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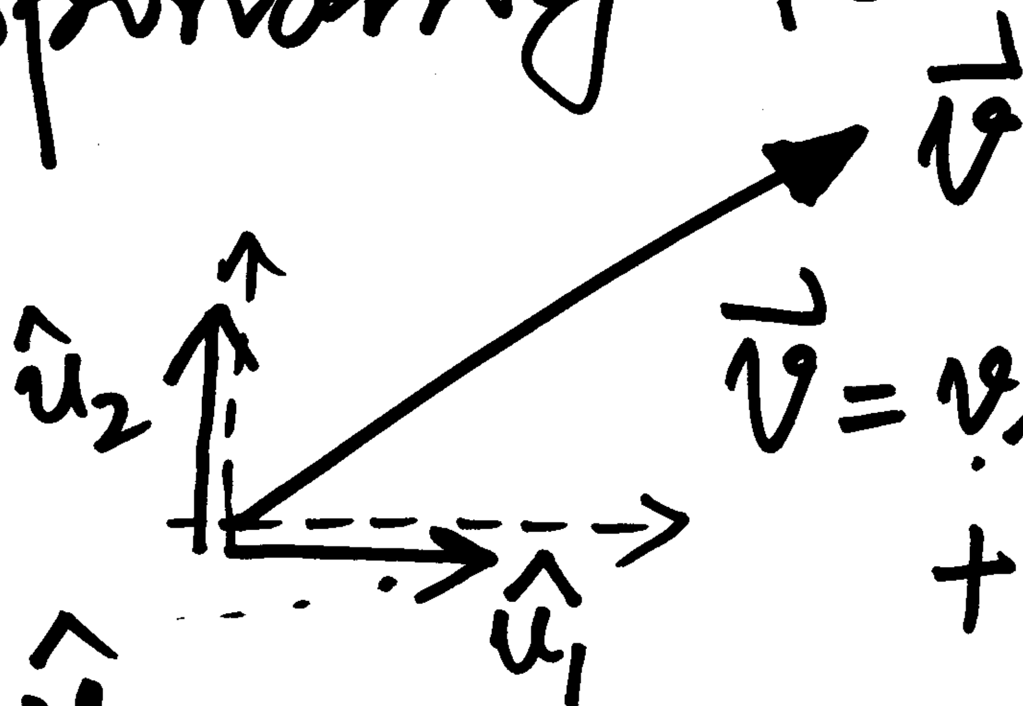
Date-12-1-10

LECTURE 4

WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

1. Functions or signals as generalized vectors
2. Connection(s) between $\mathcal{L}_2(\mathbb{R})$ functions and sequences.

2-Dimensional space corresponding to this paper:



