Lecture 46 – Some Thoughts on Wavelets: Zooming Out

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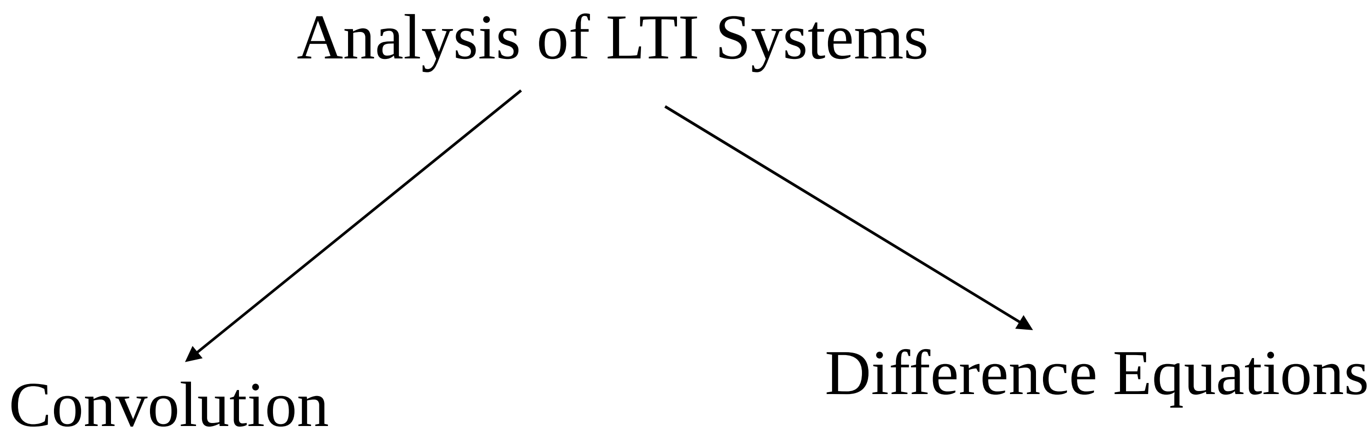
Introduction

- Wavelet Transform → Buzz word!
- Next 100 years will be of WT !
- Relatively new and efficient way of representing signals
- Multiresolution analysis helps analyze the information at multiple resolutions, simultaneously

Wavelet Transform

- Why transform?
- One serious reason – convenience!
- All prior transforms have a common thread of 'e' !

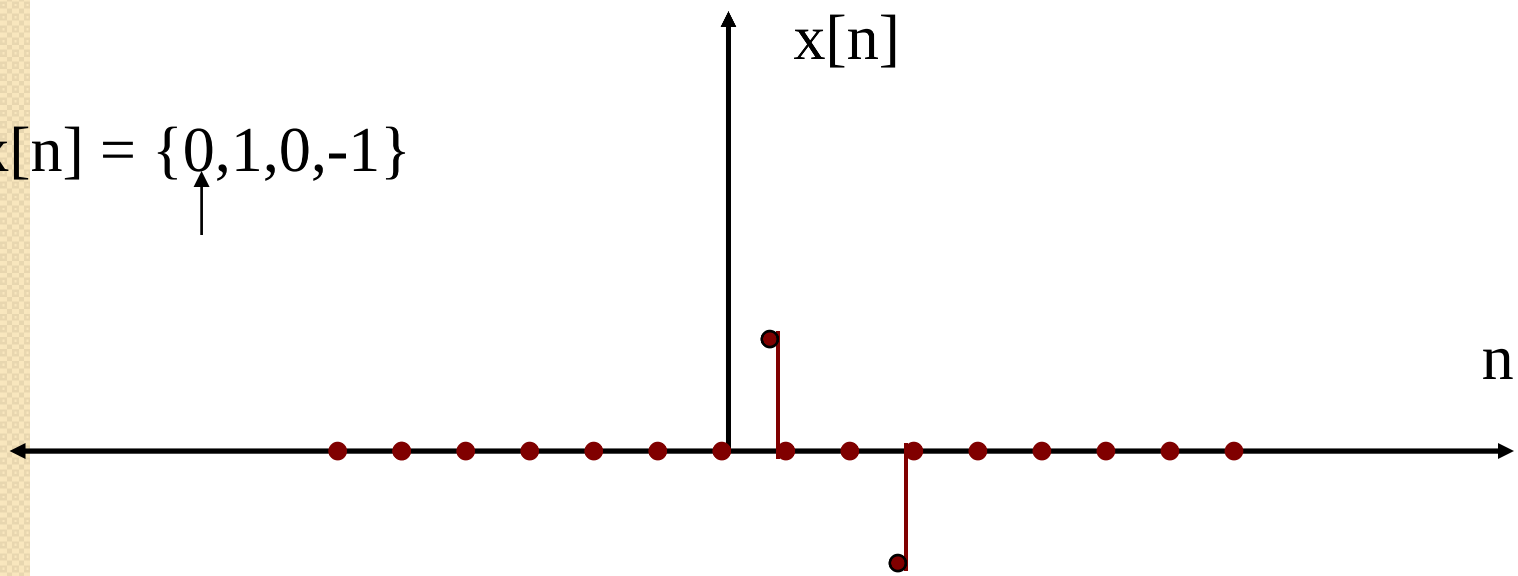
Analysis of LTI systems



HOW Convolution

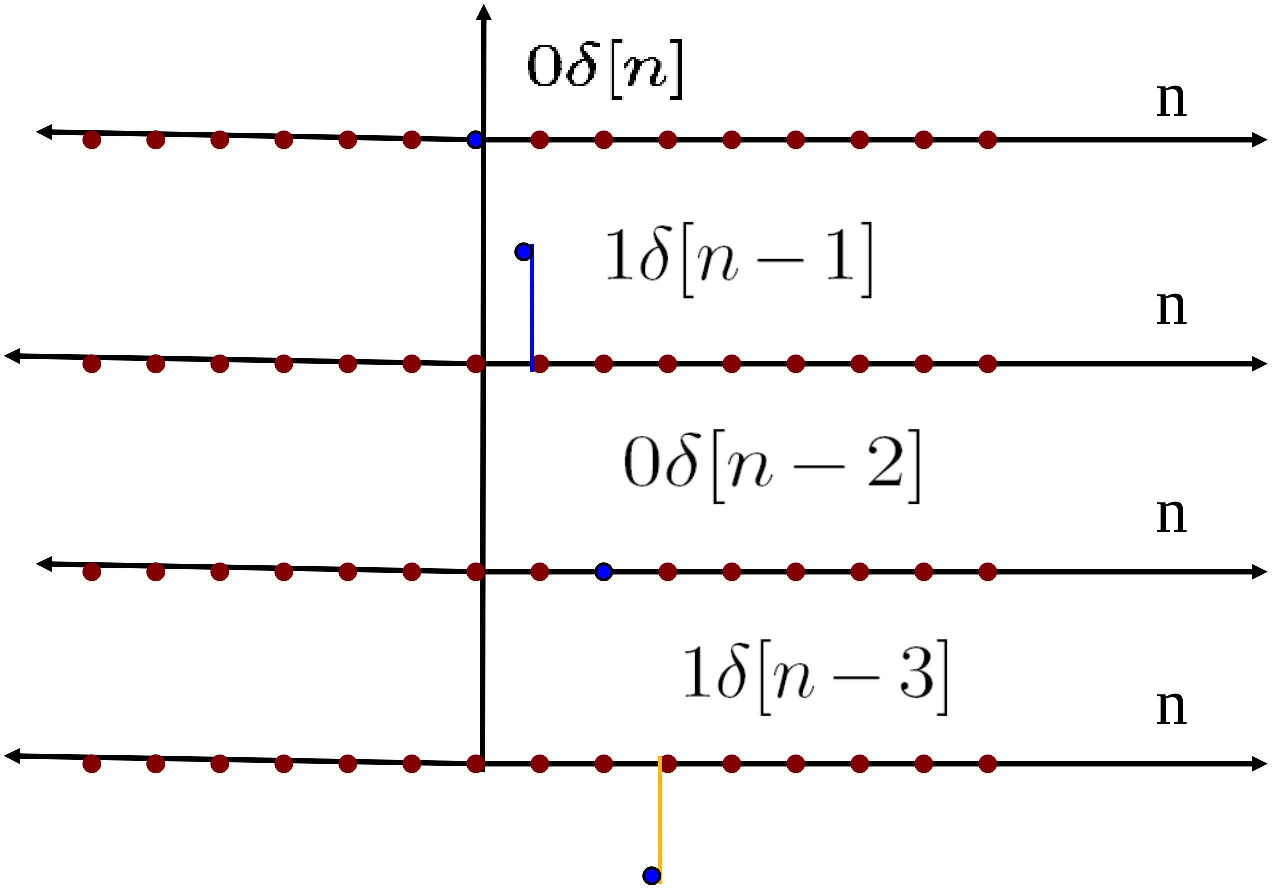
- Step I: Decompose given signal into shifted impulse sequences.

$x[n] = \{0, 1, 0, -1\}$
$\uparrow$



HOW Convolution

$$x[n] = 0\delta[n] + 1\delta[n-1] + 0\delta[n-2] + 1\delta[n-3]$$



HOW Convolution

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

HOW Convolution

- Step II:

  y[n,k] = h[n,k] = H[delta [n]]

- Step III:

$$y[n] = H[\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]]$$

→ Because of superposition (linearity)

- Step IV

$$h[n-k] = H[\delta[n-k]]$$

→ Because of Time Invariance

Thus Convolution !!

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

How Fourier Works!!

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$h[n-k]$$

How Fourier Works!!

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$x[k] \rightarrow e^{j\omega_0 n}$$

$$\delta[n-k] \rightarrow h[n-k]$$

How Fourier Works!!

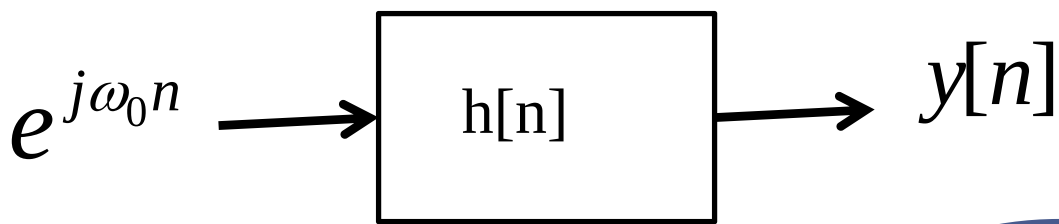
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$y[n]$

$e^{j\omega_0 n}$

$h[n-k]$

How Fourier Works!!



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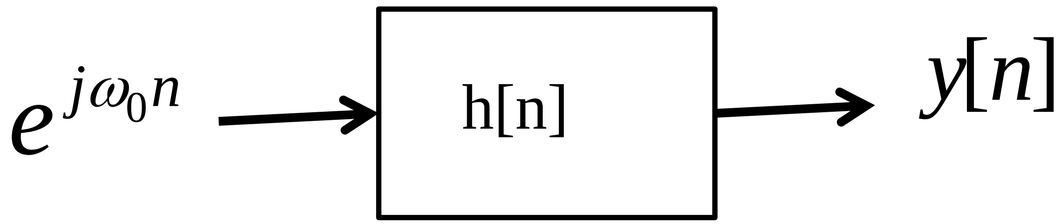
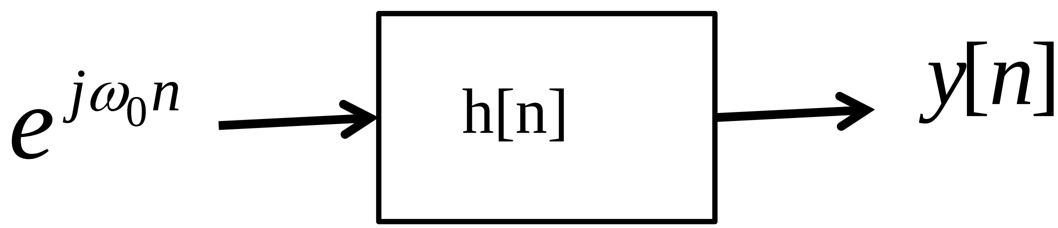
How Fourier Works!!



