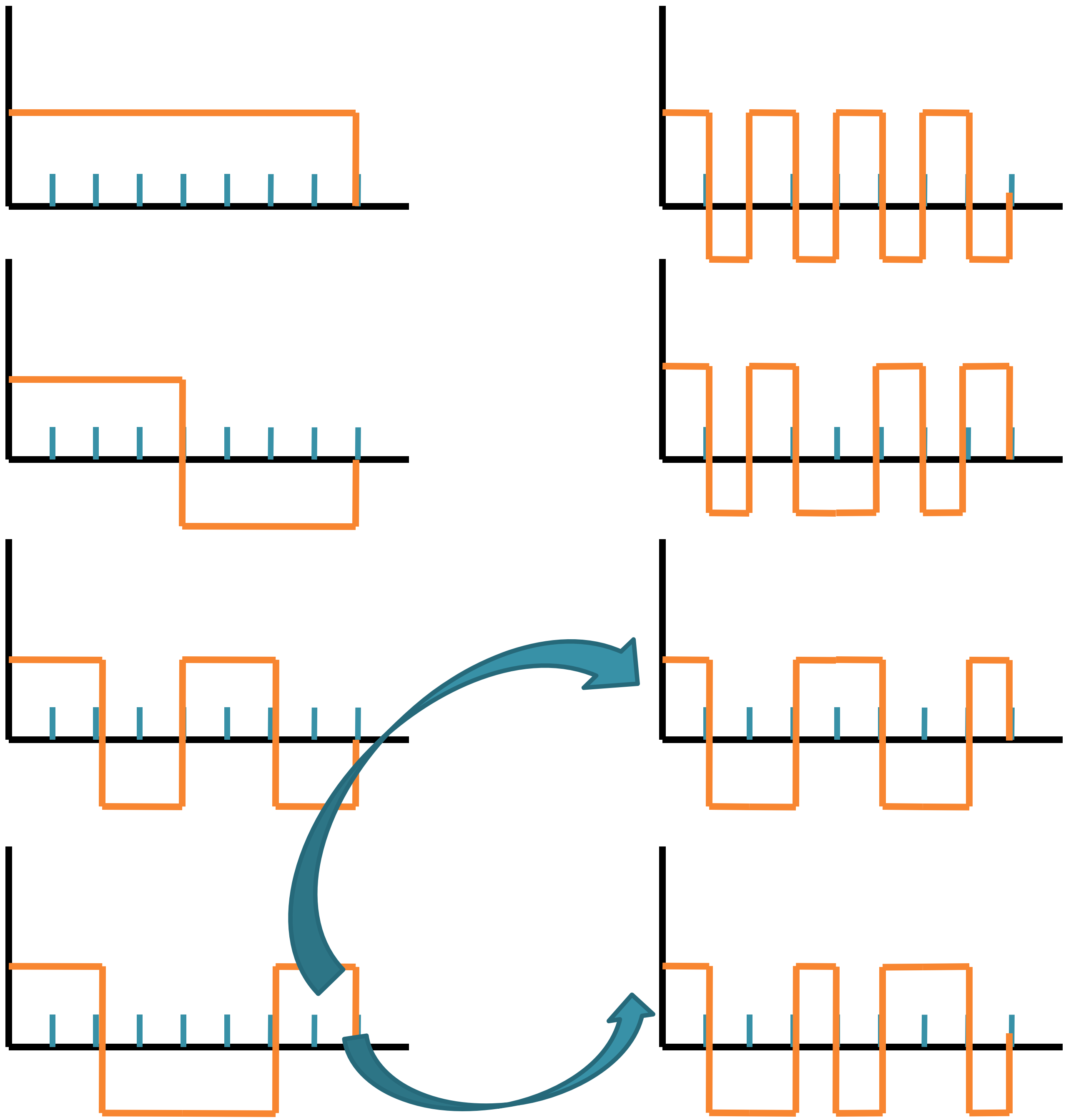

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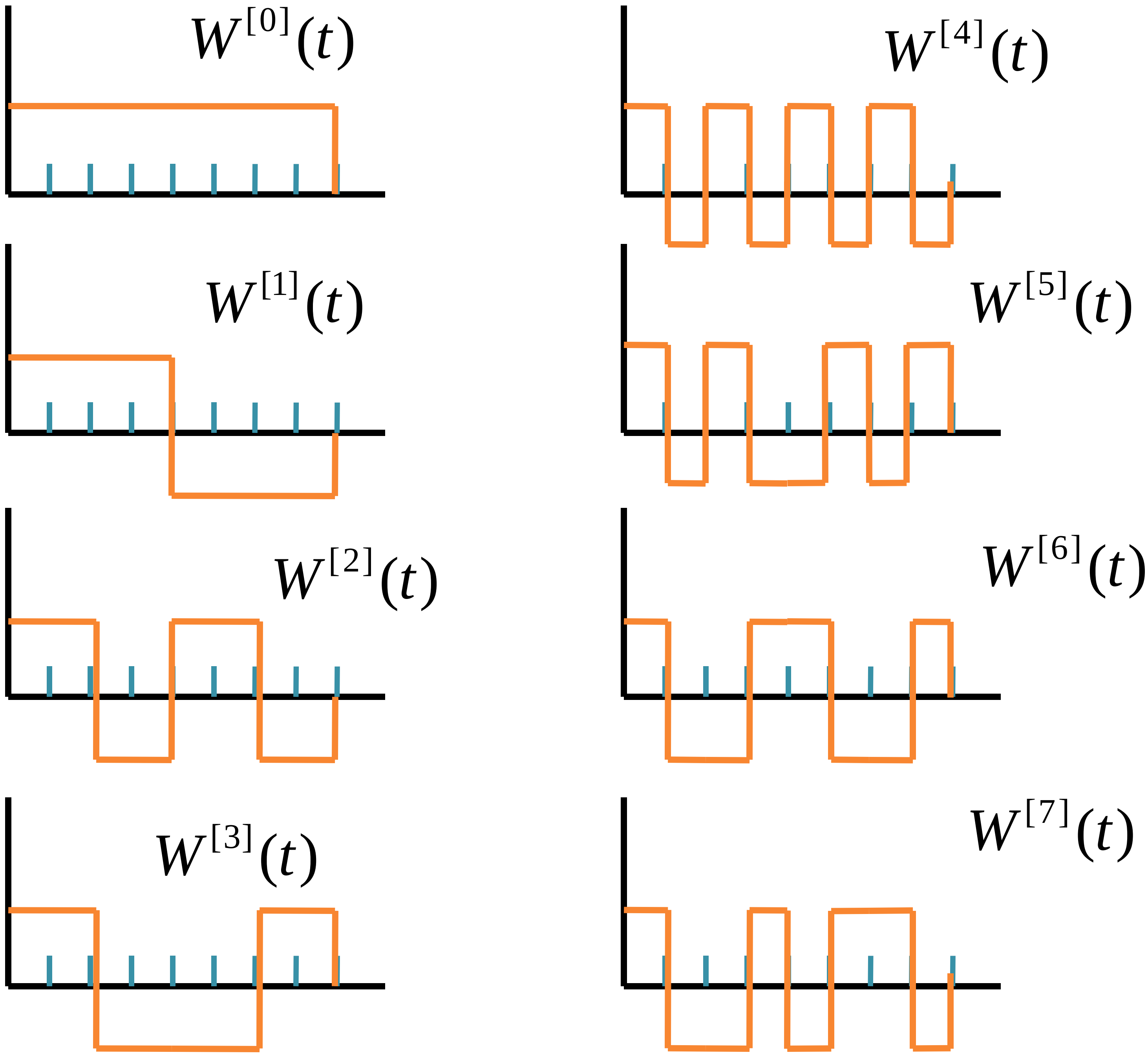

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$$h[k] = \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

$$W^{[2n+1]}(t) = \sqrt{2}\sum_{k} g[k].W^{[n]}(2t-k)$$

$$g[k] = \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$$

Example

$$x[n] = \{1, 0, -3, 2, 1, 0, 1, 2\} \in V_3$$

- Show complete decomposition using Haar Wavelet Packets till V0
- Demonstrate complete reconstruction

Example

$$x[n] = \{1, 2, 3, 4, 5, 6, 7, 8\} \in V_3$$

- Show complete decomposition using Haar Wavelet Packets till V0
- Demonstrate complete reconstruction

$$x[n] = \{1, 2, 3, 4, 0, 6, 7, 8\} \in V_3$$

Wavelet Packet Analysis

$$W^{[2n]}(t) = \sqrt{2}\sum_k h[k].W^{[n]}(2t-k)$$

$$h[k] = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$$

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Coefficients

- Who gives us coefficients of scaling equation?
- Haar

$$\psi(t) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\phi(t) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Properties of scaling coefficients

1. $$\sum h_k = \sqrt{2}$$

2. $$\sum h_{2k} = \frac{1}{\sqrt{2}}$$

3. $$\sum h_{2k+1} = \frac{1}{\sqrt{2}}$$

Properties of scaling coefficients

4. $\sum |h_k|^2 = 1$

5. $\sum h_{k-2l} h_k = \delta_{l,0}$

6. $\sum 2h_{k-2l} h_{k-2j} = \delta_{l,j}$

Thank You!

Questions ??