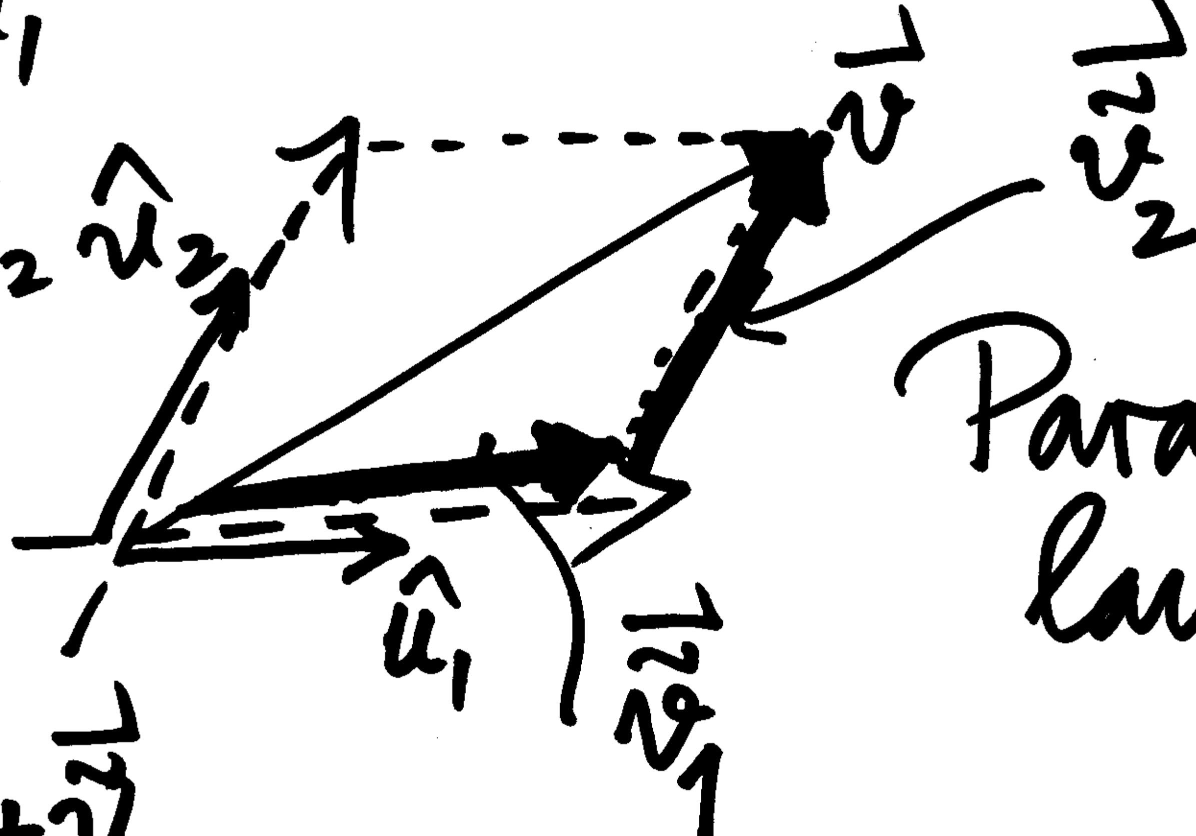
$$\vec{v} = v_1 \hat{u}_1 + v_2 \hat{u}_2$$

$$v_1 = \vec{v} \cdot \hat{u}_1$$

$$v_2 = \vec{v} \cdot \hat{u}_2$$

$\vec{v}_1 = K_1 \hat{u}_1$

$\vec{v}_2 = K_2 \hat{u}_2$



Parallelogram law

$\vec{v} = \vec{v}_1 + \vec{v}_2$

$$\vec{v} = k_1 \hat{u}_1 + k_2 \hat{u}_2$$

An infinite (countably infinite) dimensional vector is a sequence

$$x[n], \quad n \in \mathbb{Z}$$

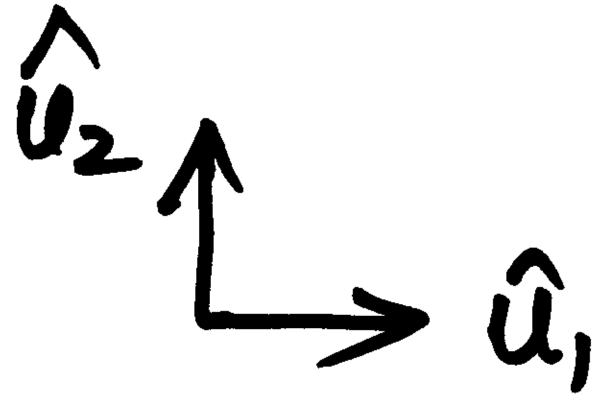
set of integers

index

sequence: a vector

each n : different 'dimension' of the vector

'Dot product' of vectors



$$\vec{e}_1 = e_{11}\,\hat{u}_1 + e_{12}\,\hat{u}_2$$
$$\vec{e}_2 = e_{21}\,\hat{u}_1 + e_{22}\,\hat{u}_2$$

$$\vec{e_1} \cdot \vec{e_2} = e_{11} e_{21} + e_{12} e_{22}$$

Sum of products of <u>corresponding</u> coordinates.