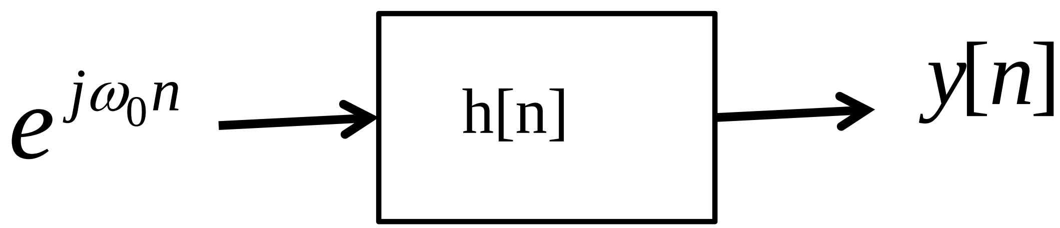
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

How Fourier Works!!



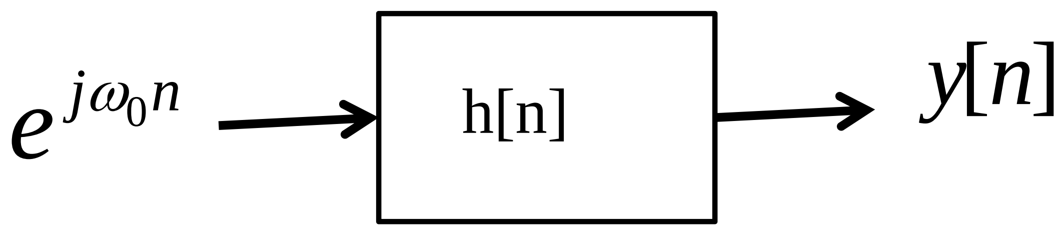
$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

How Fourier Works!!



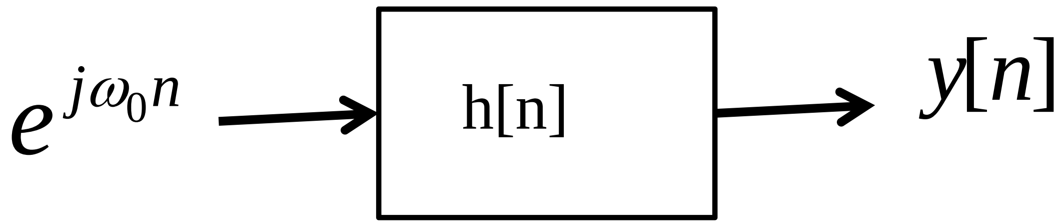
$$y[n] = \sum_{k=-\infty}^{\infty} e^{j\omega_0[n-k]} h[k]$$

How Fourier Works!!



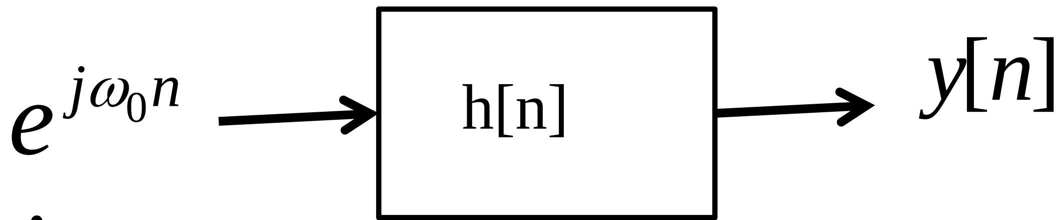
$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 n} e^{-j\omega_0 k}$$

How Fourier Works!!



$$y[n] = e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}$$

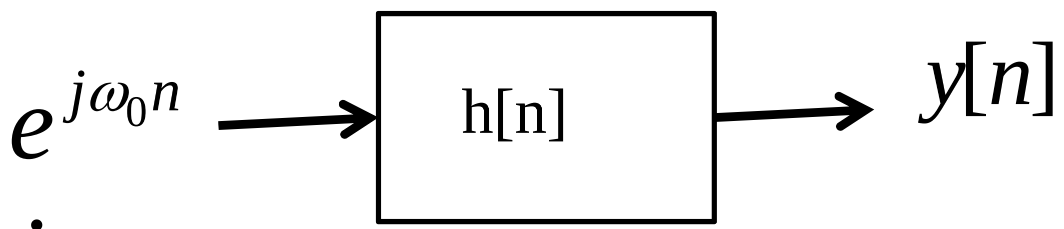
How Fourier Works!!



eigenfunction

$$y[n] = e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}$$

How Fourier Works!!



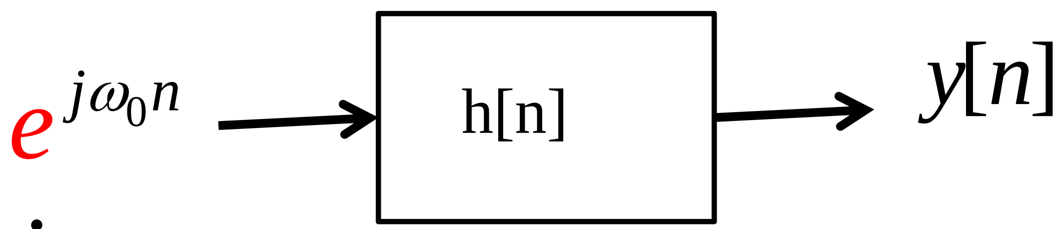
eigenfunction

DFT !!!

