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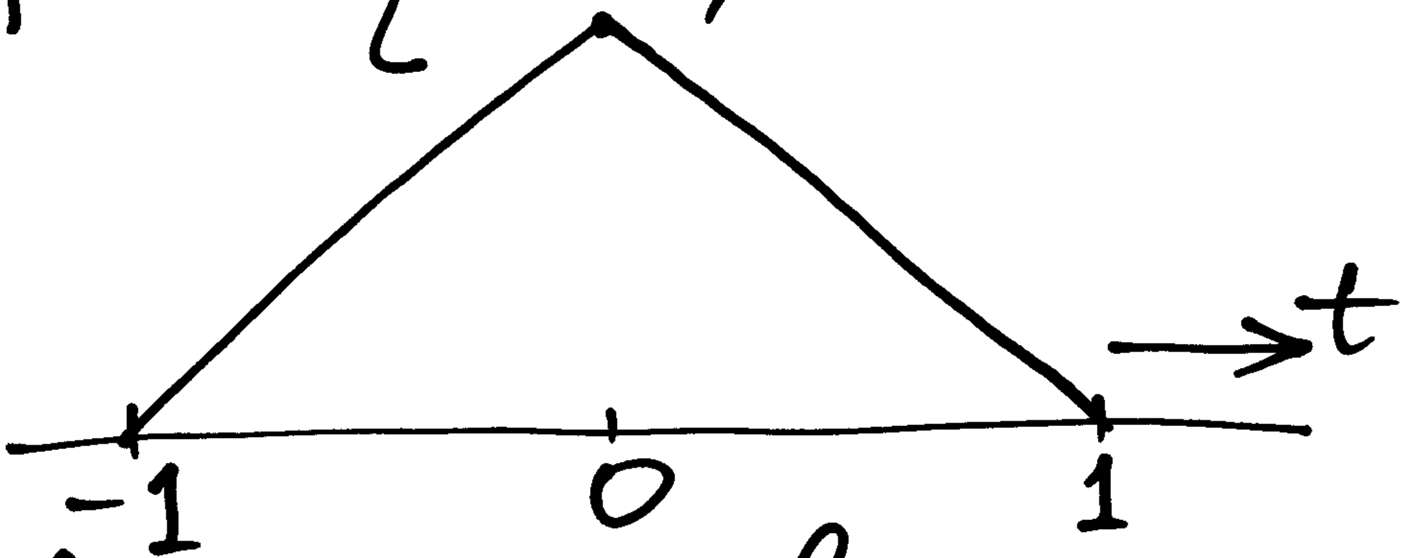
Lec-41

TUTORIAL SESSION 1

Problem:
Consider the following two functions:

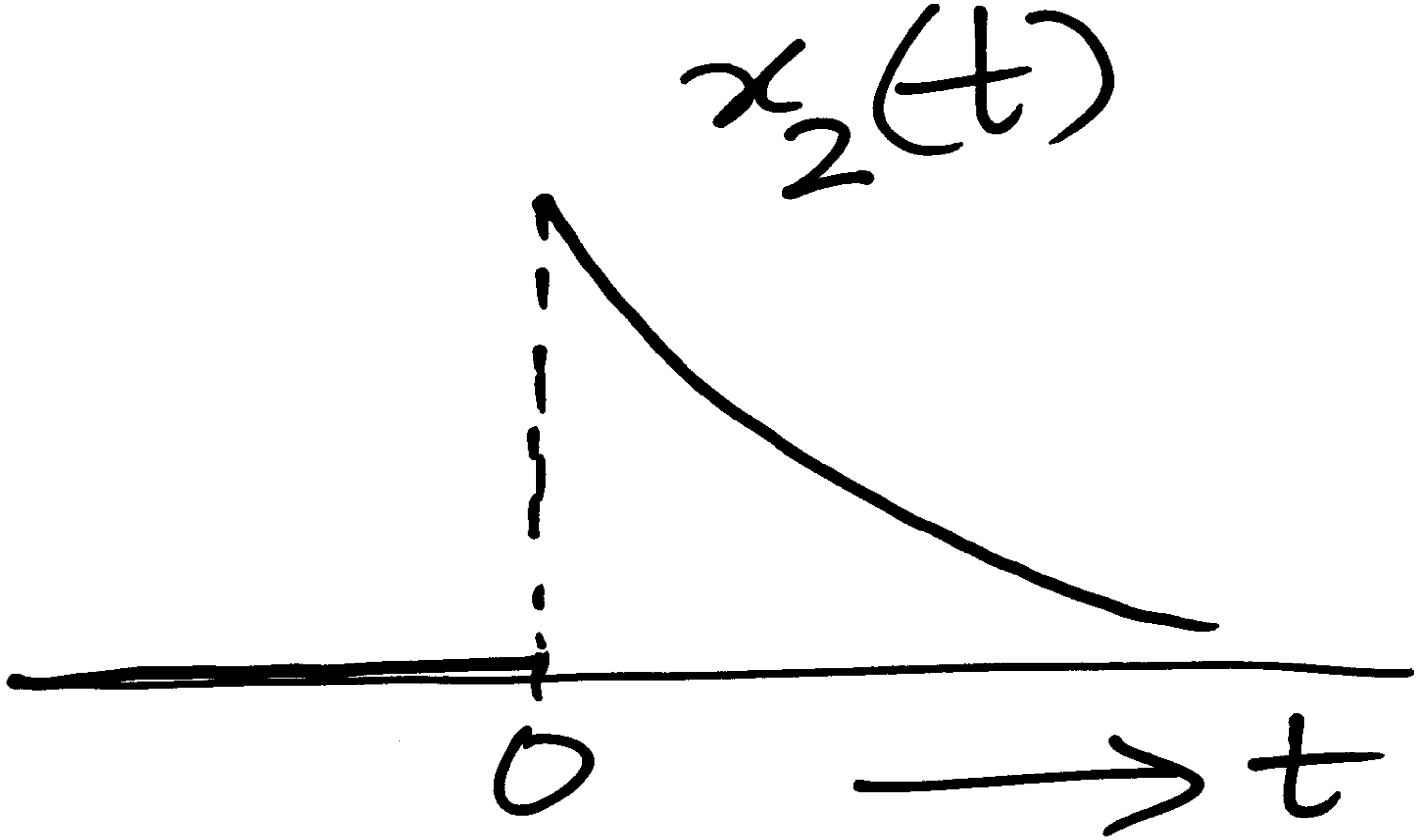
$$x_1(t) = \begin{cases} 1 - |t|, & -1 \le t \le +1 \\ 0, & \text{else} \end{cases}$$