N-dim vectors

$\vec{e}_1: \quad e_{11} \ e_{12} \ \cdots \ e_{1N}$

$\vec{e}_2: \quad e_{21} \ e_{22} \ \cdots \ e_{2N}$

$$\vec{e}_1 \cdot \vec{e}_2 = \sum_{k=1}^{N} e_{1k} e_{2k}$$

$x_1[n]$, $x_2[n]$, $n \in \mathbb{Z}$

"Dot product" or inner product

$\langle x_1, x_2 \rangle$ | Assume real sequences now.

$$\langle x_1, x_2 \rangle = \sum_{n=-\infty}^{+\infty} x_1[n]\, x_2[n]$$

We would like squared <u>norm</u> of x to be $\langle x, x \rangle$

What do we want of a 'norm'?

'Vector' x; essentially a sequence $x[n]$, $n \in \mathbb{Z}$

'norm' of sequence x

$= \|x\|$ should be

$\langle x, x \rangle^{1/2}$ and further:

$$\|x\| \geq 0$$

and $\|x\| = 0$

$\iff x = 0$.

i.e. $x[n] = 0 \quad \forall n \in \mathbb{Z}$

If x_1, x_2 are real

$$\langle x_1, x_2 \rangle = \sum_{n=-\infty}^{+\infty} x_1[n]\, x_2[n]$$

$$\langle x, x \rangle = \sum_{n=-\infty}^{+\infty} x^2[n]$$

As long as $x[n]$ real $\forall n \in \mathbb{Z}$,
this satisfies 'norm'
requirements!

The following small change for complex sequences: $\langle x_1, x_2 \rangle$

$$= \sum_{n=-\infty}^{+\infty} x_1[n] \, \overline{x_2[n]}$$

complex CONJUGATE

$$\langle x_1, x_2 \rangle = \overline{\langle x_2, x_1 \rangle}$$

(i)

Complex conjugate

(ii) linear in first argument

$$\langle a_1 x_1 + a_2 x_2, x_3 \rangle$$

$$= a_1 \langle x_1, x_3 \rangle + a_2 \langle x_2, x_3 \rangle$$

(iii) nonnegativity (positive definite)

$$\langle x, x\rangle \geq 0 \text{ and}$$

$$x = 0 \iff \langle x, x\rangle = 0$$

Standard inner product:

$$\langle x_1, x_2 \rangle = \sum_{n=-\infty}^{+\infty} x_1[n] \, \overline{x_2[n]}$$

Exercise:

Verify the properties of

i) Conjugate commutativity

$$\langle x_1, x_2 \rangle = \overline{\langle x_2, x_1 \rangle}$$

(ii) linearity in 1st argument

(iii) Positive definiteness:

$$\langle x, x \rangle = \sum_{n=-\infty}^{+\infty} x[n] \overline{x[n]}$$

$$= \sum_{n=-\infty}^{+\infty} |x[n]|^2 = 0$$

iff

$x[n] = 0 \ \forall n$

Extension to uncountably infinite dimension:

every t, $t \in \mathbb{R}$ is a different dimension

$x(t), \quad t \in \mathbb{R}$

$x(t)$ for particular t

is the 't^{th} coordinate'

"Dot product" or "inner product" between two functions

$$\langle x, y \rangle = \int_{-\infty}^{+\infty} x(t)\overline{y(t)}\,dt$$

<u>Exercise</u>: Verify the properties of

(i) Conjugate commutativity

(ii) linearity in first argument

(iii) positive definiteness

Parseval's theorem

$$x(t) \longrightarrow \hat{x}(\nu)$$

Hertz frequency variable

$$\hat{x}(\nu) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi\nu t} \, dt$$

in Hz

$$\hat{x}(\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

angular frequency variable for continuous time

$\Omega = 2\pi\nu$

Parseval's Theorem:

$$x(t) \xrightarrow{} \hat{x}(\nu)$$

$$y(t) \xrightarrow[\text{Fourier Transform}]{} \hat{y}(\nu)$$

The inner product in time

$$= \langle x, y \rangle$$

$$= \int_{-\infty}^{+\infty} x(t)\, \overline{y(t)}\, dt$$

is Equal

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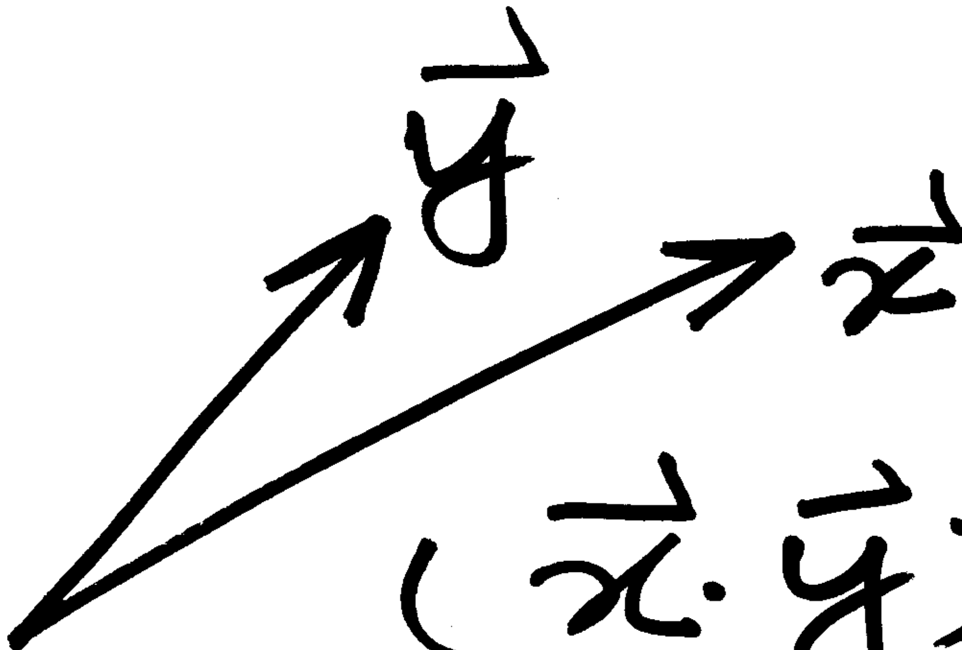
to the inner product in frequency ν

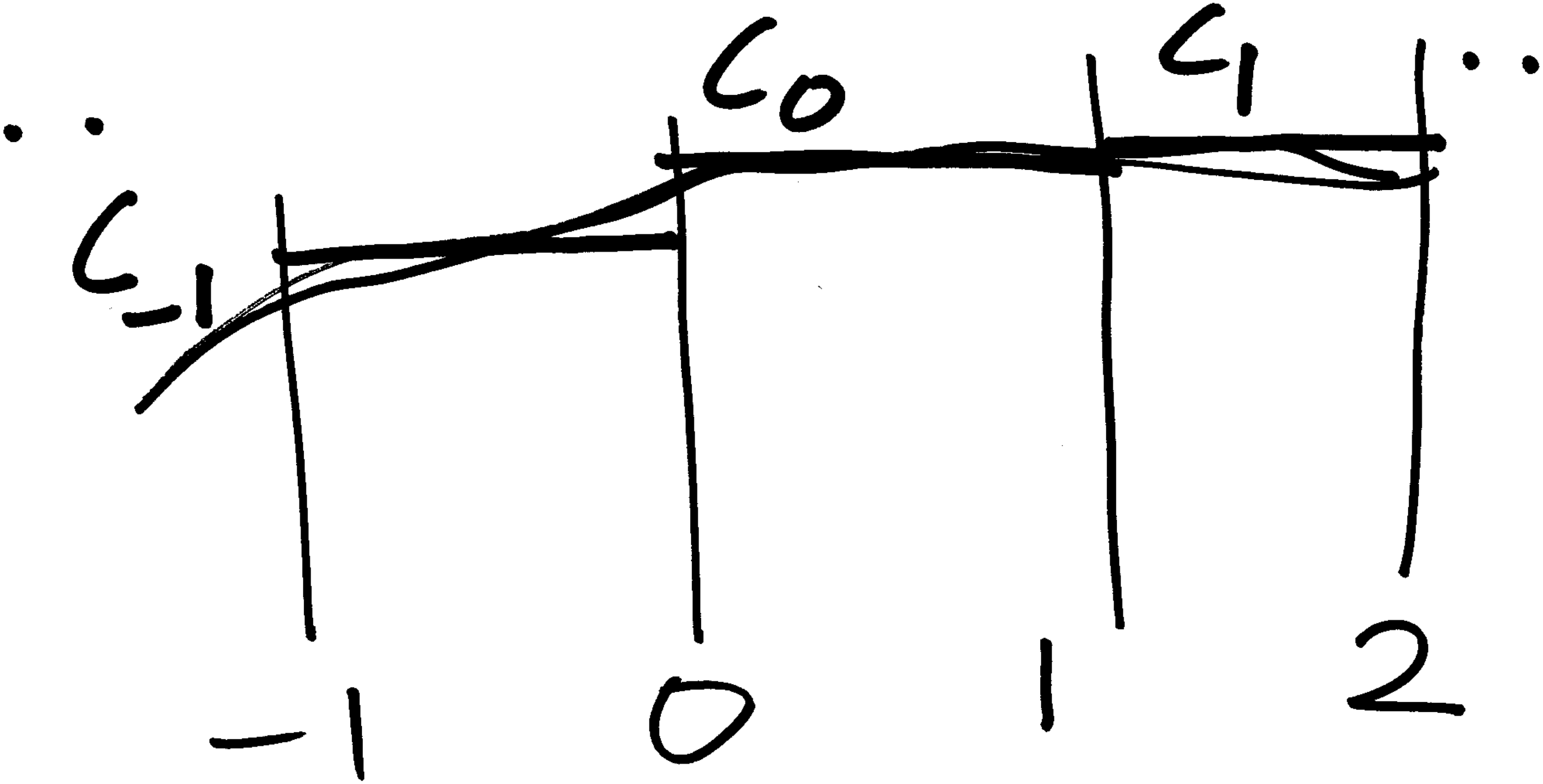
$$\langle \hat{x}, \hat{y} \rangle = \int_{-\infty}^{+\infty} \hat{x}(\nu)\, \overline{\hat{y}(\nu)}\, d\nu$$



