Prof. Gadre (student)
(Soham Basu)
Lec - 39
Date - 7-12-10

Proof that a non-zero function can not be both time- and band-limited

$$F(s) = \int_{\mathbb{R}} f(t) e^{-2\pi i s t} \, dt$$

$$f \in L_1, L_2(\mathbb{R})$$

$$f(t) = \int_{\mathbb{R}} F(s) e^{2\pi i s t} \, ds$$

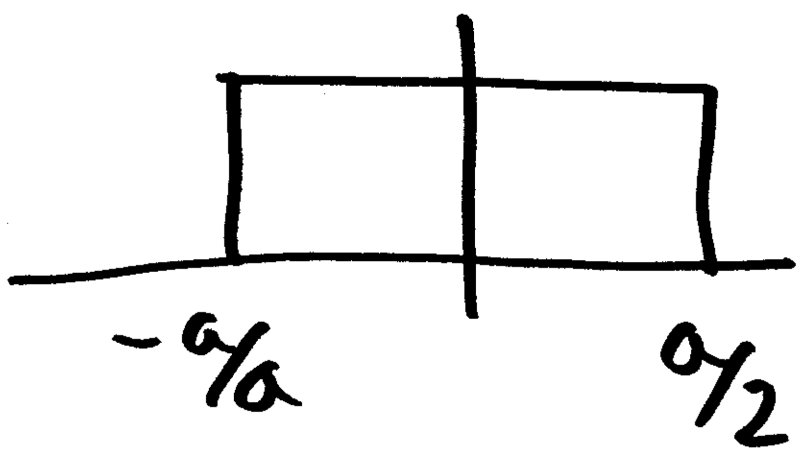
$$\sin \omega = \frac{e^{i\omega} - e^{-i\omega}}{2}$$

$$e^{i\theta} = i \sin \theta + \cos \theta$$

$$= i \sin \theta + \cos (\pi/2 - \theta)$$

$$f \in L_p(\mathbb{R})$$

$$\int_{\mathbb{R}} |f|^p \, dt < \infty$$



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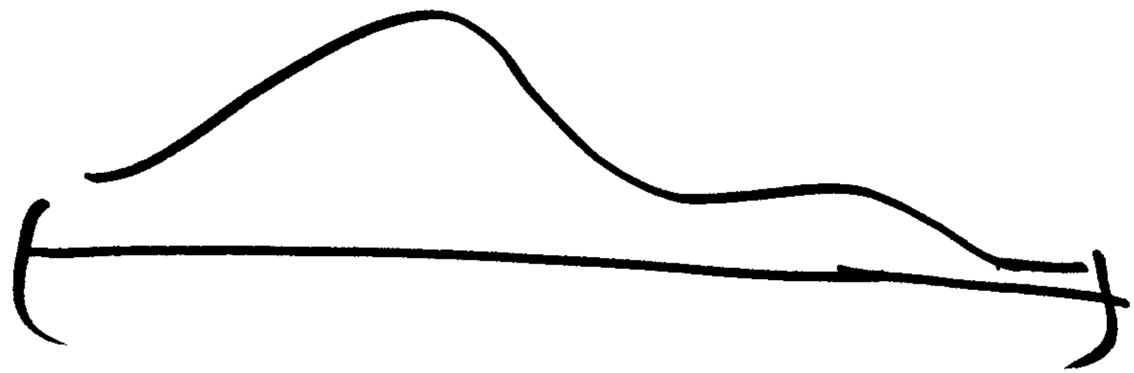
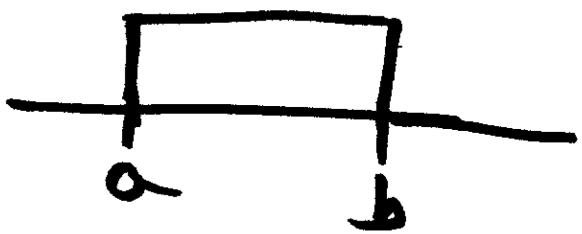
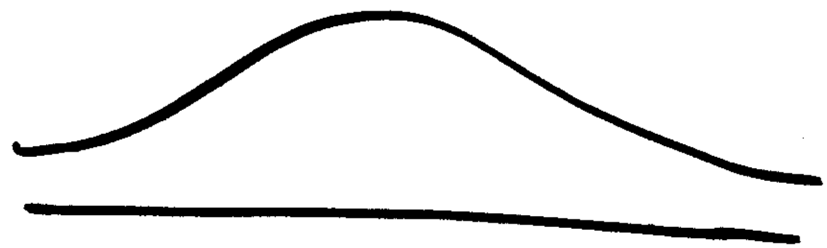
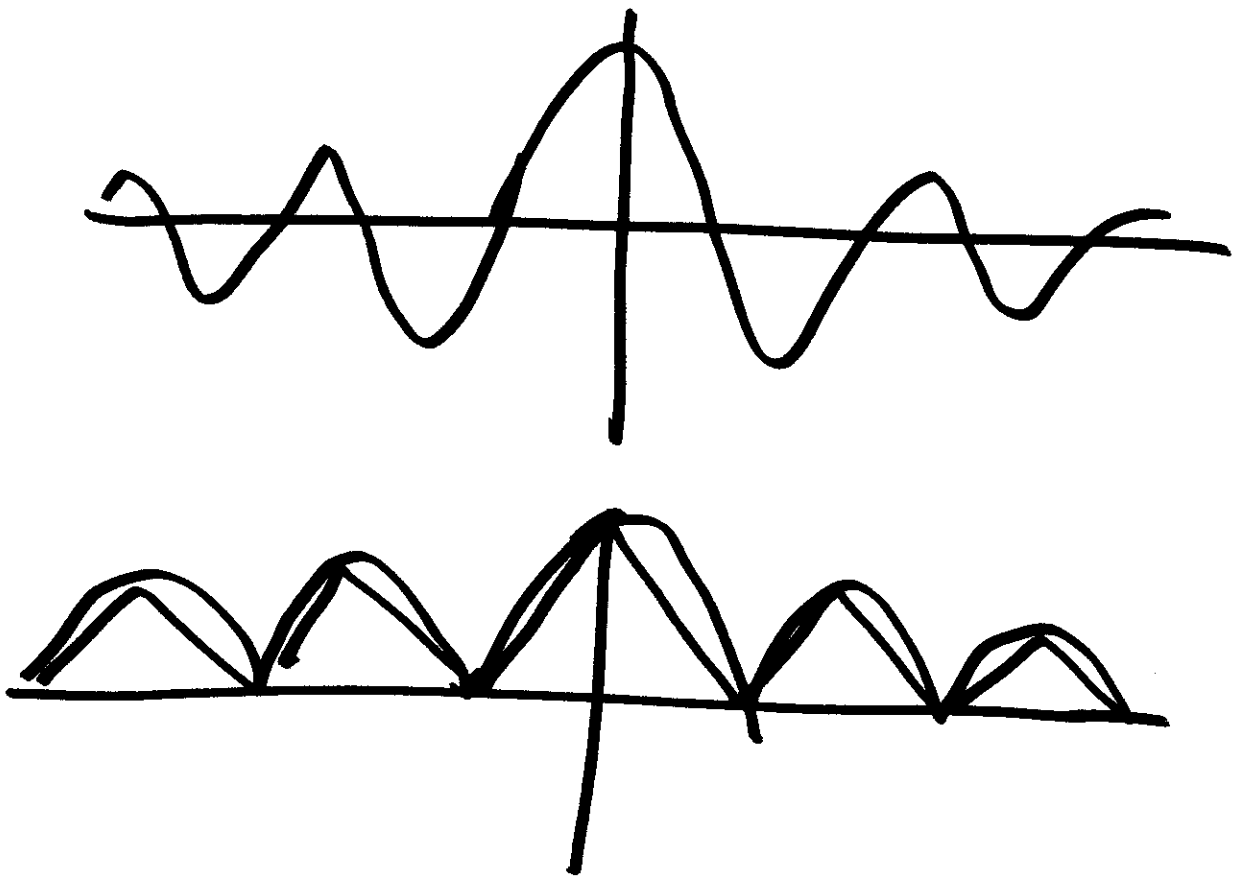