Continuous function

$$x_2(t) = e^{-t}, \; t \geq 0$$
$$= 0 \text{ else}$$

Discontinuous function


$x_2(t)$
0
t

Q1: Verify that these belong to $\mathcal{L}_2(\mathbb{R})$. Find their norms in $\mathcal{L}_2(\mathbb{R})$

Norm squared of x_1 in $\mathcal{L}_2(\mathbb{R})$ =

$$\int_{-\infty}^{+\infty} |x_1(t)|^2 \, dt$$

From symmetry,

$$= 2\int_0^1 (1-t)^2\,dt$$

$$\lambda = 1-t$$

$$= 2\int_0^1 \lambda^2 \, d\lambda$$

$$= \frac{2\lambda^3}{3}\Big|_0^1$$

$$= 2/3$$

$$\|x_1\|_2^2 = 2/3$$

$$\|x_1\|_2 = \sqrt{2/3}$$

$$\|x_2\|_2^2 =$$

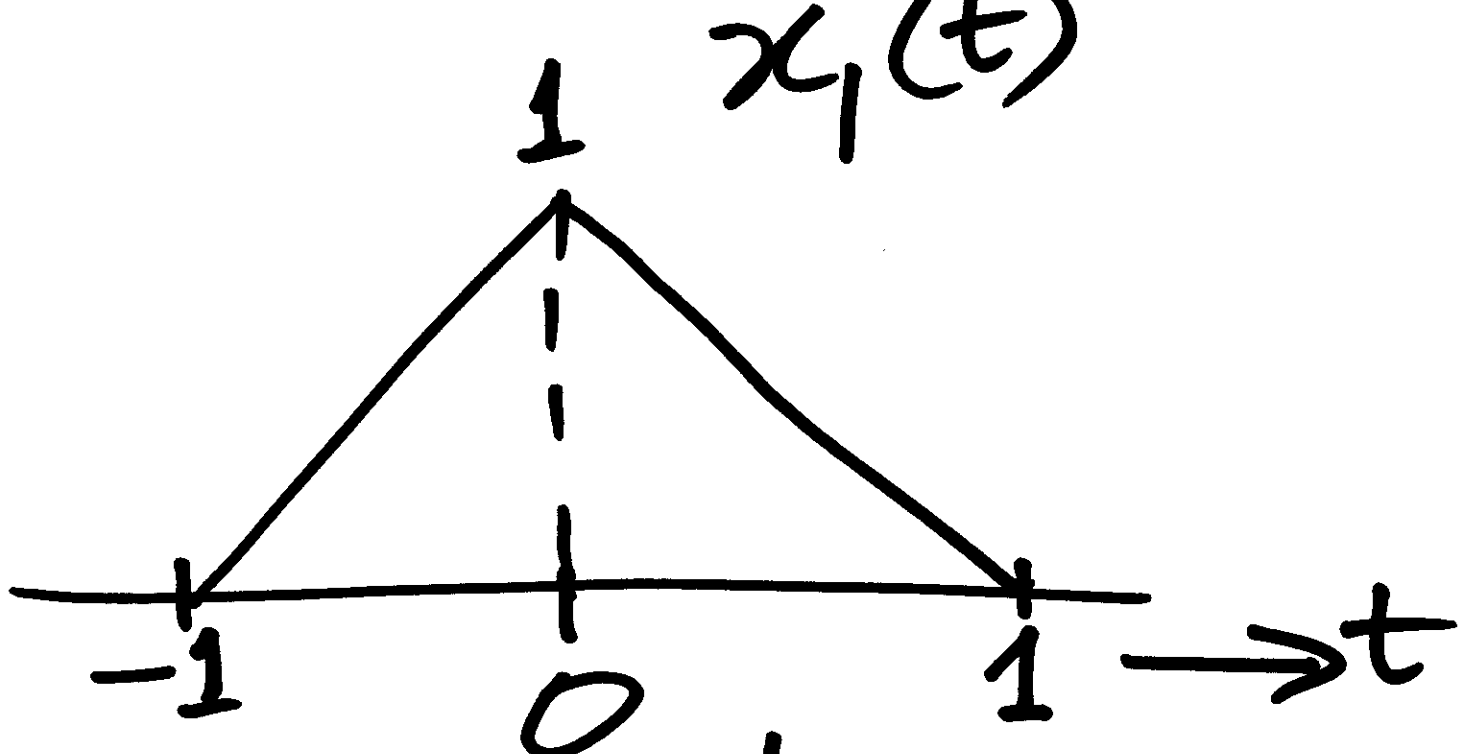
$$\int_0^\infty \left(e^{-t}\right)^2 dt$$