$(\vec{x} \cdot \vec{y})$ is independent of coordinate system

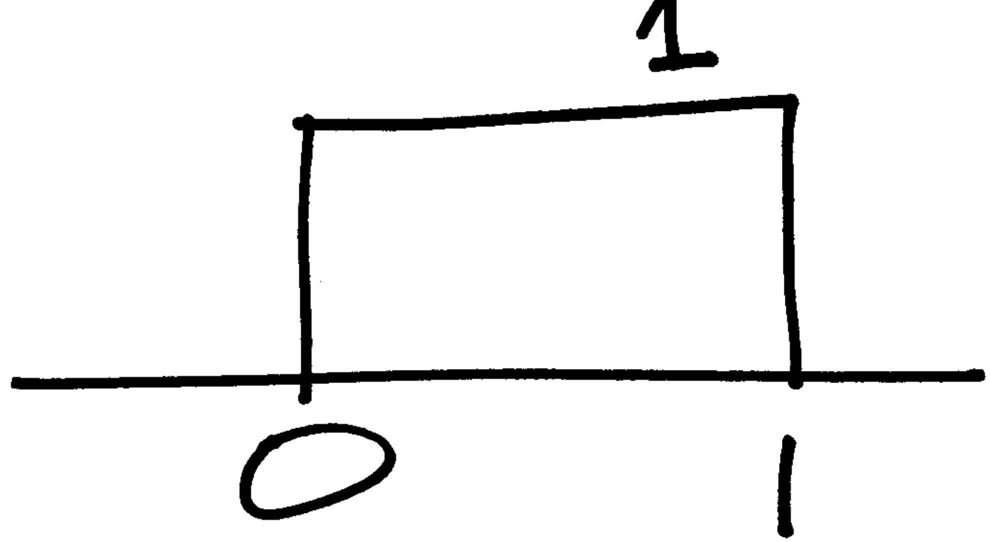
$$x(t) = \int_{-\infty}^{+\infty} \hat{x}(\nu) \cdot e^{j\nu t} \, d\nu$$

Reconstruct $x(t)$ from 'components' $\hat{x}(\nu)$.



...
c_{-1}
c_0
c_1
...
-1
0
1
2

$\phi(t)$



$$\cdots + c_{-1}\,\phi(t+1) + c_0\,\phi(t) + c_1\,\phi(t-1) + \cdots$$