$$y[n] = e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}$$

eigenvalue

Common thread in all transforms

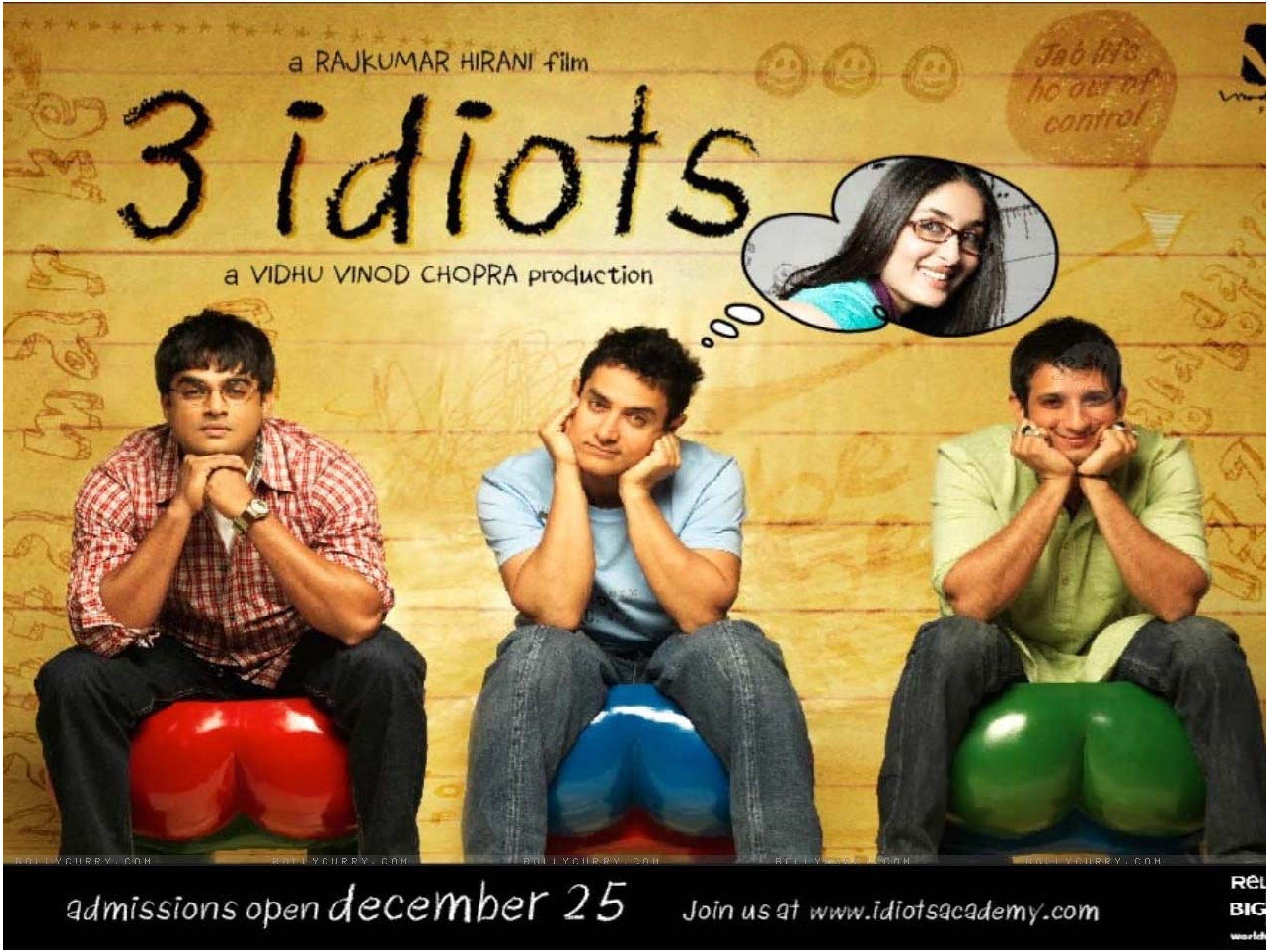


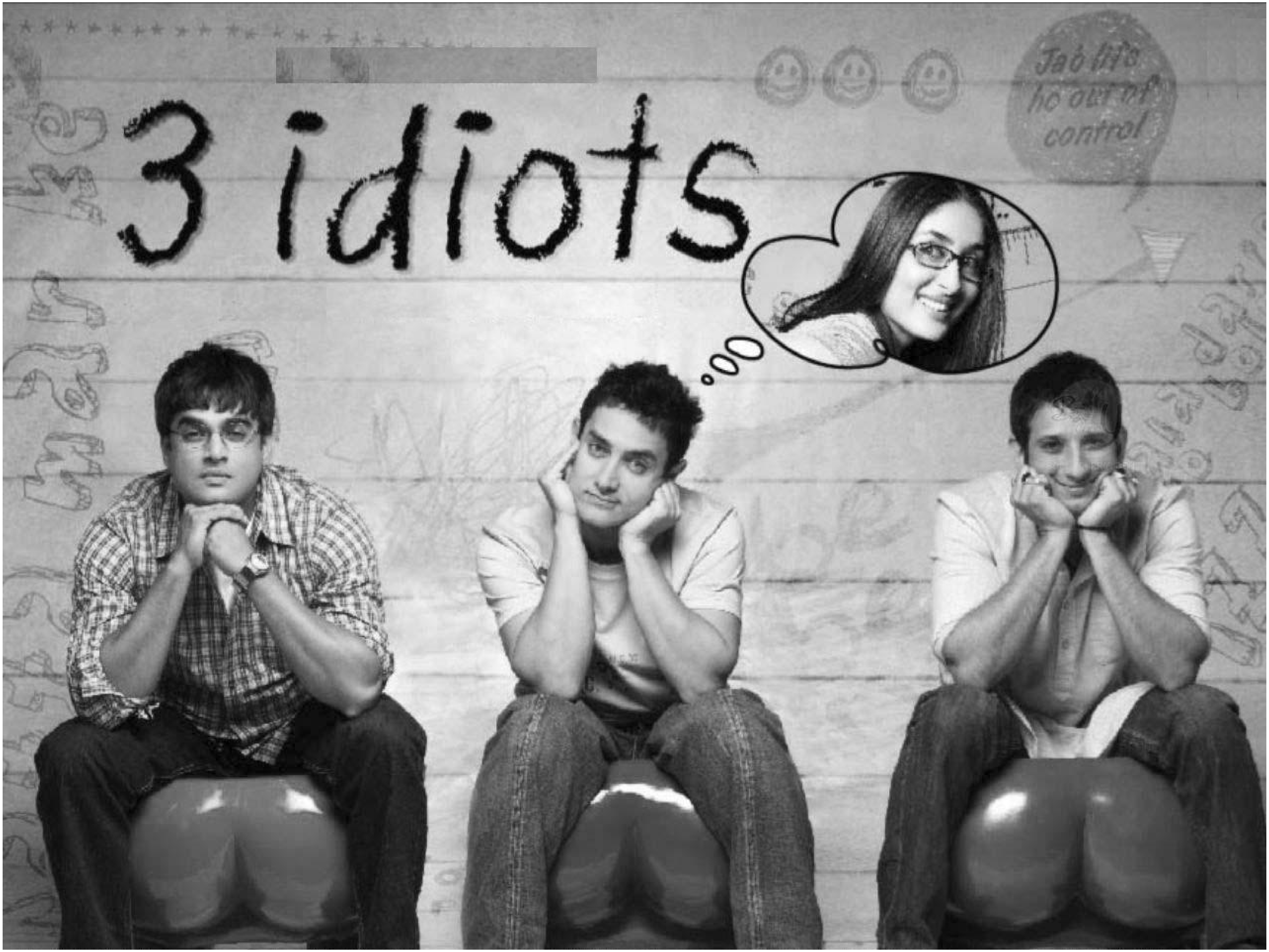
eigenfunction

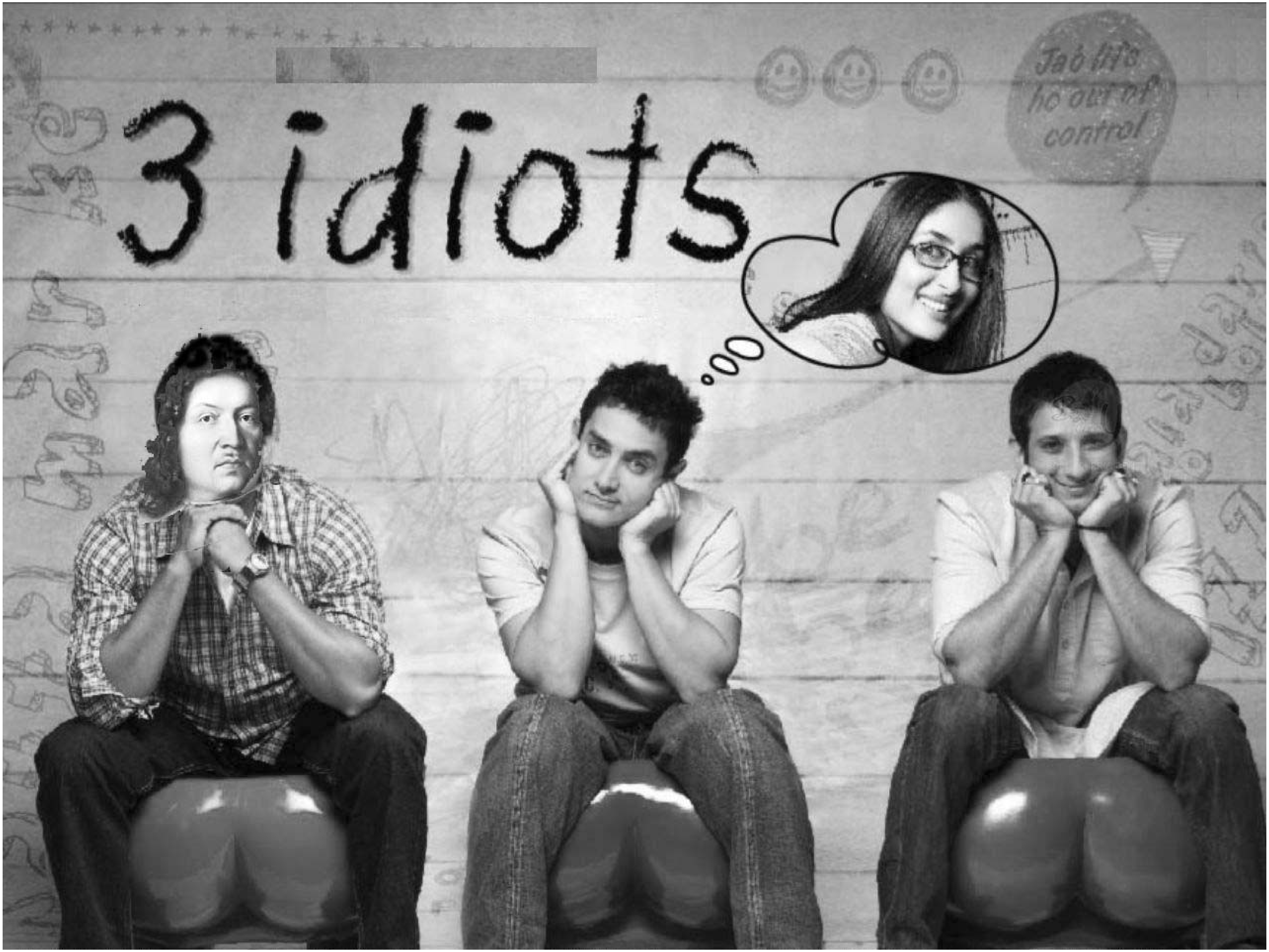
DFT !!!

$$y[n] = e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}$$

eigenvalue


a RAJKUMAR HIRANI film
3 idiots
a VIDHU VINOD CHOPRA production
Jab life ho out of control
BOLLYCURRY.COM
admissions open december 25
Join us at www.idiotsacademy.com


3 idiots
Jab life ho out of control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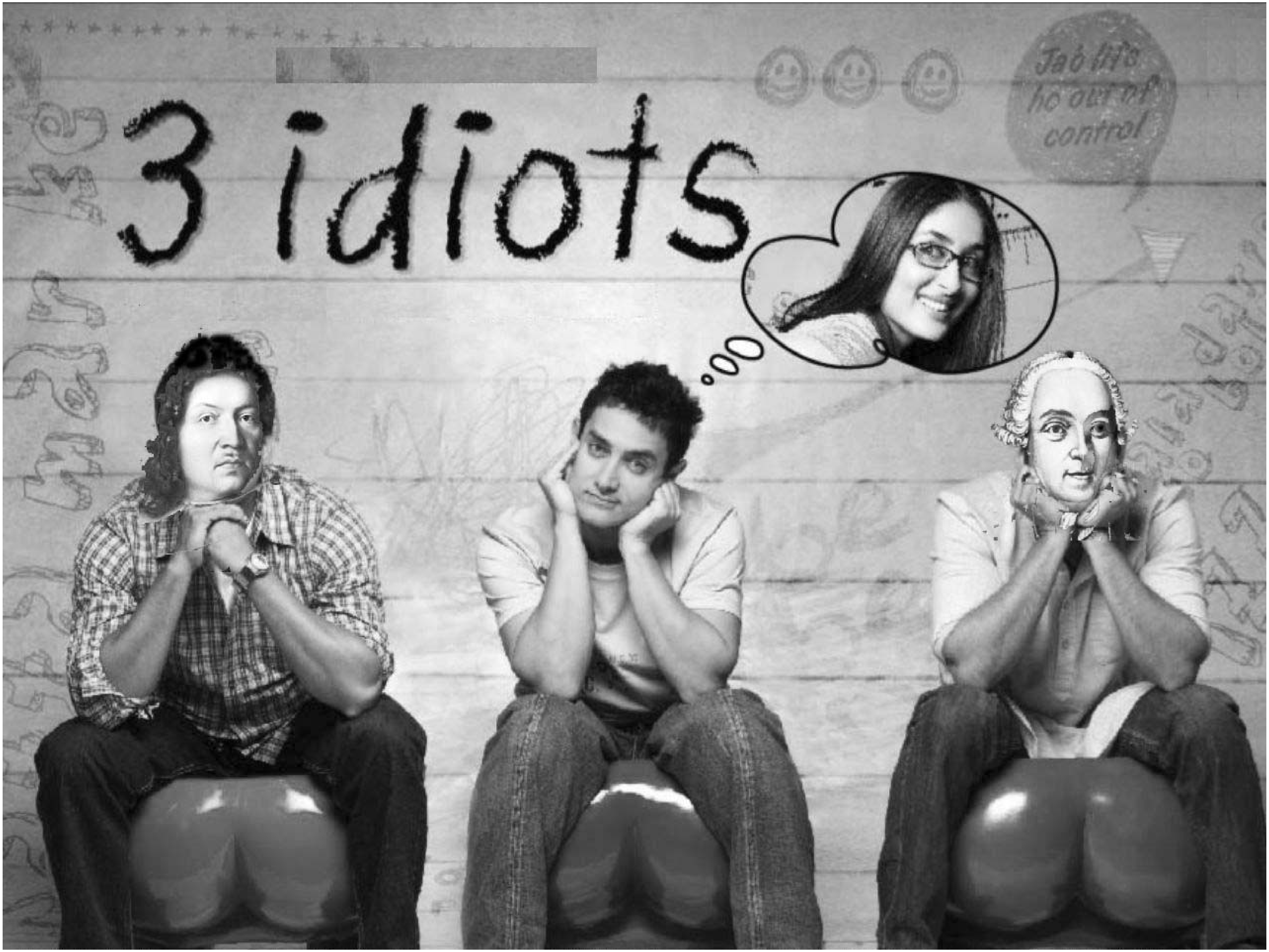
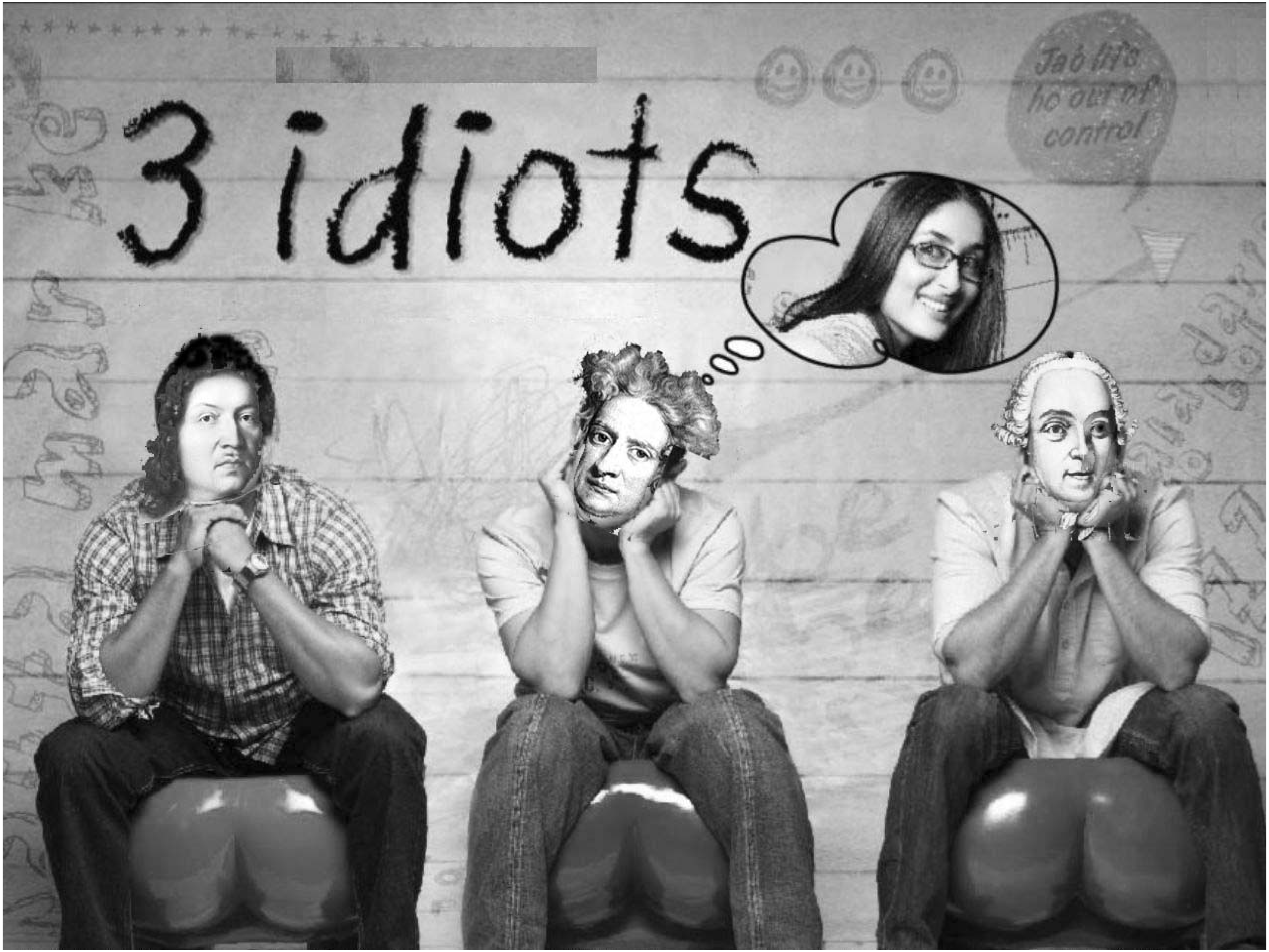