$$= \int_0^\infty e^{-2t}\, dt$$

$$= \left. \frac{e^{-2t}}{-2} \right|_0^\infty = \frac{1}{2}$$

$$\Rightarrow \|x_2\|_2 = \frac{1}{\sqrt{2}}.$$

Q2 - Obtain their projections on the space V_0 in the Haar MRA.



Nonzero projection only in $]-1,+1[$

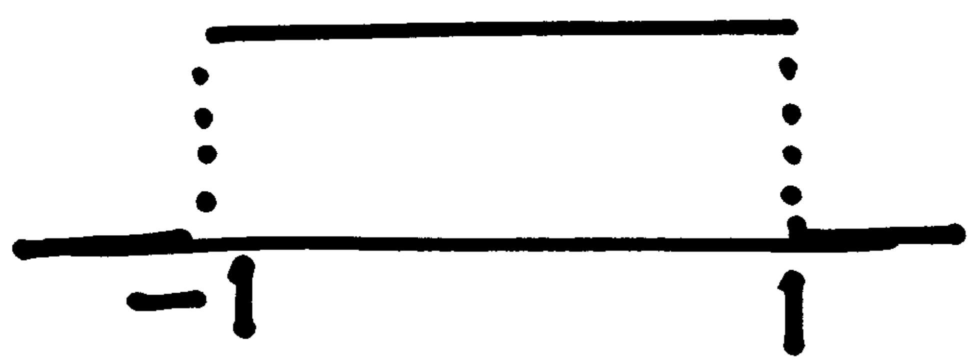
Piecewise constant approx in $]-1, 0[$ = that in $]0, +1[$ from symmetry