$$\operatorname{sinc} x = \left|\frac{\sin \pi x}{\pi x}\right|$$



$f \in L_p, p \in (1, \infty)$

$\Rightarrow \quad f \in L_1$

$$\int |fg| \, dx$$

$$\leq \left(\int |f^p| \right)^{1/p} \left(\int |g|^q \right)^{1/q}$$

Hölder's inequality

$$\int |fg| \leq \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$$

$$\frac{1}{p} + \frac{1}{q} = 1 \quad . \quad p, q \geq 0$$

$$1 \leq p, q \leq \infty$$

Young's inequality

$a, b \in \mathbb{R}^+$ $\quad ab \leq \frac{a^p}{p} + \frac{b^q}{q}$

Generalized AM-GM inequality $\quad \frac{1}{p} + \frac{1}{q} = 1$

$p + q = pq$

$$\frac{\alpha_1 a + \alpha_2 b}{\alpha_1 + \alpha_2} \geqslant \sqrt[\alpha_1+\alpha_2]{a^{\alpha_1} b^{\alpha_2}}$$

$$\frac{q a^p + p b^q}{\cancel{p+q}\ pq} \geqslant \sqrt[pq]{(a^p)^q (b^q)^p}$$

$$\frac{a^p}{p} + \frac{b^q}{q} \geqslant \sqrt[pq]{(ab)^{pq}}$$

$$\Rightarrow \frac{a^p}{p} + \frac{b^q}{q} \geqslant ab$$

$$f \in L_p(\mathbb{R})$$

$$g \in L_q(\mathbb{R})$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$f = \frac{f}{|f|_p} \qquad \int_{\mathbb{R}} |f|^p \, dt = |f|_p$$