3 idiots
control


3 idiots
Jab lif's ho out of control


3 idiots
$A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)$

Story of ‘e’

- Dr. Bernolli → Underwent an apparent accident to discover constant ‘e’

- Dr. Euler → Gave the real meaning to constant ‘e’

- Dr. Fourier → Used it for analyzing periodic / aperiodic functions/signals

Summary

- ‘e’→eigenvalue→eigenfunction→fourier transform→convolution→LTI systems→bandlimited signals→ aperiodic signals→ sampling theorem→ no aliases in reconstruction→ sparse representation → inverse FT→ convolution→ phase changes marked as directional changes→ eigenfunction→ eigenvalue→ ‘e’ !!

Wavelet Transform

- Decomposes signal into two separate series

- Single series to represent most coarse version
  - Scaling Function

- Double series to represent refined version
  - Wavelet Function

Two Questions

- Aren't conventional methods to represent signals/function good enough?

- What is strikingly special about Wavelet representation?

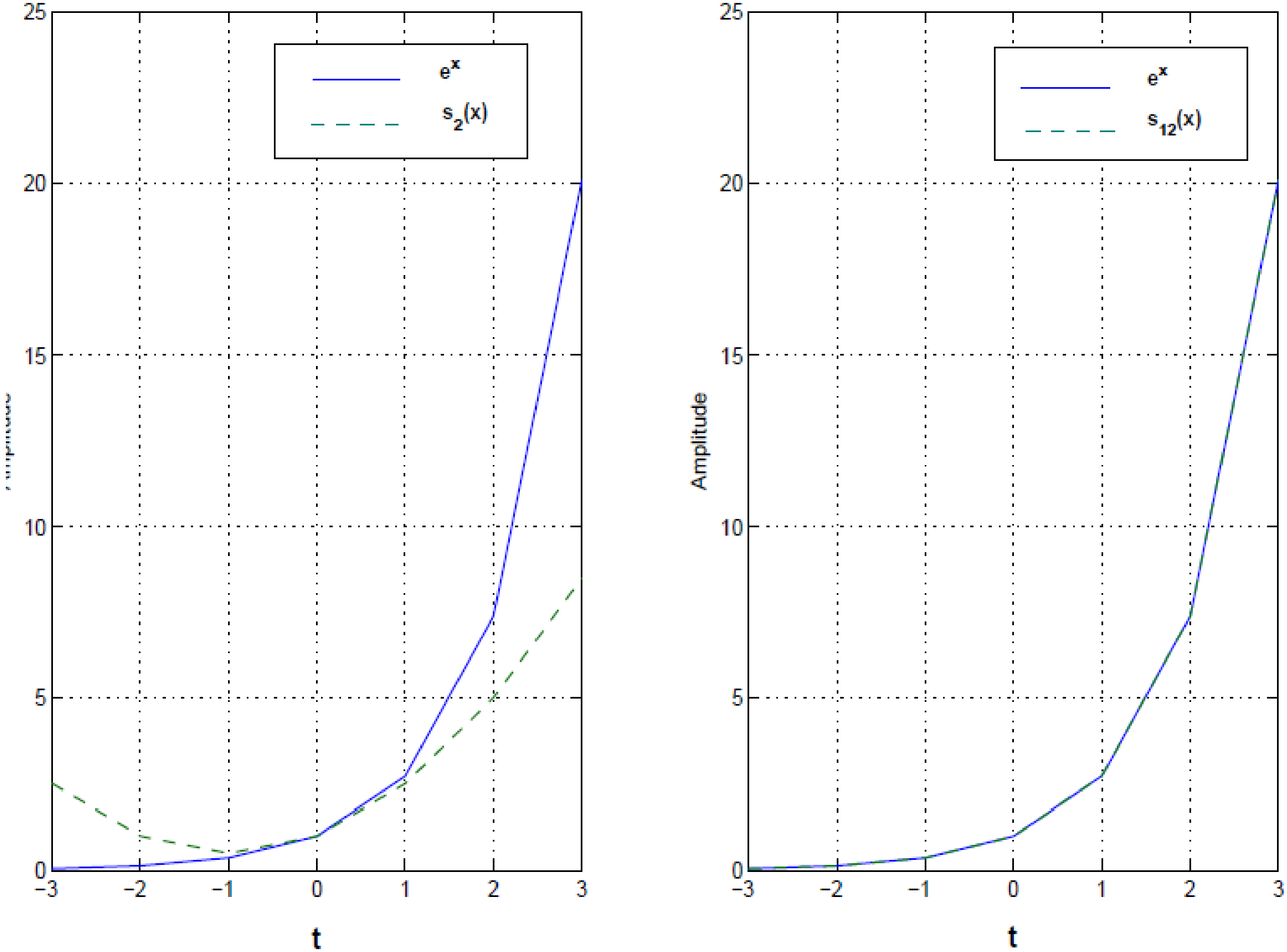
Basic representation of signals

- Known for a long time
- E.g. Taylor series expansion at x0=0

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + \ldots \qquad x \in R$$

- Decomposed pieces can be used for reconstruction

Basic representation of signals



Cooperation of series

- In Taylor series, this cooperation to build better representation is ‘rigid’
- We don’t have freedom but to add large number of terms
- In Wavelet analysis scaling function and associated wavelet function makes the representation ‘flexible’

Cooperation of series

- In Wavelet analysis the scale $\frac{1}{2^j}$ is dependent on refinement needed
- E.g. Use high value of *j* to determine spikes!
- Then, a translation $\tau_{j,k} = \frac{k}{2^j}$ can be used to focus on that part!

Fourier Series

- Noteworthy advancement of Fourier series over Taylor is set $\{1, \cos nx, \sin nx\}_{n=1}^{\infty}$ is orthogonal on $(-\pi, \pi)$, whereas powers of Taylor series, in general, are not!

$$
\begin{aligned}
f(x) &= a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx, \qquad -\pi < x < \pi \\
a_0 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) dx, \\
a_n &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin kx dx, \\
b_n &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos kx dx
\end{aligned}
$$

Fourier Series

- Special relation exists between sine and cosine parts of Fourier series
- Similar special relation exists between scaling functions and wavelet series!!

$$
\begin{aligned}
f(x) &= a_0 + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx, \qquad -\pi < x < \pi \\
a_0 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) dx, \\
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\end{aligned}
$$

Two Questions

- Aren't conventional methods to represent signals/function good enough?

- What is strikingly special about Wavelet representation?

Wavelet Transform: Speciality

- Scaling and Translation are indeed Hallmarks of Wavelet transform

- They lead us to MultiResolutioAnalysis (MRA) !!

Central Theme of MRA

- Piecewise constant approximations on UNIT intervals
- Filling in details → Zoom in OR

  Loosing details → Zoom out
- Increasing resolution → Zoom in OR

  Decreasing resolution → Zoom out
- Going arbitrarily close to the original signal!

Linear Space

$$V_0 = \begin{cases} x(t), \text{ such that} \\ \quad x(.) \in L_2(\Re) \end{cases}$$

- Space of all functions which are square integrable → $L_2(\Re)$
- And x(.) is piecewise constant on all]n,n+1[, n → integers
- Size of the interval → 2^0

Linear Space

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- Size of the interval → 2^0
- Similarly we define V_1

Linear Space

$$V_1 = \begin{cases} x(t), \text{ such that} \\ \quad x(.) \in L_2(\Re) \end{cases}$$

- Space of all functions which are square integrable → $L_2(\Re)$
- And x(.) is piecewise constant on all

  $]2^{-1}n, 2^{-1}n+1[, n \in Z$
- Size of the interval → 2^{-1}
- Similarly we define V_2

Linear Space

$$V_m = \begin{cases} x(t), \text{ such that} \\ x(.) \in L_2(\Re) \end{cases}$$

- Space of all functions which are square integrable → $L_2(\Re)$
- And x(.) is piecewise constant on all

  $]2^{-m}n, 2^{-m}n+1[, n \in Z$
- Size of the interval → 2^{-m}

Relationship

- As the spaces and spans are clear now
- Intuitionally, we observe a relationship between these spaces!

$$\ldots\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots\ldots$$

- Intuitively we can see that as we move towards right, i.e. ***up the ladder***, we are moving towards $L_2(\Re)$

Relationship

- As the spaces and spans are clear now
- Intuitionally, we observe a relationship between these spaces!

$$\ldots\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots\ldots$$

- What happens when we move in left direction i.e. ***down the ladder***?

Relationship

- As the spaces and spans are clear now
- Intuitionally, we observe a relationship between these spaces!

$$.......V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2.......$$

- The interval is going to get bigger and bigger, thus resolution shall be coarser and coarser

L2 norm

- If we require L2 norm to converge as we move in left direction, irrespective of m growing in negative direction, then,

  $\sum_{n=-\infty}^{\infty} |C_m(n)|^2$ must be zero!!!