= average of function in each of those intervals

$$= \int_0^1 (1-t)\,dt = \frac{1}{2}$$

Projection of $x_1(t)$ on V_0

$= \mathrm{Proj}_{V_0} x_1$



Consider the standard unit interval

$$]n, n+1[$$

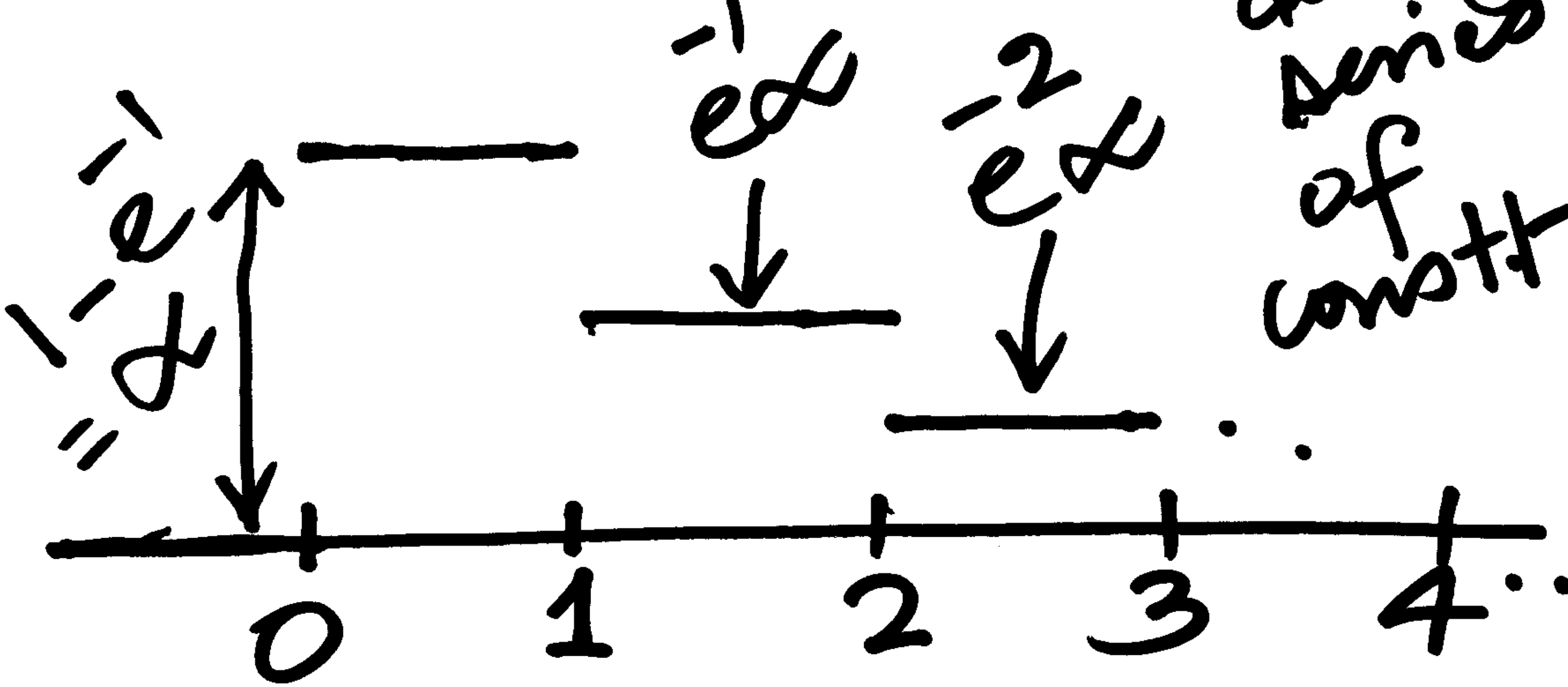
$$\int_n^{n+1} e^{-t}\,dt = \left.\frac{e^{-t}}{-1}\right|_n^{n+1}$$

$$= e^{-n} - e^{-(n+1)}$$

$$= e^{-n}\left(1 - e^{-1}\right)$$

$\mathrm{Proj}_{V_0}\, x_2$:

exponentially decaying series of consts



To verify, this again belongs to $\mathcal{L}_2(\mathbb{R})$:

$$\int_{-\infty}^{+\infty} \left| \mathrm{Proj}_{V_0} x_2(t) \right|^2 dt$$

$$\int_0^1 e^{-n}\left(1-e^{-1}\right)^2 \cdot dt$$

summed over all positive integer n

$$= \sum_{n=0}^{\infty} \left(e^{-n} (1 - e^{-1}) \right)^2$$

$$= (1 - e^{-1})^2 \sum_{n=0}^{\infty} e^{-2n}$$

Geometric series

$$= \frac{(1-e^{-1})^2}{1-e^{-2}}.$$

finite.

Q3. Obtain their projections on the space V_1 in the Haar MRA.