$$g = \frac{g}{|g|_q}$$

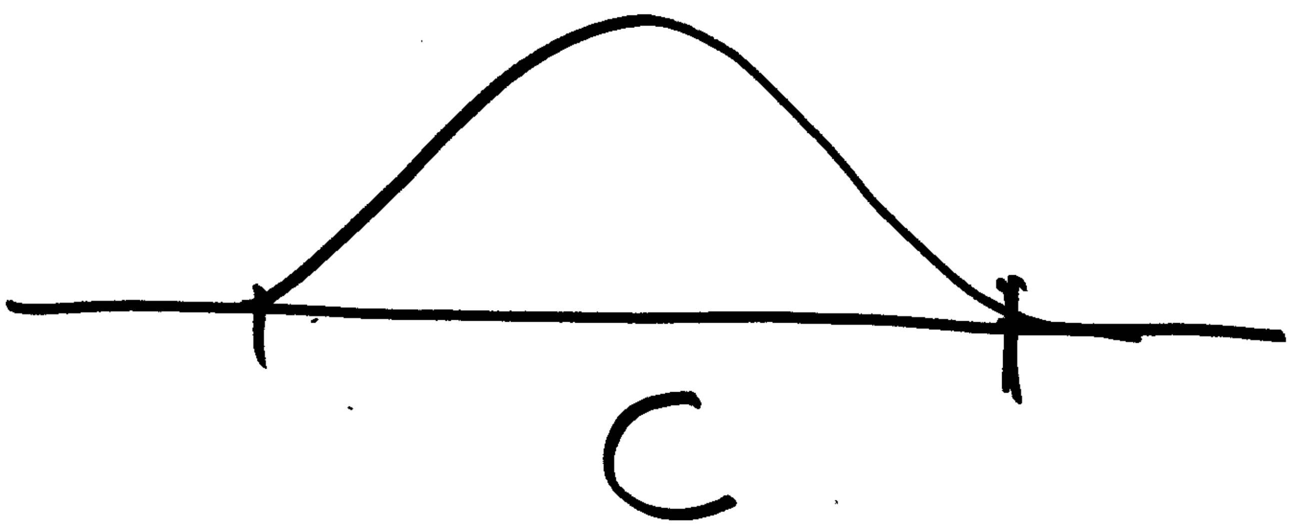
$$f(t) \qquad g(t)$$

$$\frac{|f^p(t)|}{p} + \frac{|g^q(t)|}{q} \geq |f(t)\,g(t)|$$

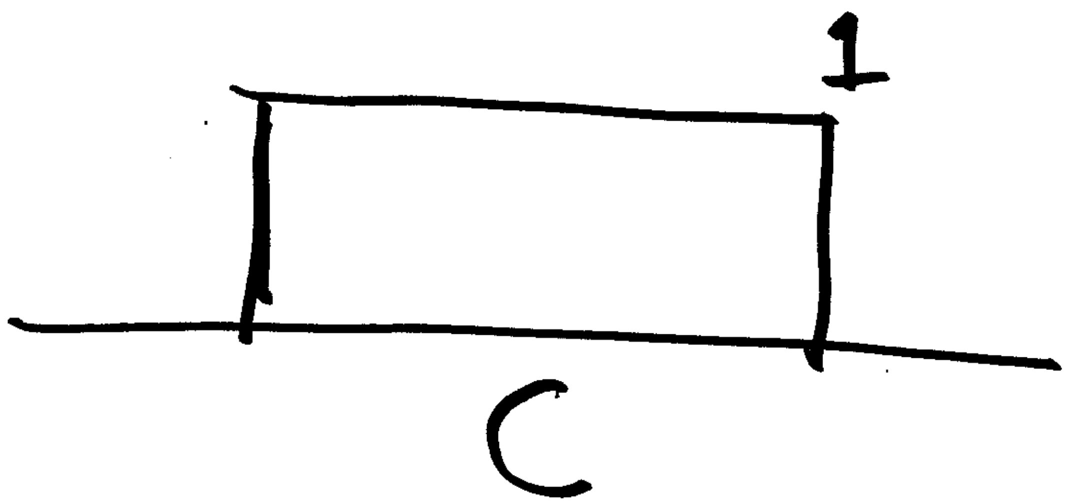
$$\int_{\mathbb{R}} \frac{|f^{\cancel{p}}(t)|^p}{p} + \int_{\mathbb{R}} \frac{|g^{\cancel{q}}(t)|^q}{q} \geq \int_{\mathbb{R}} |f(t)\,g(t)|$$

$$\frac{1}{p} + \frac{1}{q} = 1 \geq \int f(t)\,g(t)$$

$$|f|_p \, |g|_q \geq \int |fg|$$



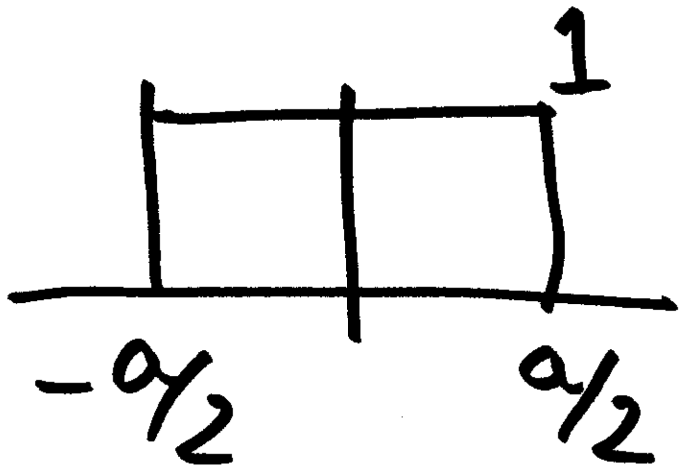
$$\int_C |fg| \leqslant \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$$



$$\int_C |f| \leqslant \left(\int_C |f|^p\right)^{1/p} \left(\int_C dt\right)^{1/q}$$

$$\searrow$$

$$|C|^{1/q}$$



$$\int_{\mathbb{R}} f(t)\cos 2\pi\omega t\,dt$$

$$\int_{-a/2}^{a/2} \cos 2\pi\omega t\,dt$$

$$= \frac{2}{\cancel{2}\pi\omega} \sin(\pi\omega a)$$

$$= 2\ \mathrm{sinc}(a\omega)$$

Vandermonde matrix

$$(a_{ij}) = (c_j^{i-1})$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a_1 & \cdots & \cdots & a_n \\ a_1^2 & & & a_n^2 \\ \vdots & & & \vdots \\ a_1^n & & & a_n^n \end{vmatrix}$$

$$A\bar{x} = \bar{y}$$

$$\bar{x} = A^{-1}\bar{y}$$

$$D(x) = \begin{array}{c} \begin{vmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ a_1 & & & x & & a_n \\ \vdots & & & x^2 & & \vdots \\ & & & \vdots & & \\ a_1^n & & & x^n & & a_n^n \end{vmatrix} \\ \begin{array}{cccccc} & & & j & & \end{array} \end{array}$$