- That is $C_m(n) = 0, \forall n$
- Hence, movement towards the left implies movement towards the trivial subspace {0}

Moving downwards

- We can write

$$\bigcap_{m \in Z} V_m = \{0\}$$

- Trivial sub-space of L2!
- It is different than null sub-space

Moving upwards

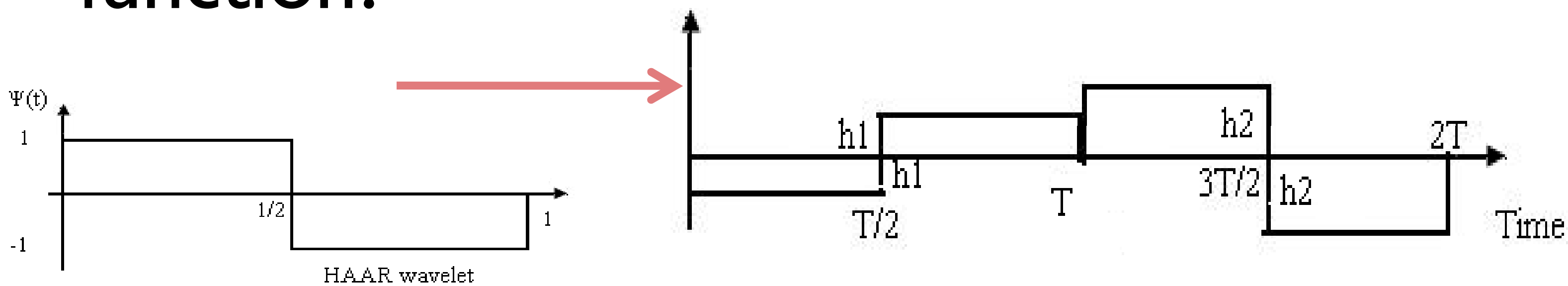
- We can write

$$\overline{\bigcup_{m \in Z} V_m = L_2(\Re)}$$

- With closure

Haar MRA – Idea of wavelets

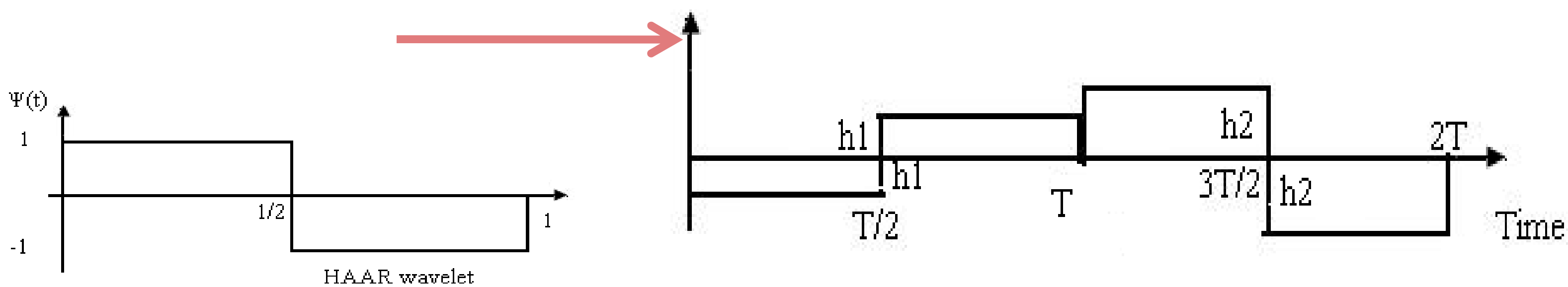
- We can construction this all using a single function!



$$f_1(t) - f_2(t) = h_1 \times \psi\left(\frac{t}{T}\right) + h_2 \times \psi\left(\frac{t-T}{T}\right)$$

Haar MRA – Idea of wavelets

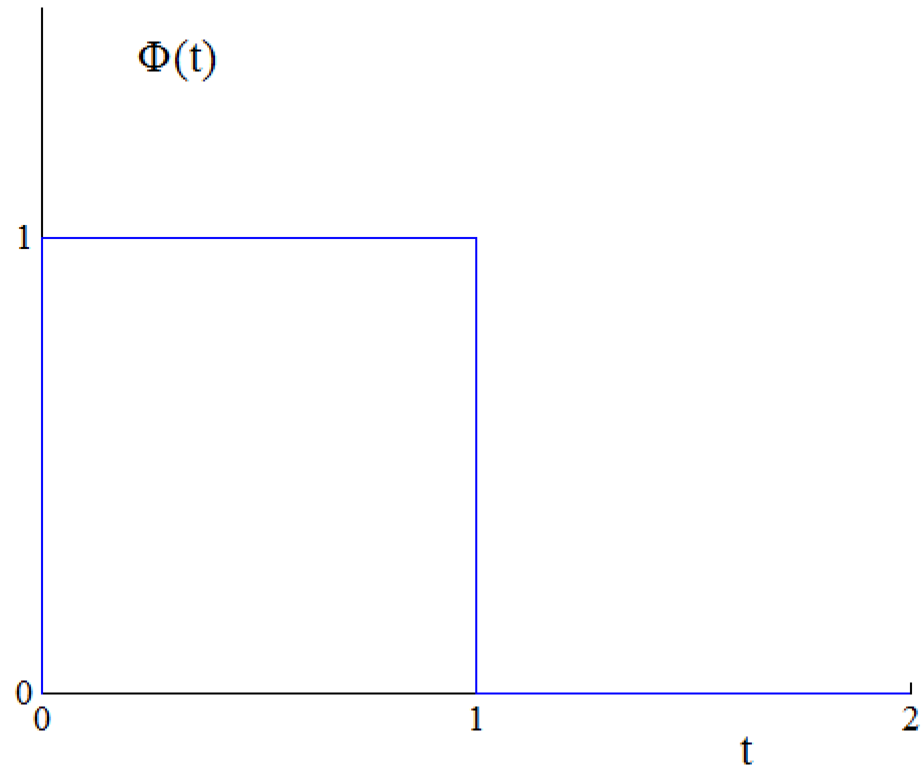
- We can construction this all using a single function! This will span W spaces



$$f_1(t) - f_2(t) = h_1 \times \psi\left(\frac{t}{T}\right) + h_2 \times \psi\left(\frac{t-T}{T}\right)$$

- What will span Vo and other spaces at that resolution??

This function !!



Scaling Function!

- Thus, any space V_m can be similarly constructed using a function $\Phi(2^m t)$

$$V_m = \underset{n \in Z}{span}\{\phi(2^m t - n)\}$$

- This will again generate ladder of subspaces!!

Axioms of MRA

- Ladder of subspaces of

  ……$V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2$…..

  are such that:

1. $$\bigcup_{m \in Z} V_m \approx L_2(\Re)$$
2. $$\bigcap_{m \in Z} V_m = \{0\}$$
3. There exists a $\Phi(t)$ such that