$\text{Proj}_{V_1} x_1$: using symmetry

we can calculate for $0 < t < 1$ only

$]0, \tfrac{1}{2}[$: $\dfrac{1}{1/2}\displaystyle\int_0^{1/2}(1-t)\,dt$

$\lambda = 1-t$

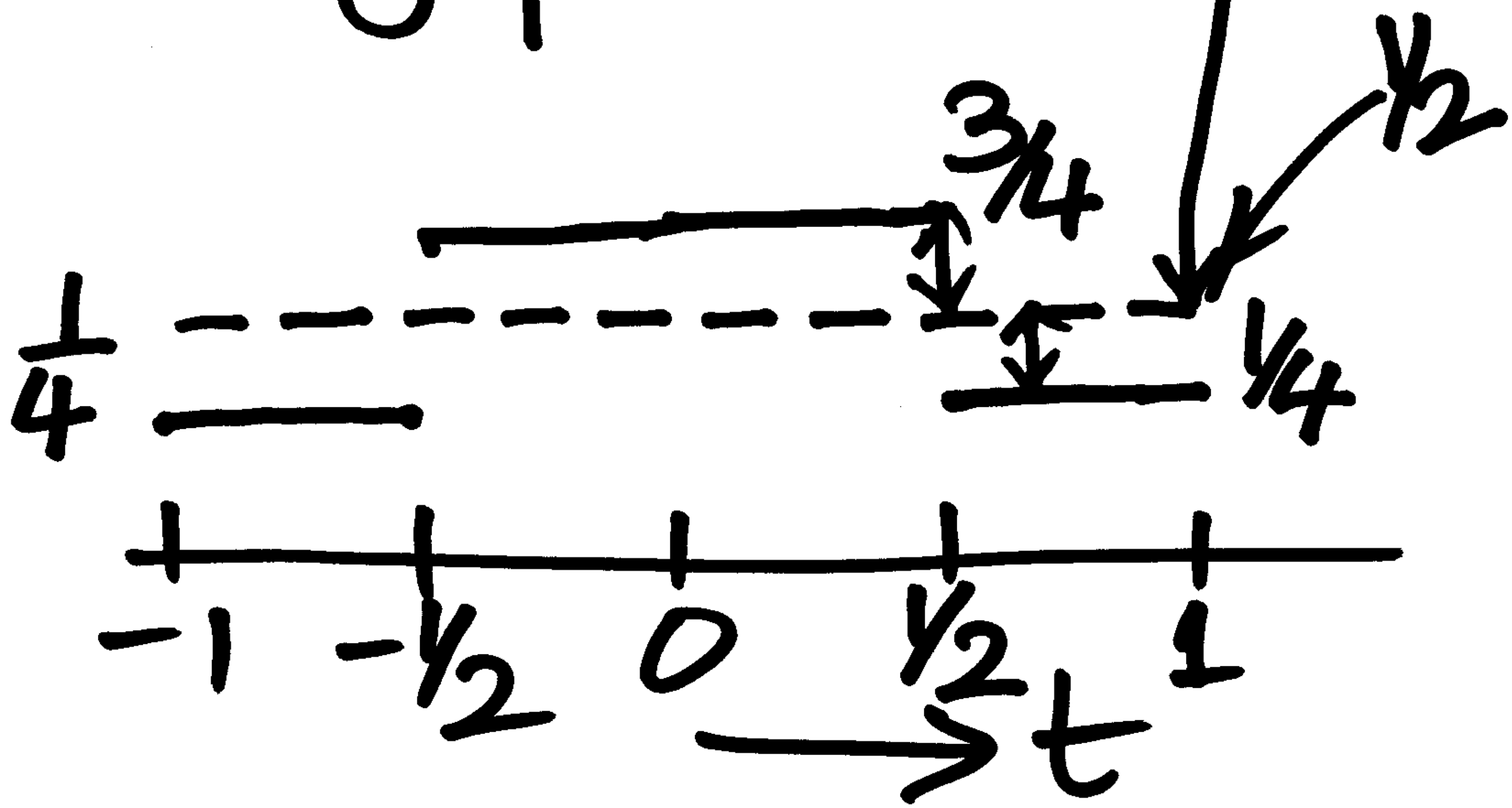
$= 2 - \displaystyle\int_1^{1/2} \lambda \, d\lambda = 2\int_{1/2}^{1} \lambda \, d\lambda$

$= \lambda^2/\cancel{2}\,\Big|_{1/2}^{1} = \cancel{\tfrac{1}{2}}\left(1 - \tfrac{1}{4}\right) = \tfrac{3}{4}$

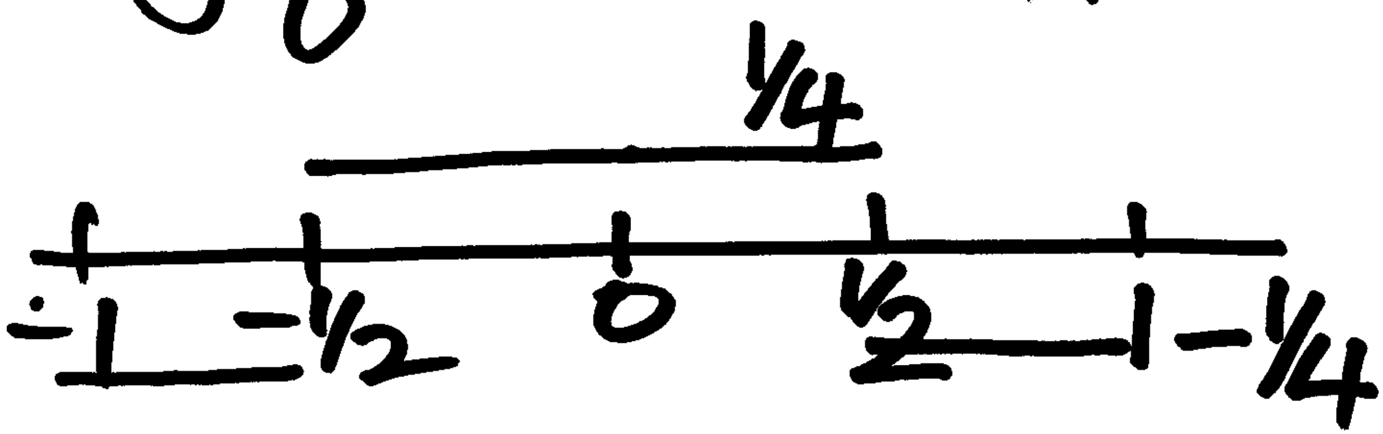
$]\frac{1}{2}, 1[$: $\dfrac{1}{1/2} \displaystyle\int_{1/2}^{1} (1-t)\,dt$