$$D(x) = (-1)^s \prod_{i=j} (a_j - a_i)$$

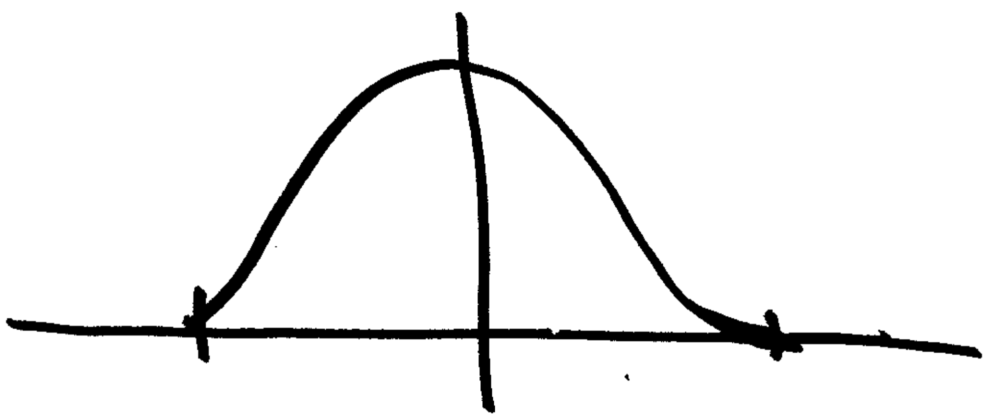
$\cancel{R(}$

$$(-1)^s \prod_{i<j} (a_i - a_j)$$

$$|M| \neq 0$$

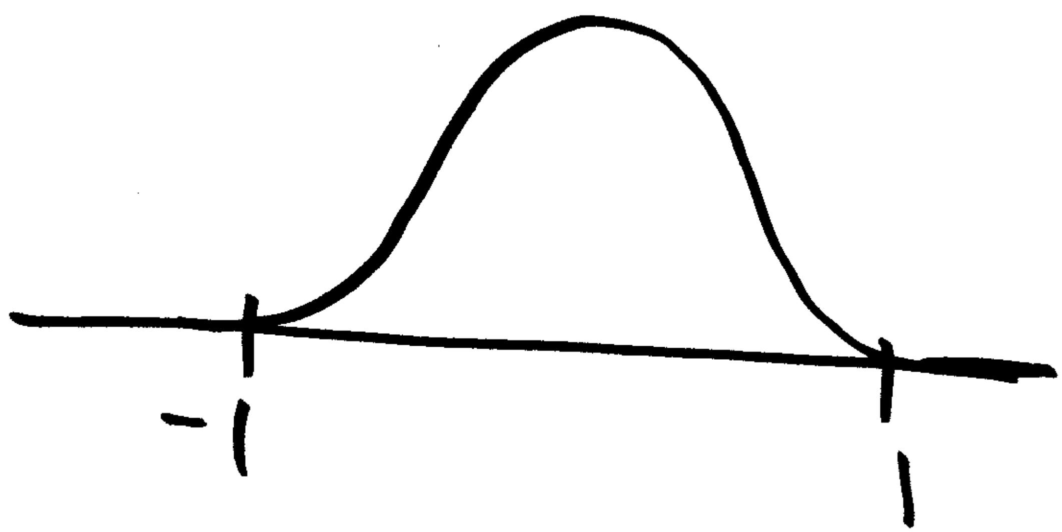
$$C(x) = e^{-\frac{1}{1-x^2}} \qquad \rightarrow \infty \qquad |x| < 1$$

$$= 0 \qquad \text{otherwise}$$



$$C^p(x) \cdot \frac{2x}{(1-x^2)^2} \underbrace{e^{-\underbrace{\frac{1}{1-x^2}}_{\to \infty}}}_{\to 0}$$

$p \in \mathbb{N}$

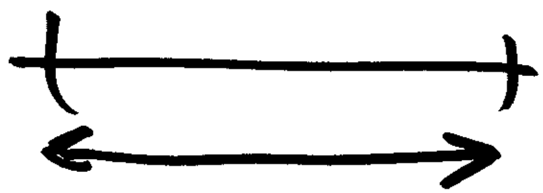


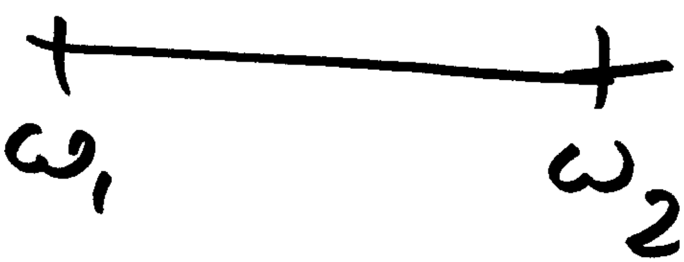
$$F(s) = \int f e^{-2\pi i \omega t}\, dt$$

$$|F(s)| \leq \int |f|\, \underbrace{|e^{-2\pi i \omega t}|}_{\to 1}\, dt$$

$$\leq \int |f|\, dt = L_1(f)$$

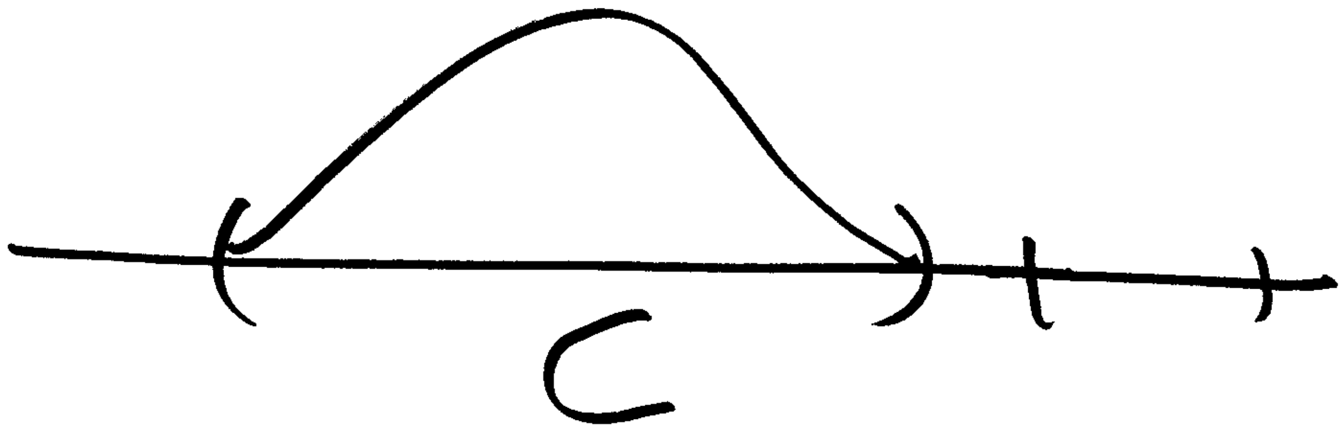
$$f'(t) \quad F(\omega) = \int_{C} i\omega f e^{-2\pi i \omega t} dt$$