$$V_m = \underset{n \in Z}{span}\{\phi(2^m t - n)\}$$

Axioms of MRA

4. $\phi(t-n)_{n\in Z}$ Is an orthogonal set

5. If $f(t) \in V_m$

   then, $f(2^{-m}t) \in V_0, \forall m \in Z$

6. If $f(t) \in V_0$

   then, $f(t-n) \in V_0, \forall n \in Z$

MRA Theorem

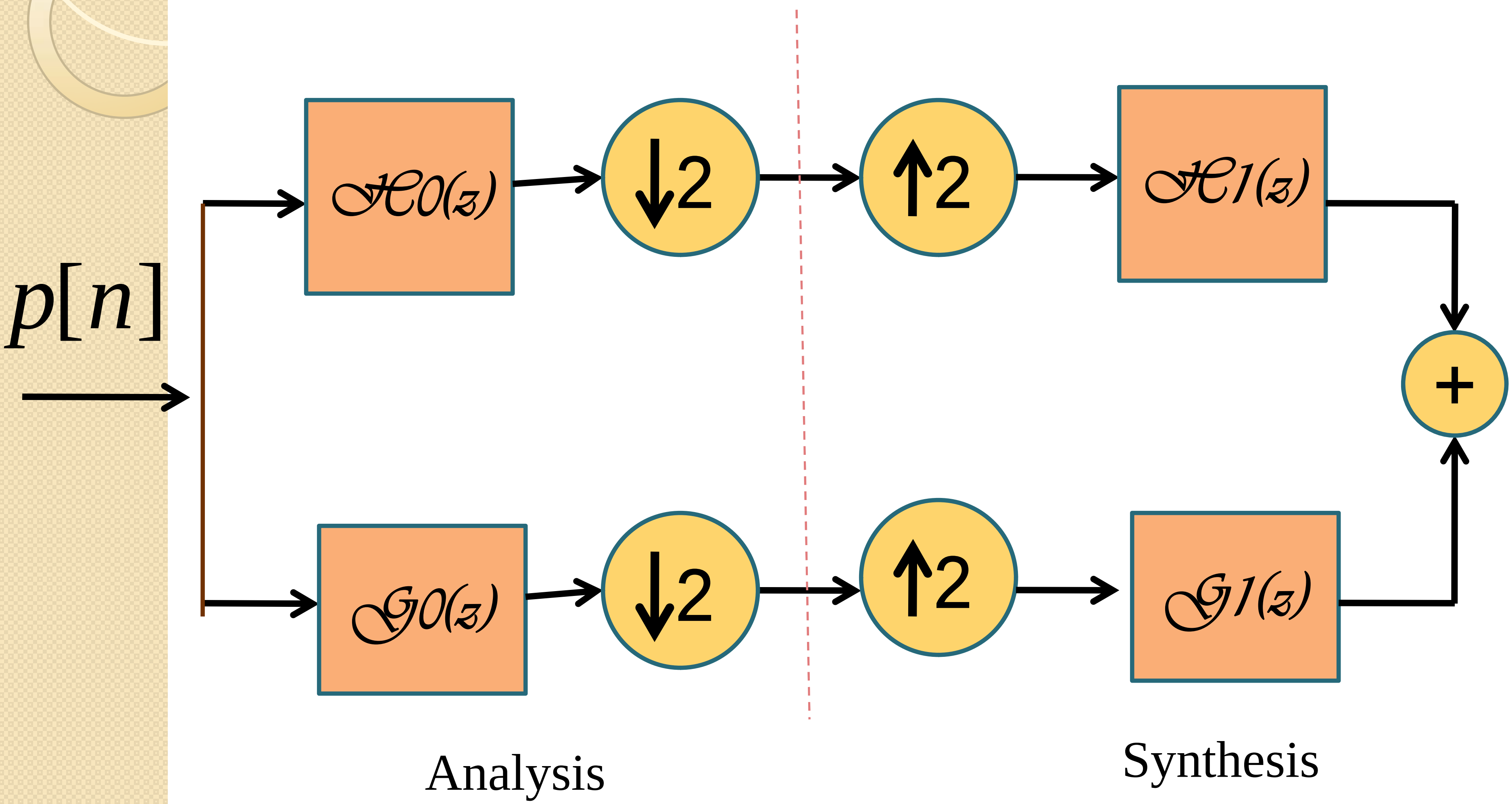
- Given these axioms, there exists a

$$\psi(.) \in L_2(\Re)$$

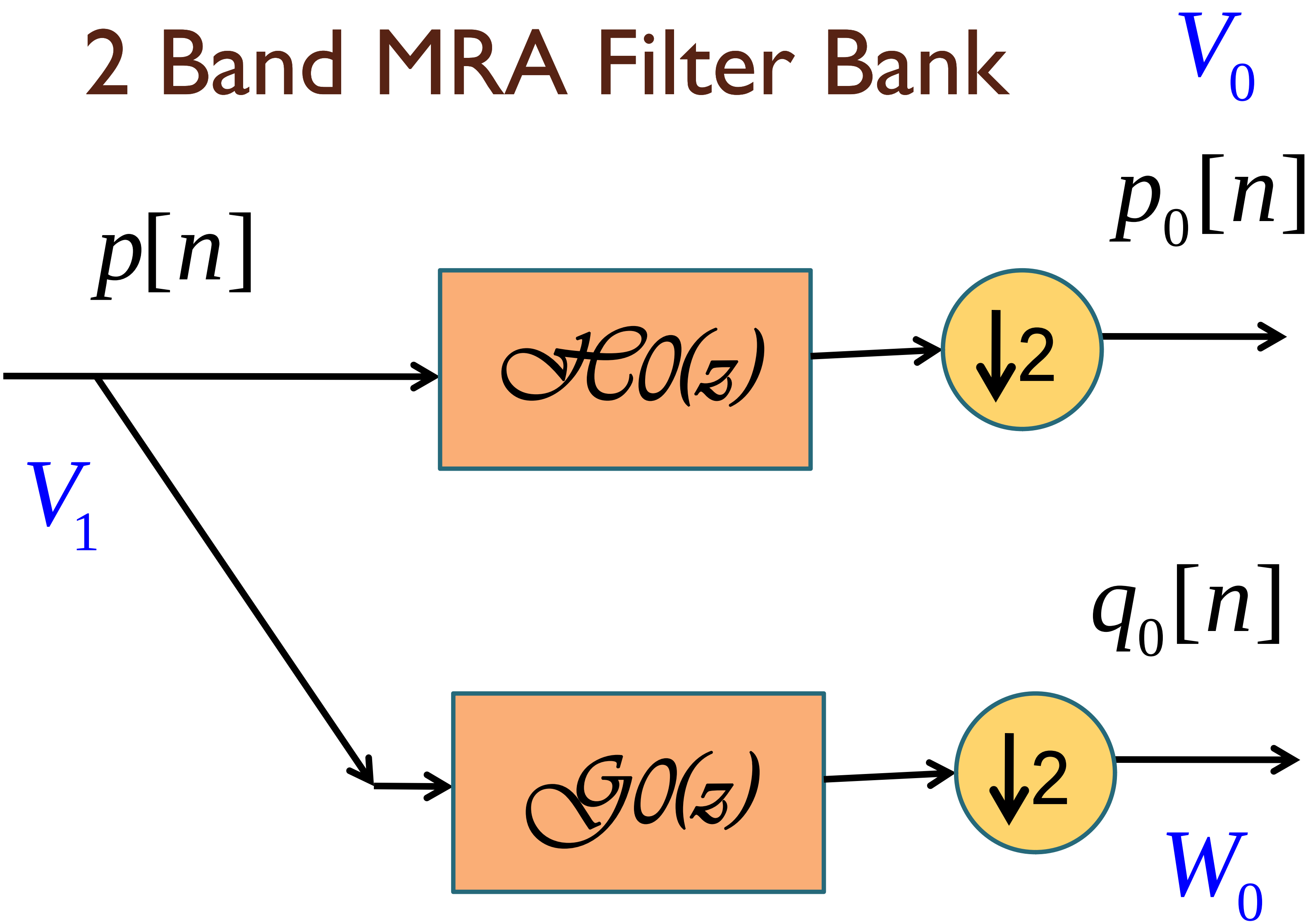
so that $\psi\{2^{-m}t - n\}_{m \in Z, n \in Z}$

span $L_2(\Re)$

2 Band MRA Filter Bank



2 Band MRA Filter Bank



Wavelet Transform

- Decomposes signal into two separate series

- Single series to represent most coarse version
  - Scaling Function

- Double series to represent refined version
  - Wavelet Function

Application

- Detecting hidden jump discontinuity
- Consider function

$$g(t) = \begin{cases} t, 0 \le t < \dfrac{1}{2} \\ t - 1, \dfrac{1}{2} \le t < 1 \end{cases}$$

- Clear jump at $t=0.5$

Application

- Detecting hidden jump discontinuity
- Let's integrate

$$h(t) = \int g(t)dt = \begin{cases} \dfrac{t^2}{2}, 0 \le t < \dfrac{1}{2} \\ \dfrac{t^2}{2} - t + \dfrac{1}{2}, \dfrac{1}{2} \le t < 1 \end{cases}$$

- Cusp jump at *t=0.5*

Application

- Detecting hidden jump discontinuity
- Let's integrate again

$$f(t) = \int h(t)dt = \begin{cases} \dfrac{t^3}{6}, 0 \le t < \dfrac{1}{2} \\ \dfrac{t^3}{6} - \dfrac{t^2}{2} + \dfrac{t}{2} - \dfrac{1}{8}, \dfrac{1}{2} \le t < 1 \end{cases}$$

- Appears smooth to eye

Wavelet Packet Analysis

$$W^{[2n]}(t) = \sqrt{2}\sum_{k} h[k].W^{[n]}(2t-k)$$

$$W^{[2n+1]}(t) = \sqrt{2}\sum_{k} g[k].W^{[n]}(2t-k)$$

Wavelet Packet Analysis

$$W^{[2n]}(t) = \sqrt{2}\sum_{k} h[k].W^{[n]}(2t-k)$$

$$h[k] = \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

$$W^{[2n+1]}(t) = \sqrt{2}\sum_{k} g[k].W^{[n]}(2t-k)$$

$$g[k] = \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$$

Wavelet Packet Analysis

$$n = 0$$

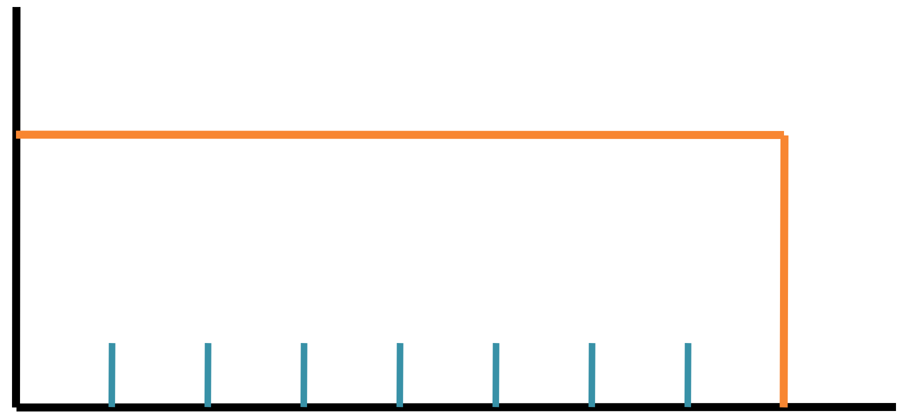
$$W_{(j,k)}^{[0]}(t) = \phi(2^j t - k)$$

$$W_{(j,k)}^{[1]}(t) = \psi(2^j t - k)$$

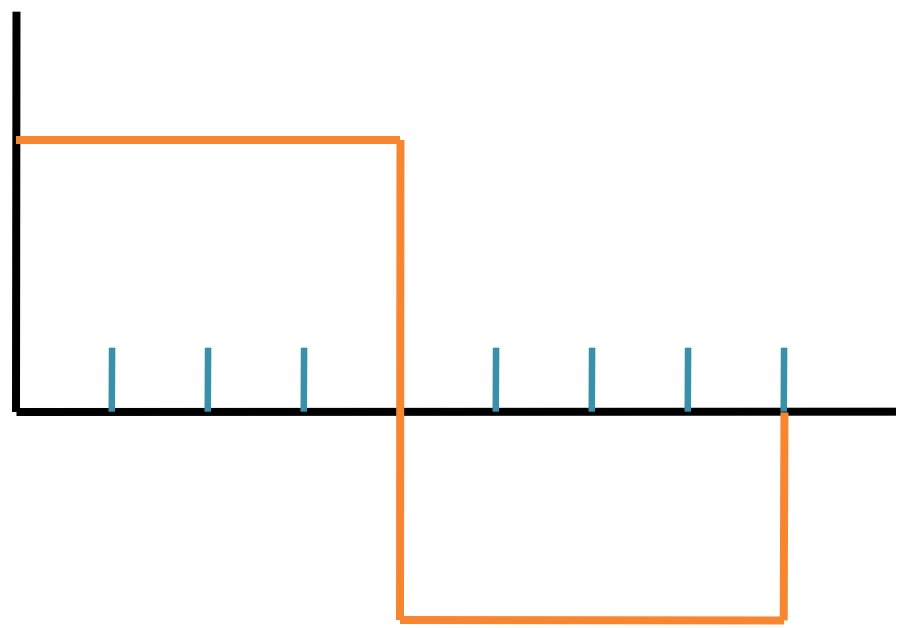
Wavelet Packet Analysis

$$n = 0$$

$$W_{(j,k)}^{[0]}(t) = \phi(2^j t - k)$$



$$W_{(j,k)}^{[1]}(t) = \psi(2^j t - k)$$



Wavelet Packet Analysis

$$n = 1$$

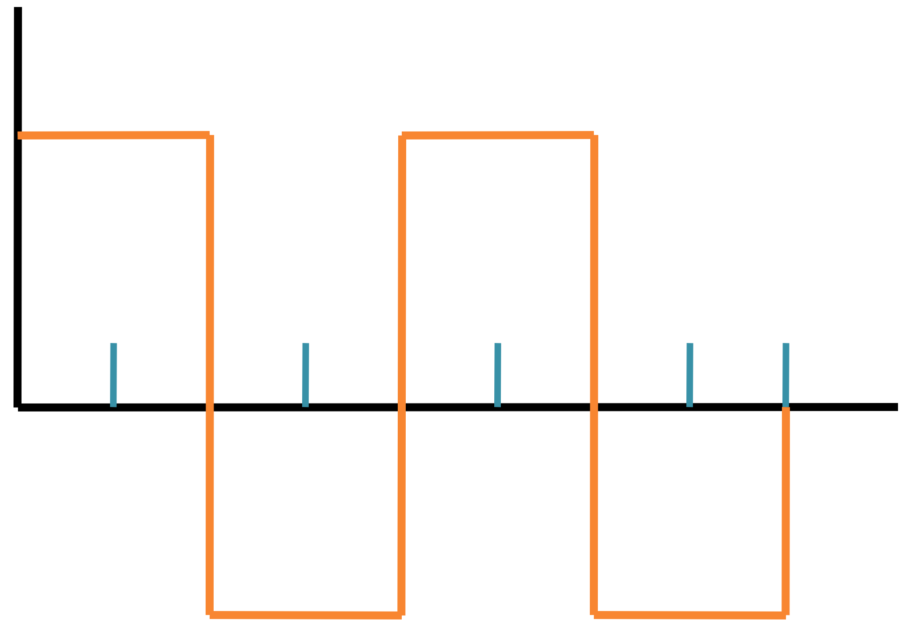
$$W^{[2]}(t) = \sqrt{2}\sum_{k} h[k].W^{[1]}(2t - k)$$