$= 1/4.$

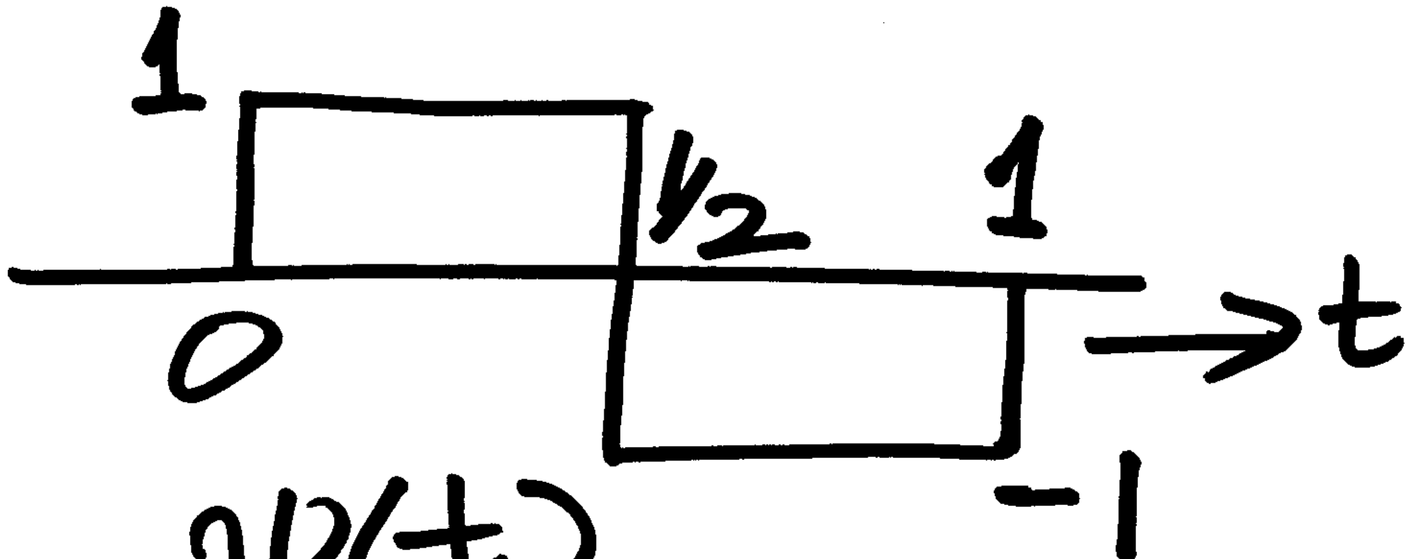
$\mathrm{Proj}_{V_1} x_1$ $\mathrm{Proj}_{V_0} x_1$



$$\mathrm{Proj}_{W_0} x_1 = -\mathrm{Proj}_{V_0} x_1 + \mathrm{Proj}_{V_1} x_1$$



$$= \frac{1}{4}\psi(t) - \frac{1}{4}\psi(t+1)$$



$\psi(t)$

Haar wavelet

$\mathrm{Proj}_{v_1} x_2$ $n \geq 0$

$]n \cdot \frac{1}{2}, (n+1) \cdot \frac{1}{2}[$

$$= \frac{1}{\sqrt{2}} \int_{n/2}^{(n+1)/2} \cdot e^{-t} \, dt$$

$$= 2 \cdot \int_{n/2}^{(n+1)/2} e^{-t}\, dt$$

$$= \frac{2 e^{-t}}{-1} \Bigg|_{n/2}^{(n+1)/2}$$

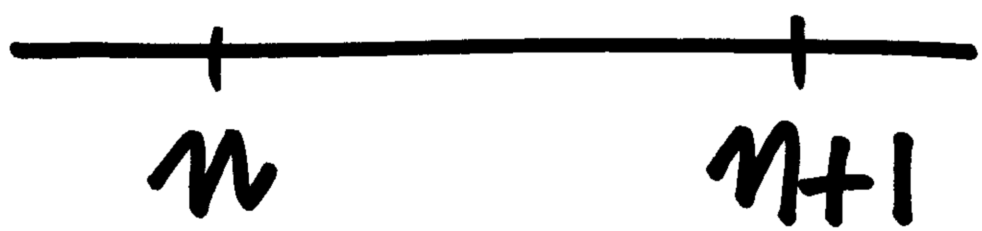
$$= \left(e^{-n/2} - e^{-(n+1)/2}\right)^2$$

$$= 2 \cdot e^{-\frac{n}{2}}\left(1 - e^{-1/2}\right)$$

again an exponential sequence.