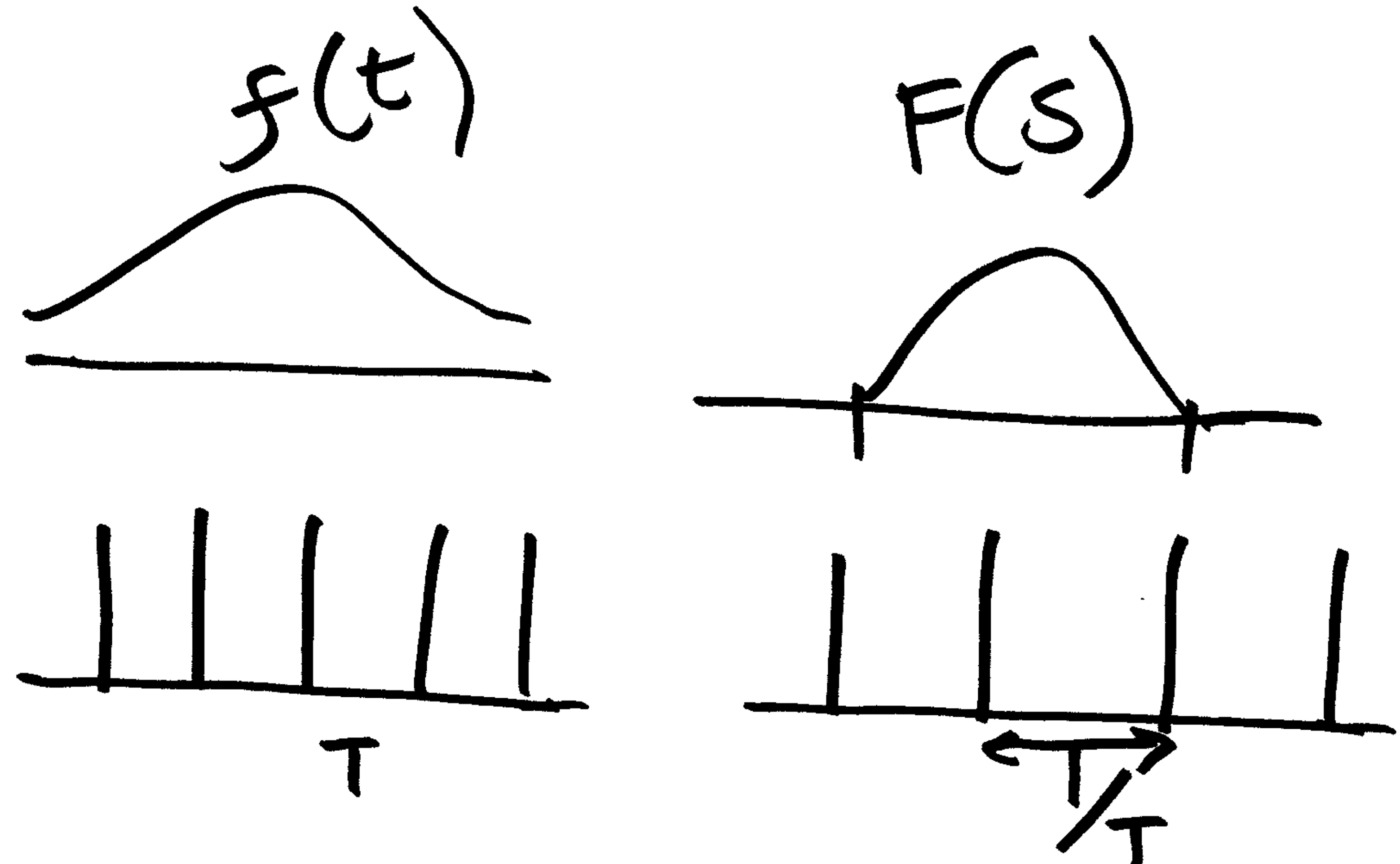
$$f(t) = f(0) + t\,f'(0) + \frac{t^2}{2!} f''(0) + \cdots$$

