Prof. Gadre
Date - 24-1-11
Lec - 43

SESSION 43

TUTORIAL ON UNCERTAINTY PRODUCT

Uncertainty Principle

Bound on simultaneous time and frequency localization

For a time centered function $x(t)$,

Time variance

$$\sigma_t^2(x) = \frac{\|t\,x(t)\|_2^2}{\|x(t)\|_2^2}$$

$$\frac{|x(t)|^2}{\|x(t)\|_2^2}$$

Constitutes a prob density function

Time center = t_0

$$\int t \cdot \frac{|x(t)|^2}{\|x(t)\|_2^2} \, dt$$

Time centered,
we mean
$t_0 = 0$.
Similarly freq

$$x(t) \xrightarrow[\text{Transform}]{\text{Fourier}} \hat{x}(\Omega)$$

$$\frac{|\hat{x}(\Omega)|^2}{\|\hat{x}(\Omega)\|_2^2} \text{ is}$$

FREQUENCY DENSITY

Frequency mean Ω_0
or frequency 'centre'

$$= \frac{\int \Omega \, |\hat{x}(\Omega)|^2 \, d\Omega}{\|\hat{x}(\Omega)\|_2^2}$$

Frequency centred

$\Rightarrow \Omega_0 = 0$.

Frequency variance

$$= \frac{\|\Omega \hat{x}(\Omega)\|_2^2}{\|\hat{x}(\Omega)\|_2^2}$$

for $\Omega_0 = 0$

For an $\mathcal{L}_2(\mathbb{R})$ function

Time variance ×

Frequency variance

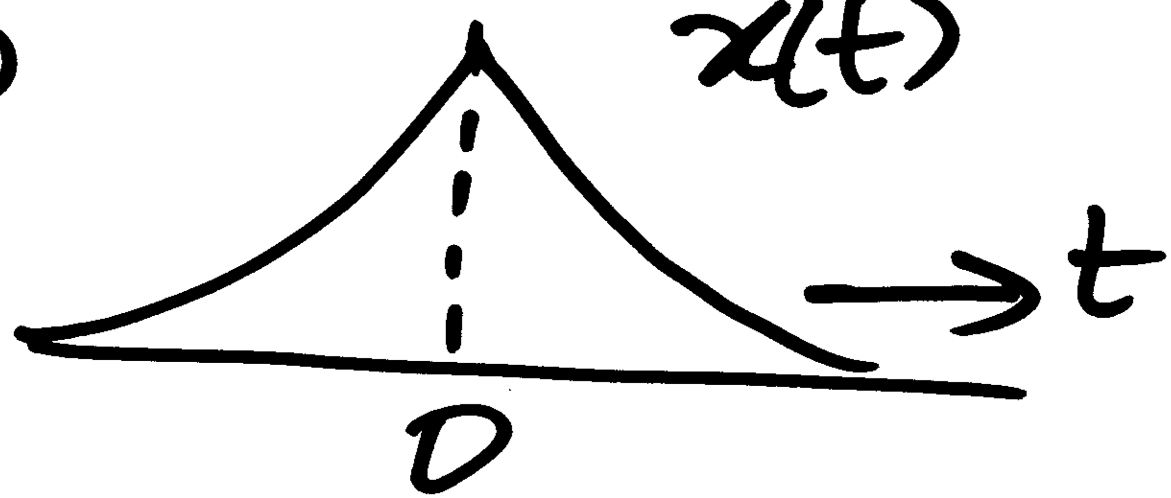
lower bounded (by 0.25)

Example 1 Calculate:

Uncertainty product of $e^{-|t|}$ for all t.

$$x(t) = e^{-|t|}$$

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$



This function $x(t)$
is real and even:
hence: both time
and frequency
centered.