$\mathrm{Proj}_{W_0}\, x_2$:

a multiple of $\phi(t-n)$



The Constant by which we must multiply $\psi(t-n) =$

Average over $]n, n+\frac{1}{2}[$

$-$ Average over $]n, n+1[$

$$]n, n+\tfrac{1}{2}[\; =$$

$$\int_{n}^{n+\frac{1}{2}} e^{-t}\, dt$$

$$= \left. \frac{e^{-t}}{-1} \right|_{n}^{n+\frac{1}{2}}$$

$$= e^{-n}\left(1 - e^{-\frac{1}{2}}\right)$$

multiplying factor for

$\psi(t-n) =$

$$e^{-n}(1-e^{-1/2}) - e^{-n}(1-e^{-1})$$

$$= e^{-n}\left(1 - e^{-1/2} - 1 + e^{-1}\right)$$

$$= \underbrace{e^{-n}\left(e^{-1} - e^{-1/2}\right)}_{d_n}$$

$$\mathrm{Proj}_{W_0} x_2(t) = \sum_{n=0}^{\infty} d_n \psi(t-n)$$

For exponentially decaying one sided functions: the projections on V_m, $m \in \mathbb{Z}$ - -

and projections on W_m, $m \in \mathbb{Z}$
are all exponentially decaying piecewise constants

Exercise:

Show that d_n can also be obtained

$$\langle x_2(t), \psi(t-n) \rangle$$

$$\langle x_2(t), \psi(t-n) \rangle$$

$$= \int_{n}^{n+1/2} e^{-t}\, dt - \int_{n+1/2}^{n+1} e^{-t}\, dt$$

$$= \frac{e^{-t}}{-1} \Bigg|_{n}^{n+\frac{1}{2}} - \Bigg|_{n+\frac{1}{2}}^{n+1}$$

$$= e^{-n} - e^{-\left(n+\frac{1}{2}\right)}$$

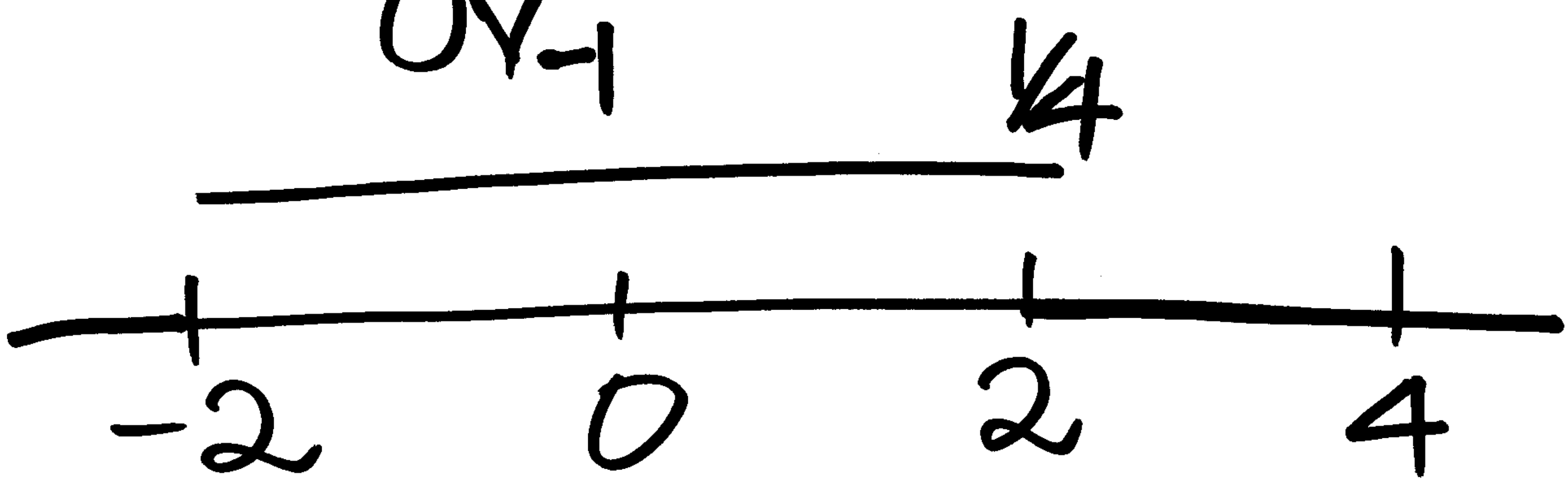
$$- \left(e^{-\left(n+\frac{1}{2}\right)} - e^{-(n+1)} \right)$$