$$= \sum_{i=0}^{\infty} a_i t^i$$



$$f = 0$$

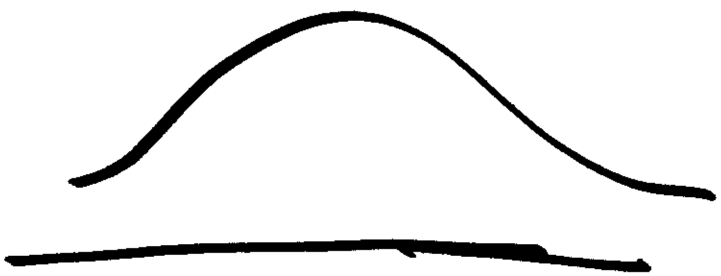
Identity theorem

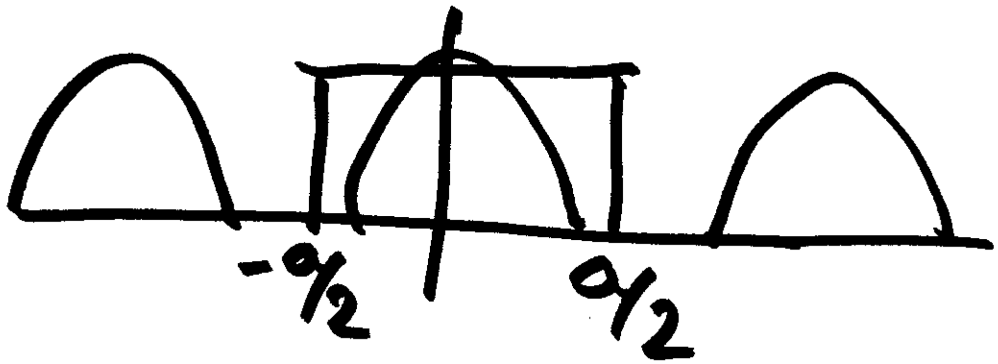


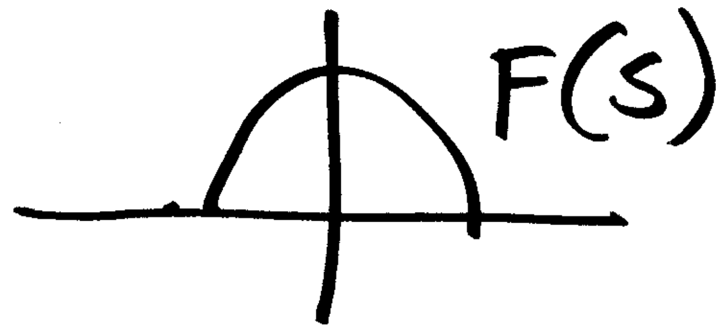
$$\int_{\mathbb{R}} \delta(t)\,dt = 1$$

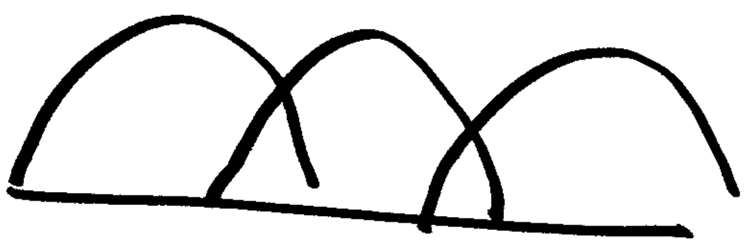
$$\delta(t) = 0 \;\forall t \neq 0$$

f F

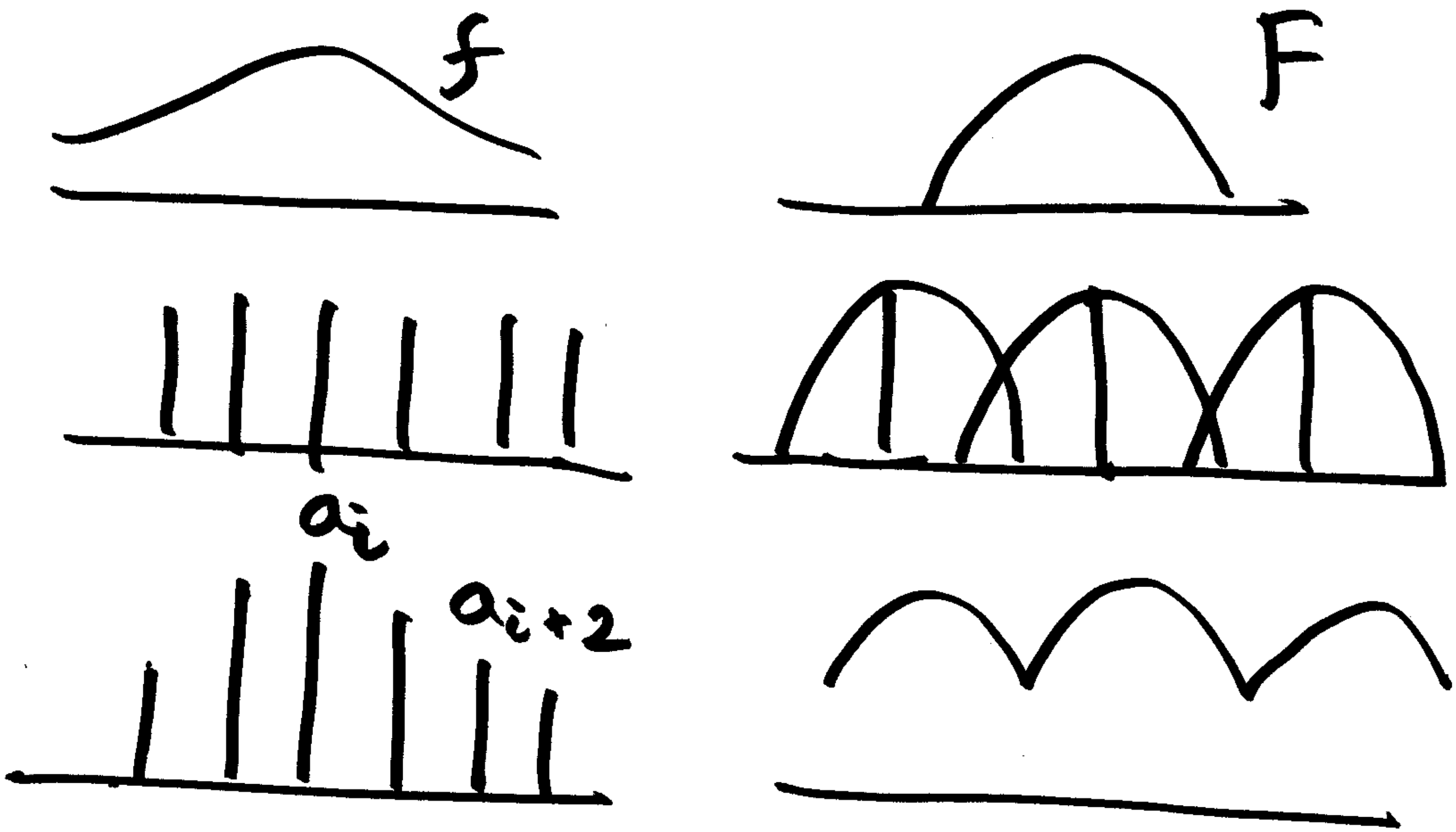






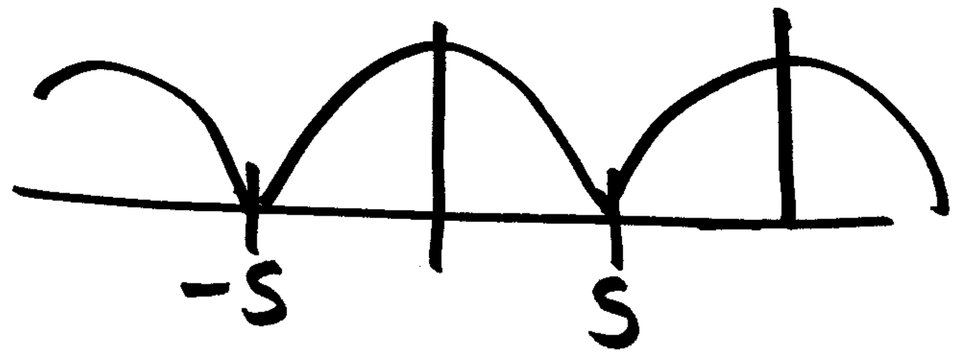


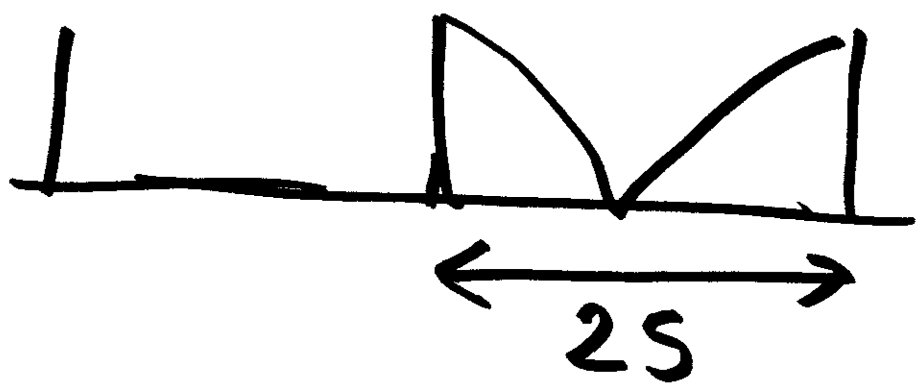
$$f \cdot g \rightarrow F * G$$



F

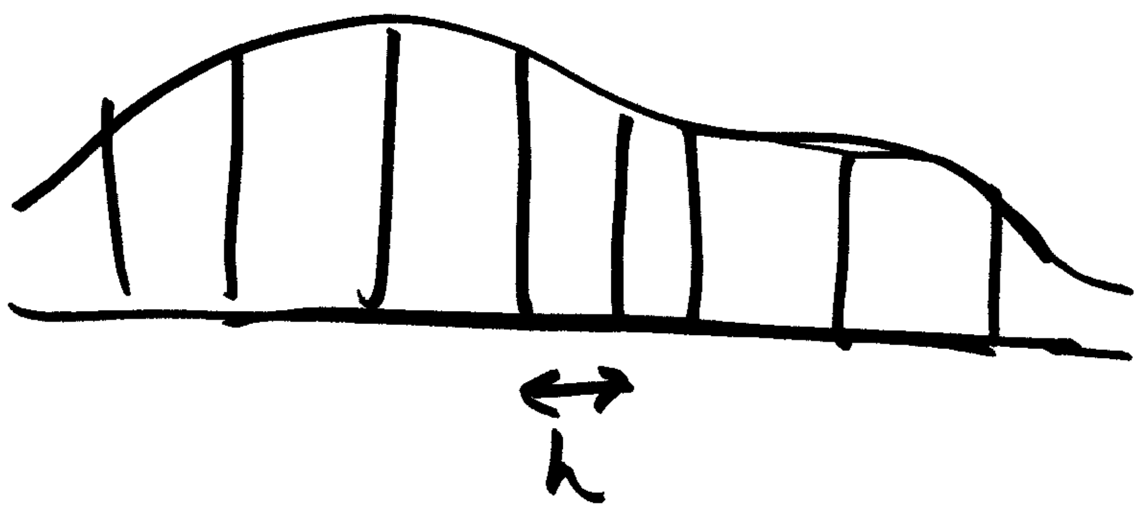
t



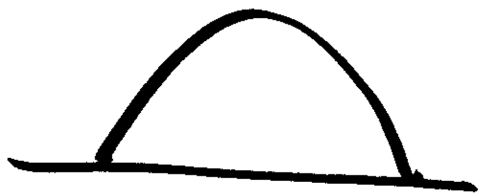