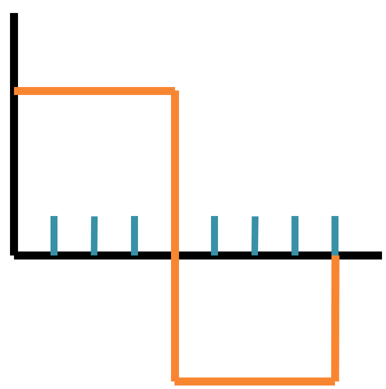
$$W^{[3]}(t) = \sqrt{2}\sum_{k} g[k].W^{[1]}(2t - k)$$

Wavelet Packet Analysis

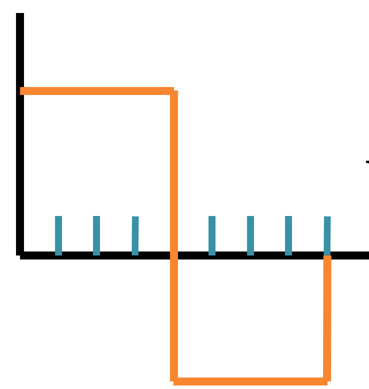
$$n = 1$$

$$W^{[2]}_{(j,k)}(t)$$





$$+$$



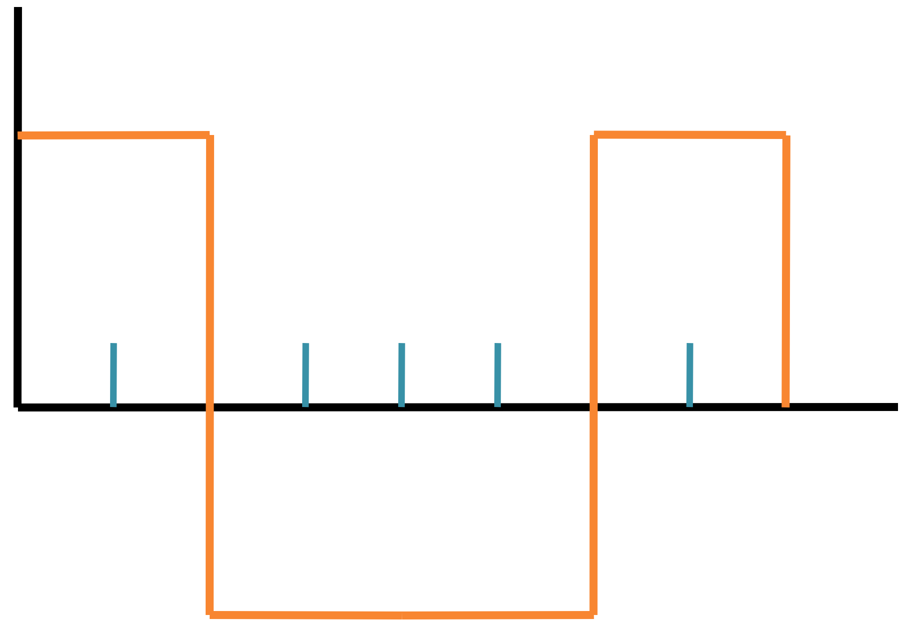
$$W^{[1]}(2t-1)$$

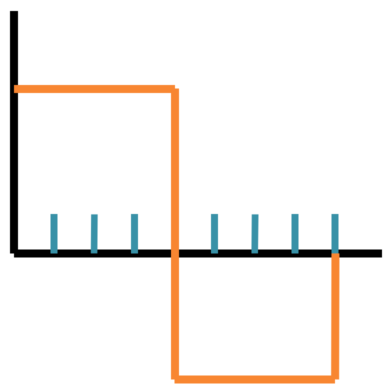
$$W^{[1]}(2t)$$

Wavelet Packet Analysis

$$n = 1$$

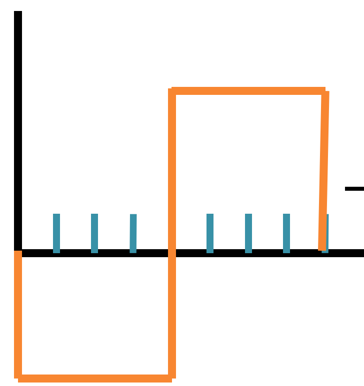
$$W^{[3]}_{(j,k)}(t)$$





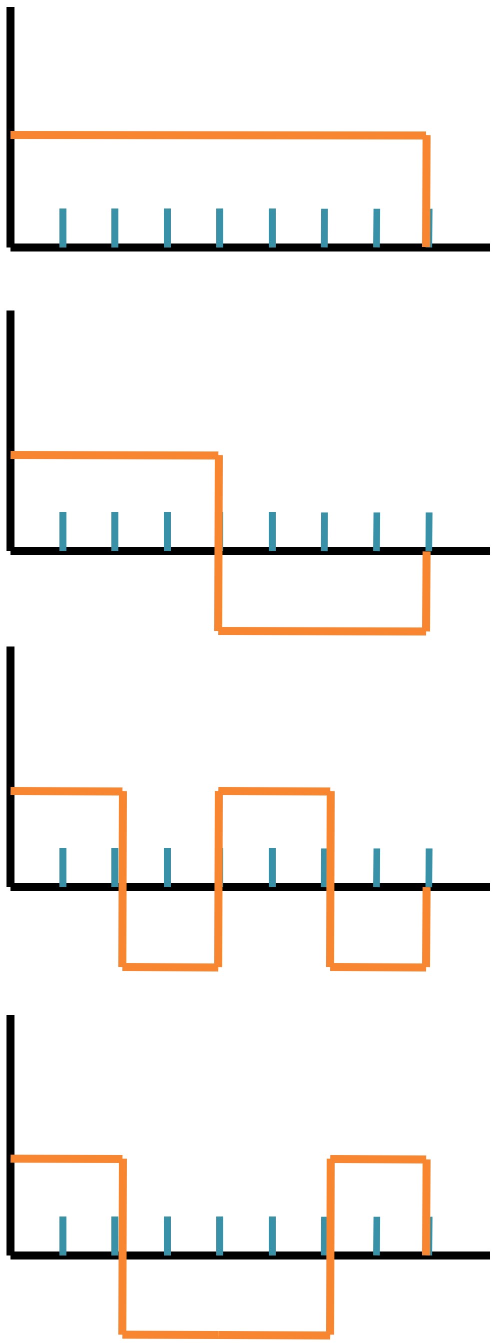
$$W^{[1]}(2t)$$

$$+$$



$$-W^{[1]}(2t-1)$$

Bases



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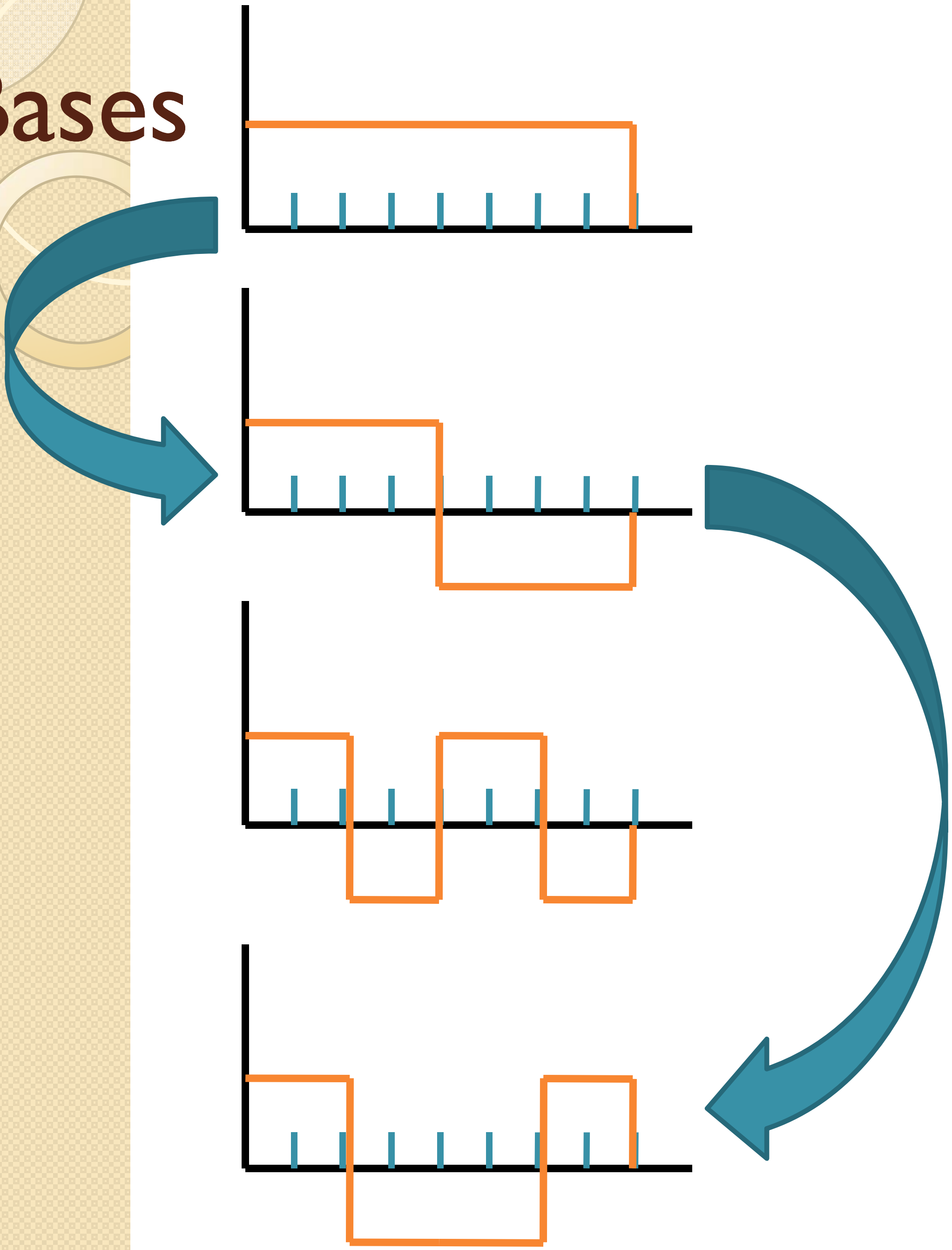