Time variance:

$$\|x\|_2^2 = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

$$= 2\int_0^{\infty} e^{-2t} \, dt = \left. \frac{2e^{-2t}}{-2} \right|_0^{\infty}$$

$$= \frac{2}{2} = \underline{\underline{1}}$$

$$\|x\|_2^2 = 1$$

L_2 norm^2 of $t\,x(t)$

$$= \|t\,x(t)\|_2^2$$

$$\int_{-\infty}^{+\infty} |t x(t)|^2 \, dt$$

$$= 2 \int_{0}^{\infty} \left(t e^{-t^2}\right) dt$$

Consider

$$\int (te^{-t})^2 \, dt$$

$$= \int t^2 e^{-2t} \, dt$$

$$= t^2 \cdot \frac{e^{-2t}}{-2}$$

$$- \int 2t \cdot \frac{e^{-2t}}{-2}\, dt$$

$$\int_{\cancel{0}}^{\cancel{\infty}} t\, e^{-2t}\, dt$$

$$= t \cdot \frac{e^{-2t}}{-2} - \int \frac{e^{-2t}}{-2}\, dt$$

limits $0 \to \infty$

$$\cdots -\frac{t}{2}\cdot e^{-2t} + \frac{1}{2}\int e^{-2t}\,dt$$

Indefinite integral

$$\int t^2 e^{-2t}\,dt$$

$$= \frac{t^2}{-2} e^{-2t} + \cdots$$

Thus $\int_0^\infty t^2 e^{-2t}\,dt$

$$= \frac{1}{2}\int_0^\infty e^{-2t}\,dt$$

$$= \frac{1}{2} \cdot \frac{e^{-2t}}{-2} \Big|_0^{\infty}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot$$

Time variance

$$= 2\int_0^{\infty} t^2 e^{-2t}\,dt$$

$$= 2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

Frequency Variance

$$= \frac{\|\Omega \hat{x}(\Omega)\|_2^2}{\|\hat{x}(\Omega)\|_2^2}$$

$$= \frac{\| j\Omega \hat{x}(\Omega) \|_2^2}{\| \hat{x}(\Omega) \|_2^2}$$

Parseval's theorem

$\Longrightarrow$

$$= \frac{\left\| \frac{dx(t)}{dt} \right\|_2^2}{\|x(t)\|_2^2}$$

$$x(t) = e^{-|t|}$$

$$\frac{dx(t)}{dt} = \begin{cases} (-1)e^{-t} & t > 0 \\ e^{t} & t < 0 \end{cases}$$