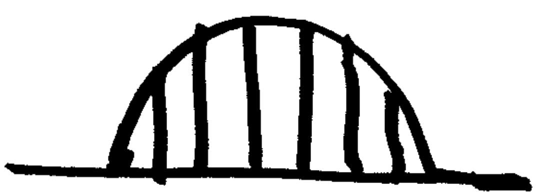
$$\frac{1}{T} \geq 2s \qquad T \leq \frac{1}{2s}$$

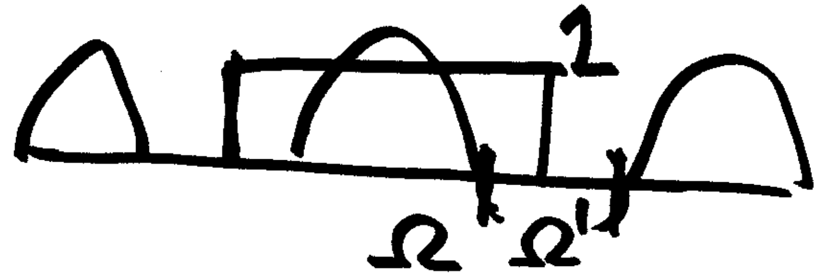
$$f(t) = \sum_{k \in \mathbb{Z}} f(hk)\, \mathrm{sinc}\left(\frac{t-kh}{h}\right)$$



$f(t)$ $F(s)$







$h < \frac{1}{2s}$

$m = 2h\Omega'$

$$f(t) = \sum_{k=-\cancel{\infty}\,n}^{\cancel{\infty}\,n} \cancel{m}\, f\left(\overset{0}{\cancel{hk}}\right) \mathrm{sinc}\left(\cancel{m}\,\frac{t-hk}{h}\right)$$

$$\downarrow$$

$$\frac{\sin\left(\pi m \frac{t-kk}{h}\right)}{\pi \cancel{m}\, \frac{t-hk}{h}}$$