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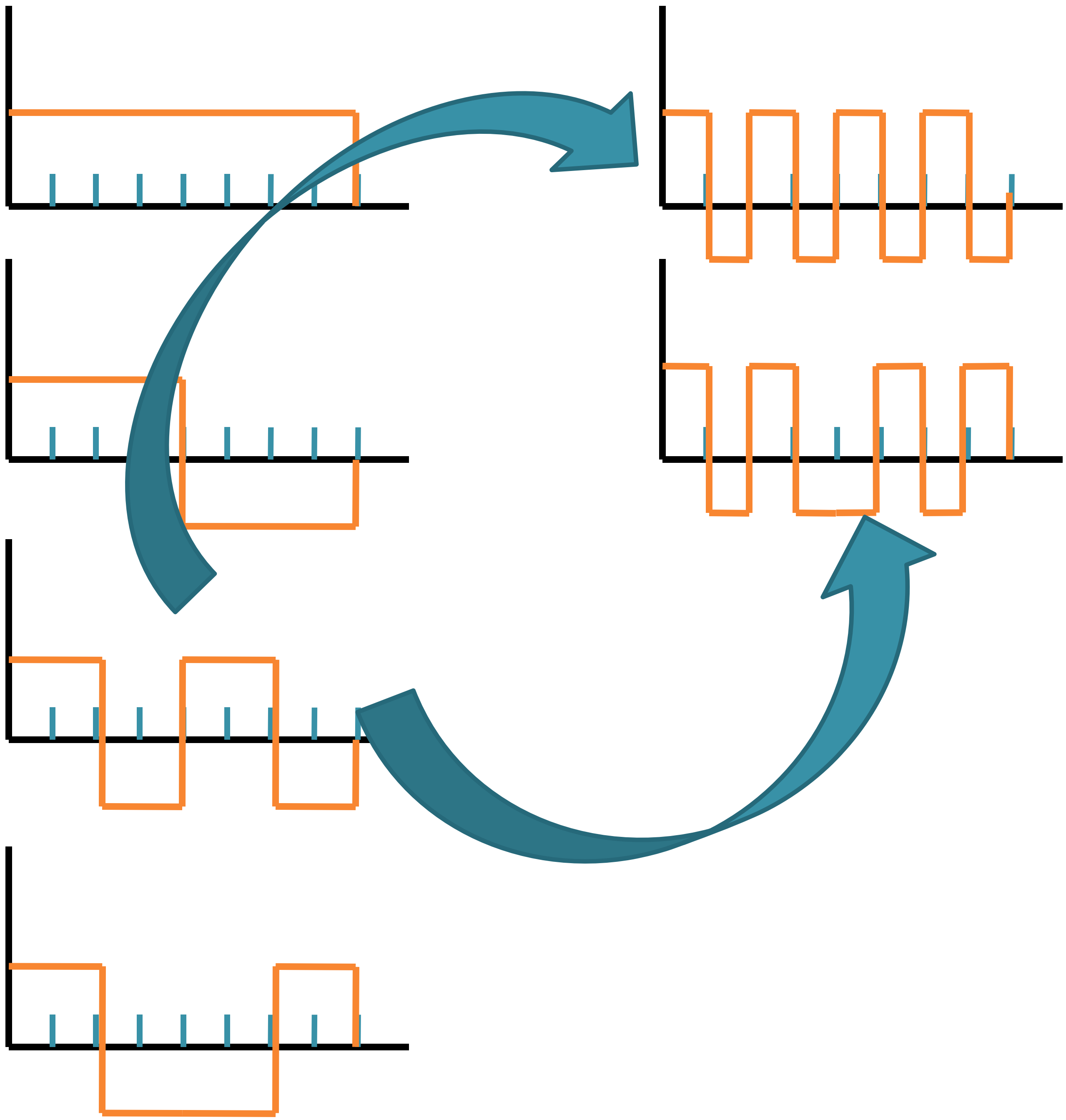
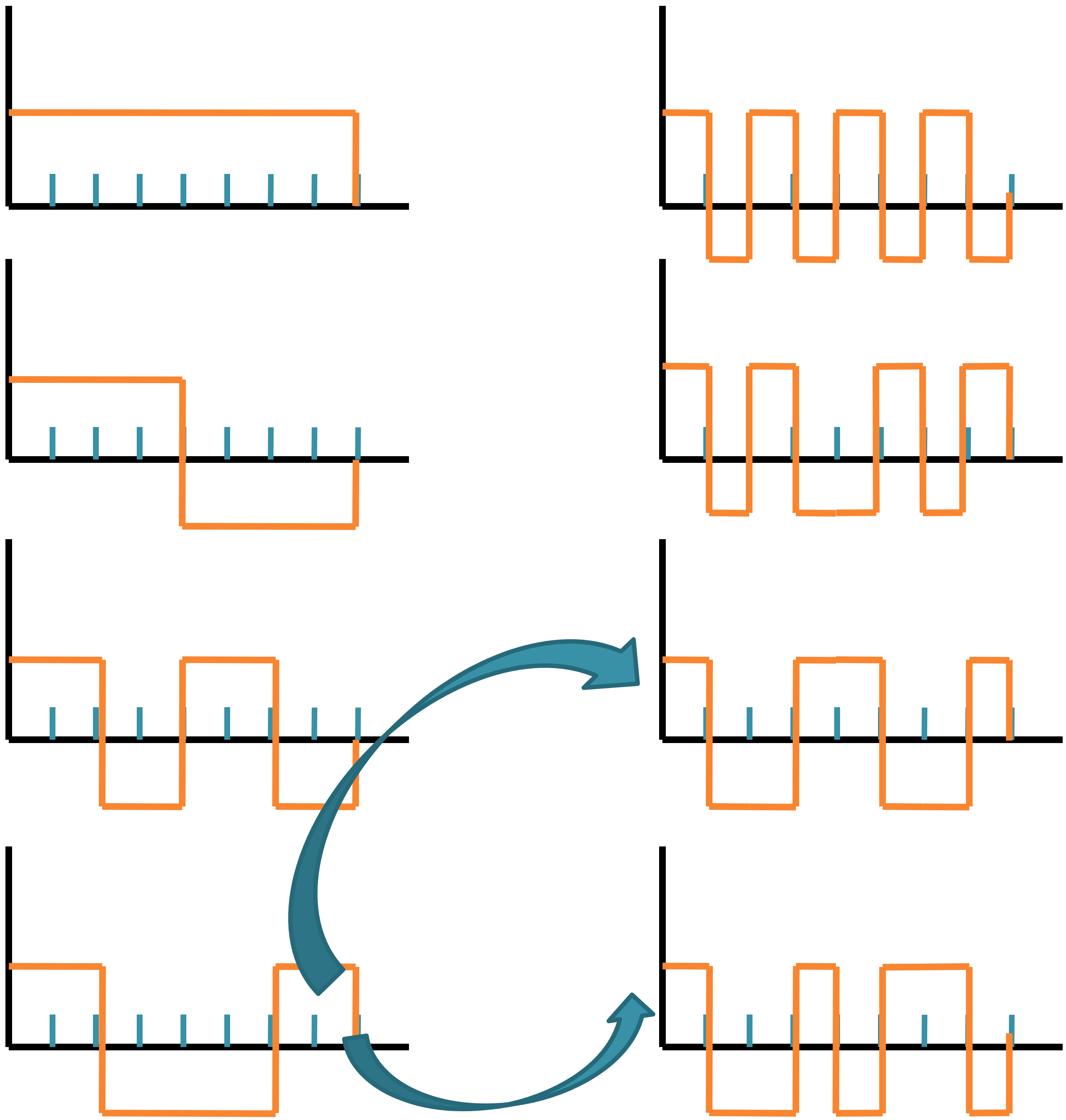


Bases



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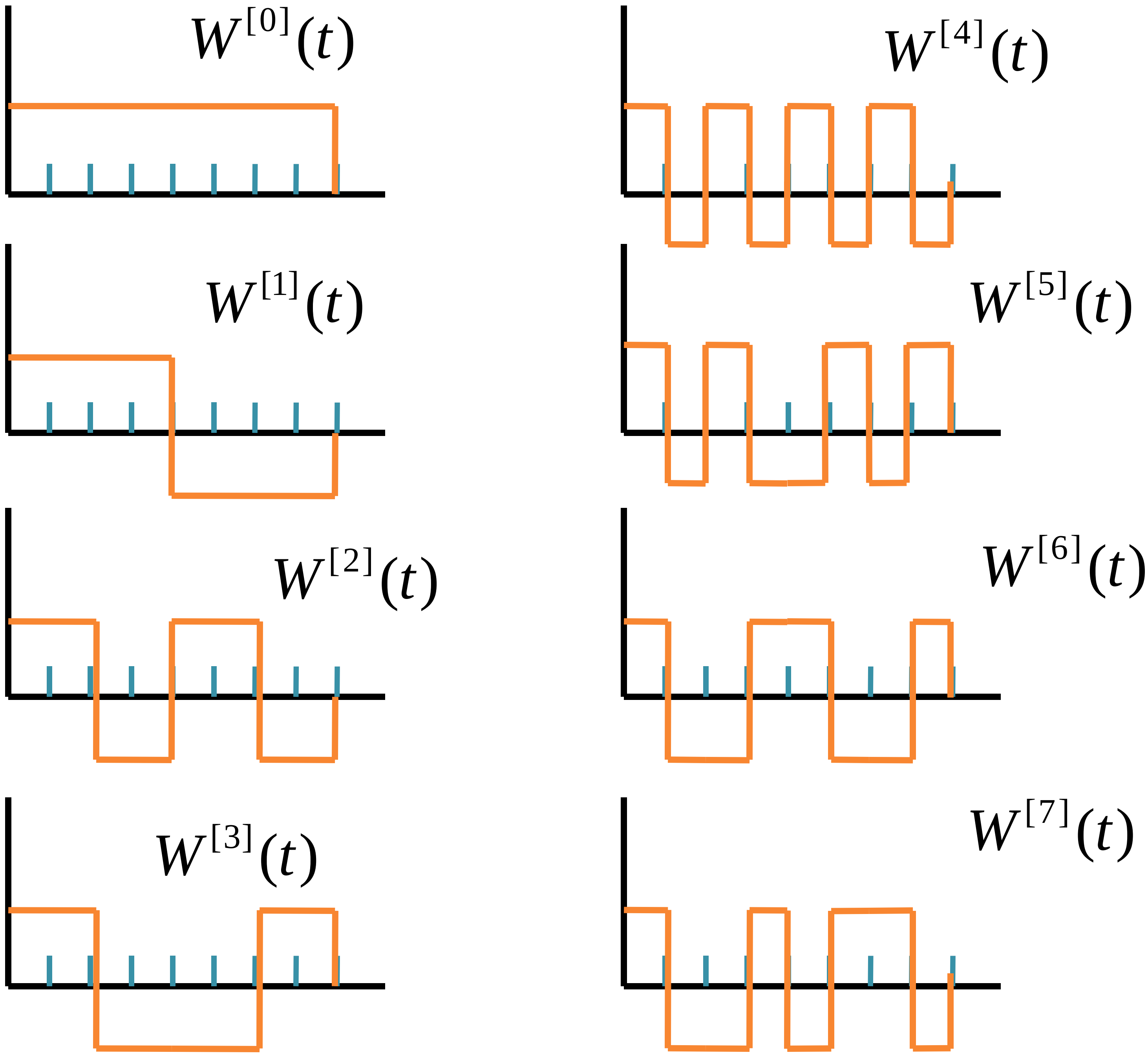

Wavelet Packet Analysis

$$W^{[2n]}(t) = \sqrt{2}\sum_{k} h[k].W^{[n]}(2t-k)$$

$$h[k] = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}$$

$$W^{[2n+1]}(t) = \sqrt{2}\sum_{k} g[k].W^{[n]}(2t-k)$$

$$g[k] = \left\{ \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\}$$

Example

$$x[n] = \{1, 0, -3, 2, 1, 0, 1, 2\} \in V_3$$

- Show complete decomposition using Haar Wavelet Packets till V0
- Demonstrate complete reconstruction

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$$x[n] = \{1, 2, 3, 4, 5, 6, 7, 8\} \in V_3$$

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Coefficients

- Who gives us coefficients of scaling equation?
- Haar

$$\psi(t) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\phi(t) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Properties of scaling coefficients

1. $\sum h_k = \sqrt{2}$

2. $\sum h_{2k} = \dfrac{1}{\sqrt{2}}$

3. $\sum h_{2k+1} = \dfrac{1}{\sqrt{2}}$

Properties of scaling coefficients

4. $\sum |h_k|^2 = 1$

5. $\sum h_{k-2l} h_k = \delta_{l,0}$

6. $\sum 2 h_{k-2l} h_{k-2j} = \delta_{l,j}$

Thank You!

Questions ??