Rearrange to get d_n

Next Q: Obtain in general $\text{Proj}_{V_m} x_1$

$m < 0$, V_{-1}, V_{-2} so on

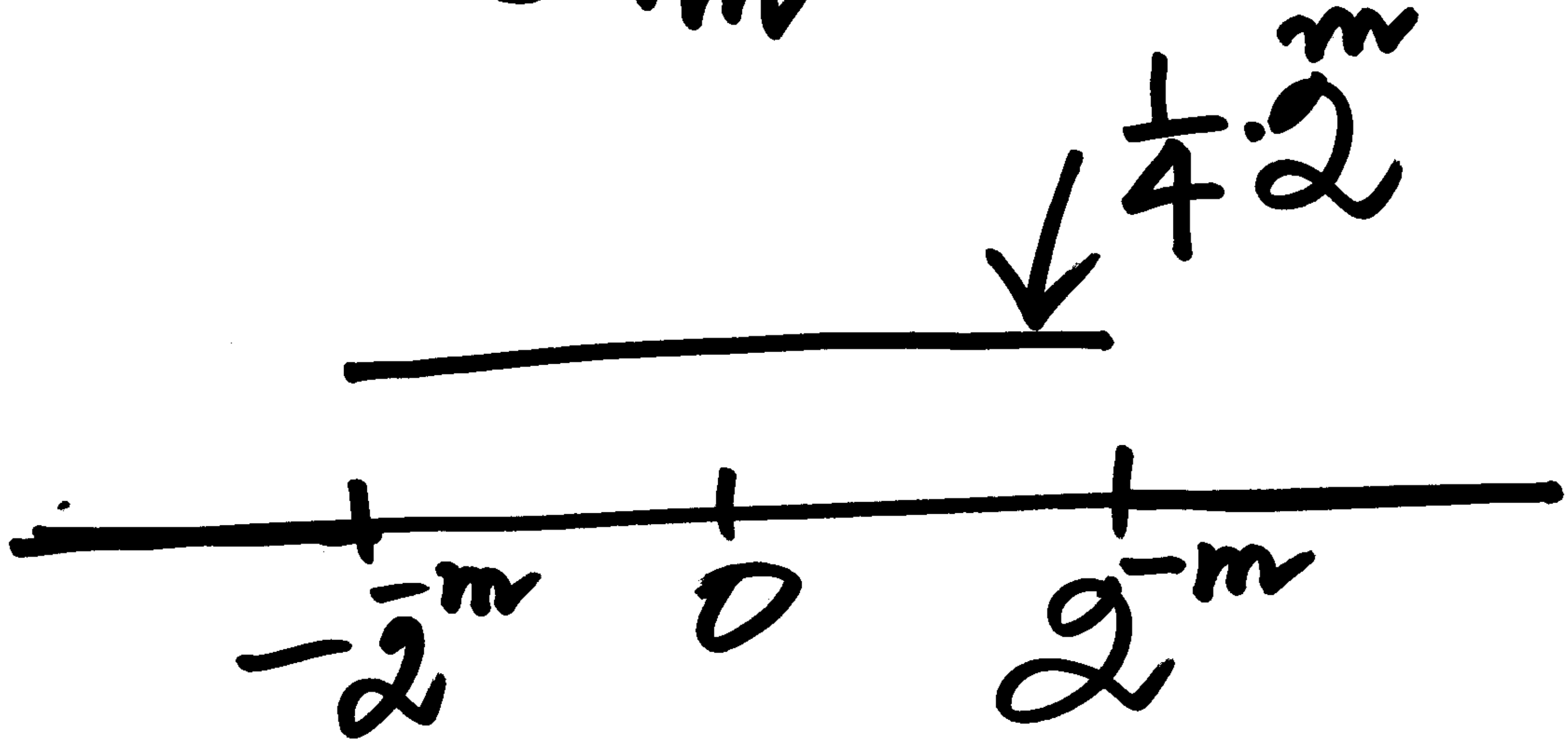
$\mathrm{Proj}_{V_{-1}} x_1$



$$\frac{1}{2}\int_0^2 x_1(t)\,dt$$

$$=\frac{1}{2}\int_0^1 (1-t)\,dt = \frac{1}{4}$$

$\mathrm{Proj}_{V_m} x_1 \qquad m<0$



Exercises:

1. Continue this process to obtain $\mathrm{Proj}_{W_m} x_1$, $m < 0$

2. Repeat for $x_2(t)$

$\mathrm{Proj}_{V_m} x_2$ $m>0$ $m<0$ in general

3. Obtain

$\text{Proj}_{W_m} x_2$

$m<0$ and $m>0$
separately