$$\left\| \frac{dx(t)}{dt} \right\|_2^2$$

$$= \|x(t)\|_2^2 = 1$$

in this case

Frequency variance

$= 1$

Time variance

$= 1/2$

Uncertainty product

$$= \frac{1}{2} \times 1 = \frac{1}{2}$$

$$= 0.5 > 0.25$$

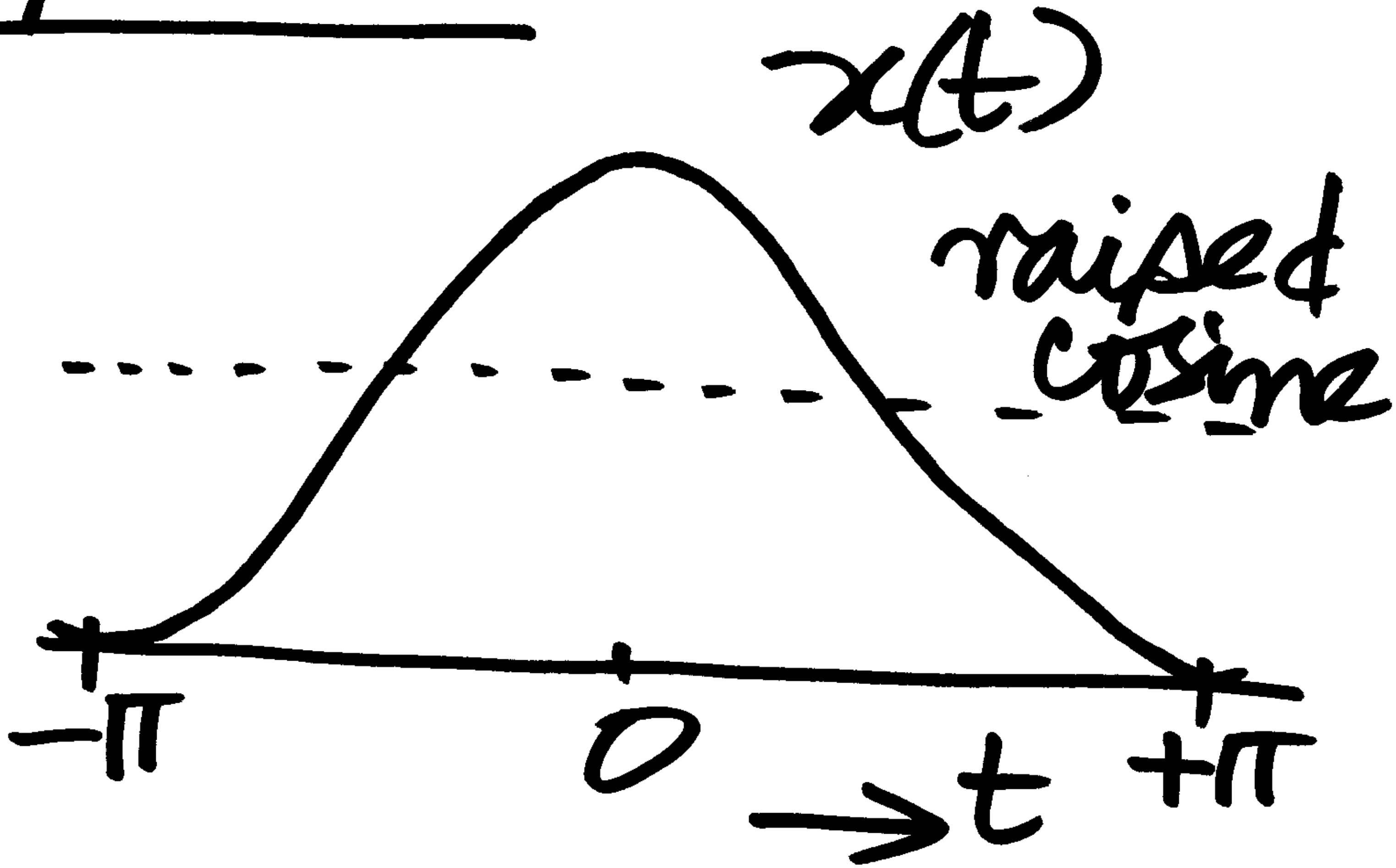
Example 2

'Raised cosine' function

$$x(t) = 1 + \cos t \qquad -\pi \leq t \leq +\pi$$

$$0 \quad \text{else}$$

Sketch



Its L_2 norm^2

$$= 2\int_0^{\pi} |x(t)|^2 \, dt$$

(real and even)

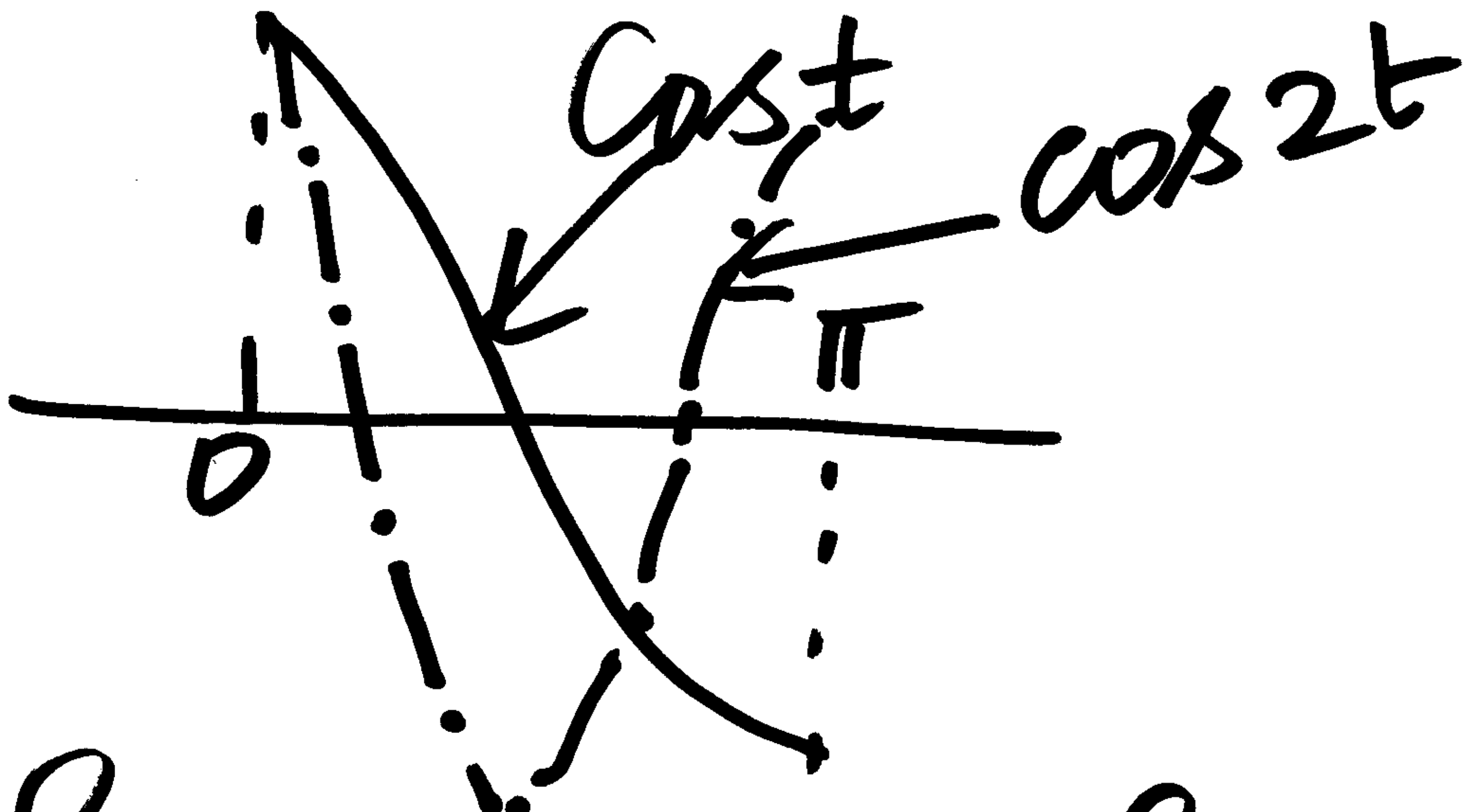
$$= 2\int_0^{\pi} (1+\cos t)^2 \, dt$$

$$= 2\int_0^{\pi} (1+\cos^2 t + 2\cos t) \, dt$$

Consider

$$\int_0^{\pi}\left(1+\cos^2 t+2\cos t\right)dt$$

$$=\int_0^{\pi}\left\{1+\frac{1+\cos 2t}{2}+2\cos t\right\}dt$$



have 0 integral over]0, π[

$$= \int_0^{\pi} \left(1 + \frac{1}{2}\right) dt$$

$$= \underline{\underline{\frac{3}{2}\pi}}$$

$$\|x\|_2^2 = 2 \times \frac{3}{2}\pi$$

$$= \underline{\underline{3\pi}}.$$

The function is already time and frequency centered.
(real and even)

Frequency variance:

$$\left\| \frac{dx(t)}{dt} \right\|_2^2$$

$$\frac{dx(t)}{dt} = \frac{d}{dt}(1+\cos t)$$

$$= \frac{-\sin t}{2\int_0^{\pi} (\sin^2 t)\, dt}$$

$$= 2 \cdot \int_0^{\pi} \frac{1 - \cos 2t}{2} \, dt$$

$$= \int_0^{\pi} dt = \pi$$

Freq. Variance $= \frac{\pi}{2\pi}$

To calculate the time variance we need

$$\int t^2 |x(t)|^2 dt$$

$$= \int t^2 (1+\cos t)^2 dt$$

$$\int t^2(1+\cos t)^2\,dt$$

$$= \int t^2\left(1+\frac{1+\cos 2t}{2}+2\cos t\right)dt$$

$$\int_0^{\pi} t^2 \cos 2t \, dt \quad \text{and} \quad \int_0^{\pi} t^2 \cos t \, dt$$

are required.

$$\int t^2 \cos mt \, dt$$

$$= t^2 \frac{\sin mt}{m} - \int 2t \frac{\sin mt}{m} \, dt$$

$$\int t \sin mt \, dt$$

$$\int t \sin mt \, dt =$$

$$-\frac{t \cos mt}{m} + \int \frac{\cos mt}{m} dt$$

$$= -\frac{t}{m} \cos mt + \frac{\sin mt}{m^2}$$

Thus: $\int t^2 \cos mt \, dt$

$$= \frac{t^2}{m} \cdot \sin mt - \frac{2}{m}\left\{-\frac{t}{m}\cos mt + \frac{\sin mt}{m^2}\right\}$$

We need only consider the term

$$\frac{2t}{m^2}\cos mt\Big|_0^{\pi}$$

$n = 1$

$$2t\cos t \Big|_0^{\pi}$$

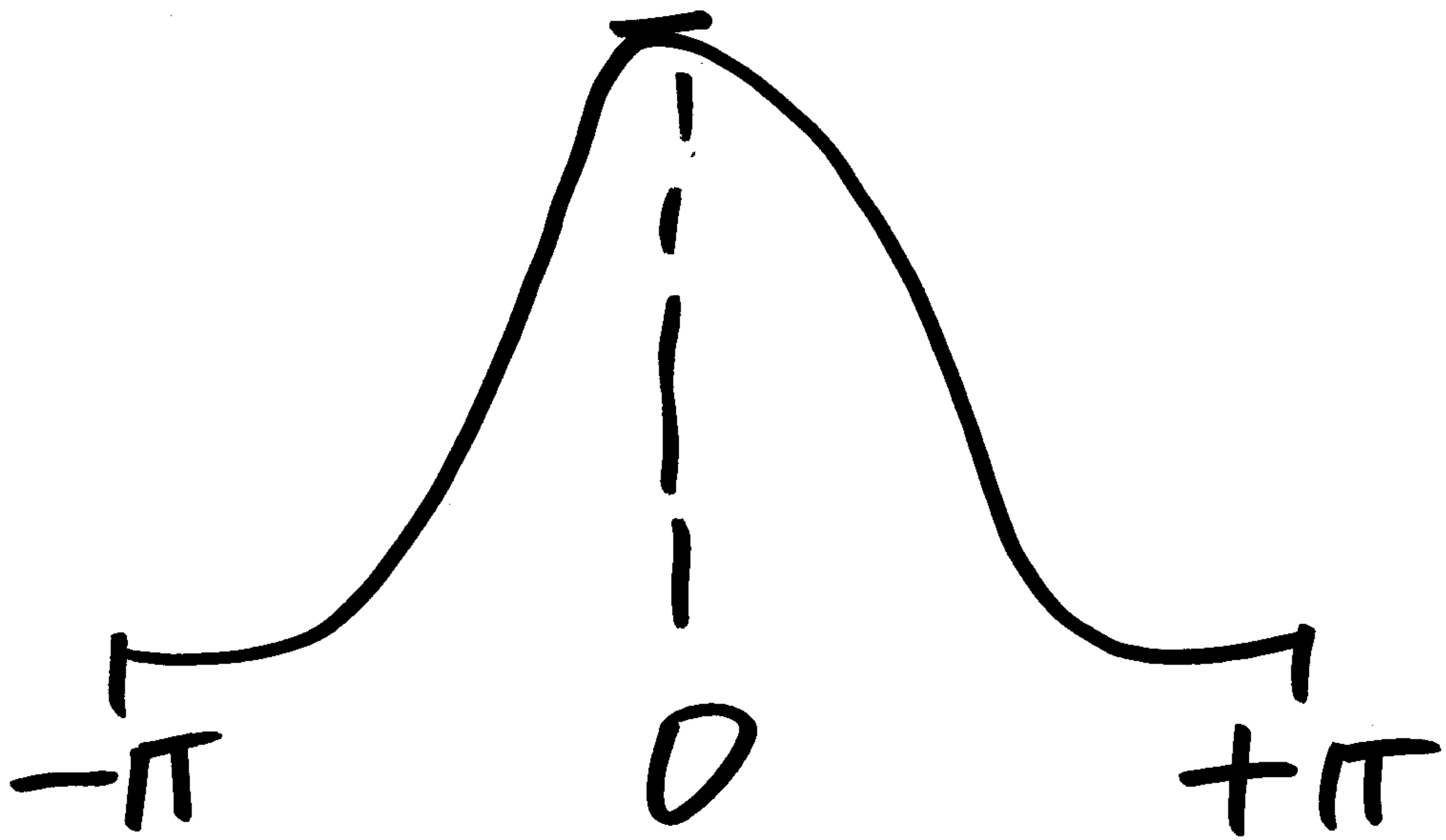
$$= 2\pi \cos \pi$$

$$= -2\pi$$

$m = 2:$

$$\frac{2t}{4}\cos 2t\Big|_0^{\pi}$$

$$\frac{2}{4}\cdot\pi\cdot(1) = \frac{\pi}{2}$$



$-\pi$
0
$+\pi$