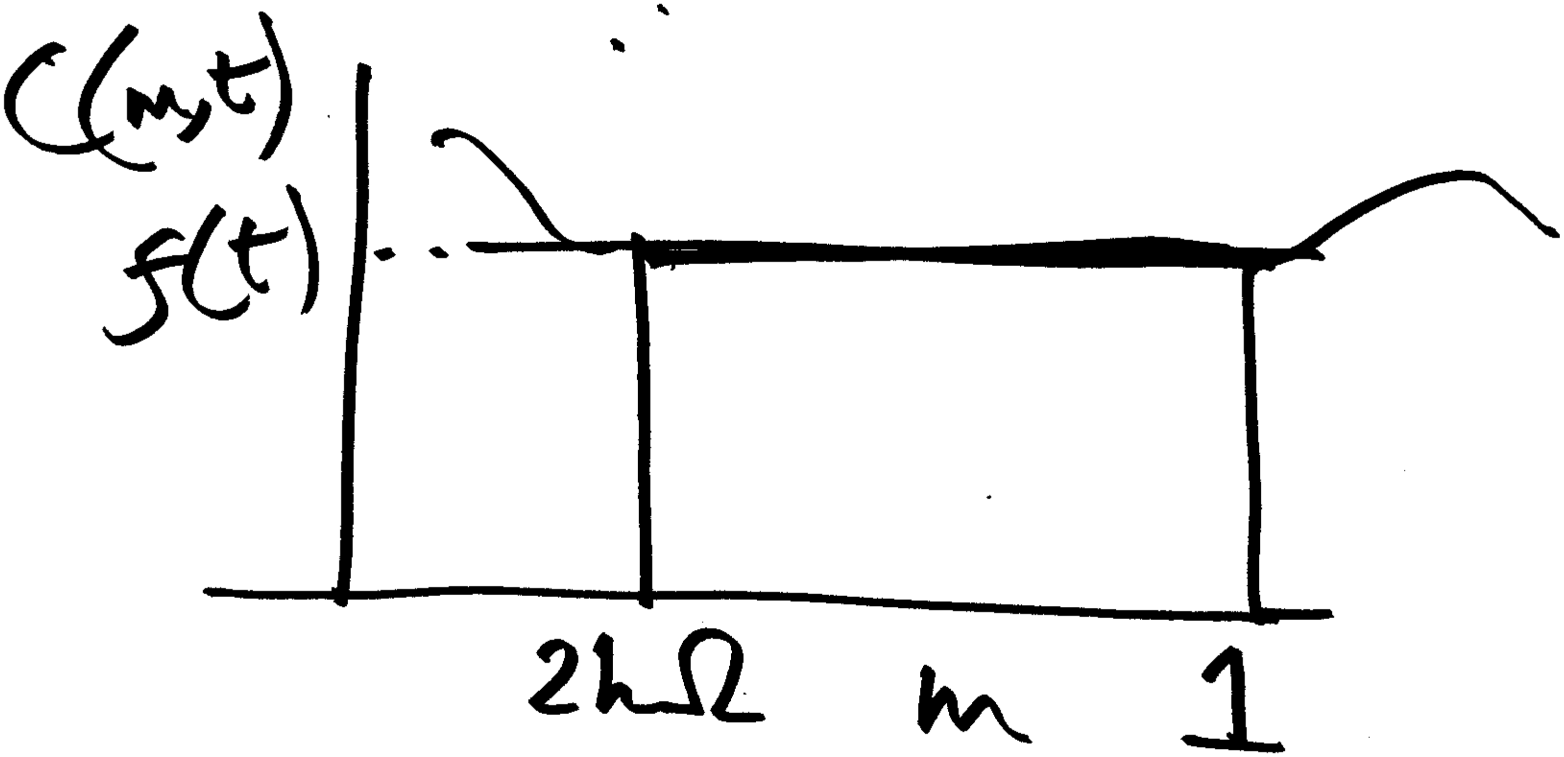
$$\boxed{1 \geqslant m \geqslant 2h\Omega}$$

$$f(t) = \sum_{k=-n}^{n} f(hk)\,\frac{\sin\left(m\pi \boxed{\cancel{\frac{t-hk}{h}}}\right)}{\pi\left(\boxed{\cancel{\frac{t-hk}{h}}}\right)}$$

$$= C(m, \cancel{t})$$

$$C'(m) = \sum_{k=-n}^{n} f(hk)\cos\left(m\pi\,\frac{t-hk}{n}\right)$$

$$0 = c'(m)$$



$$C^{1}(m) = \sum_{k=-n}^{n} f(nk) \cos \pi m \left(\frac{t-kh}{h}\right)$$

$$C^{(2i+1)}(m) = \pi \left(\frac{t-kh}{h}\right)^{2i} \cos\left(\pi m \left(\frac{t-kh}{h}\right)\right)$$

$$= 0$$

$$1 \geq m \geq 2m\Omega$$

$-n \quad 0 \quad n$

$$\psi_{ik} = \left(\frac{\pi(t-hk)}{h}\right)^{2i}$$

$$C_k = f(hk)\cos\left(m\pi\left(\frac{t-kh}{h}\right)\right)$$

$$VC = 0$$

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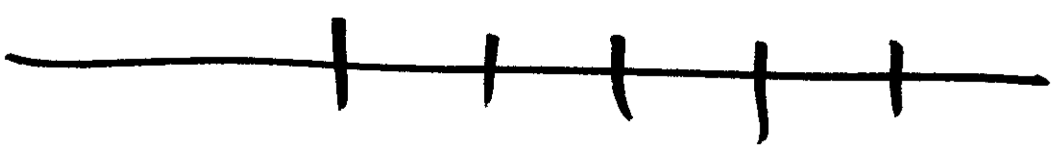
$$(t - hk_i)^2 = (t - hk_j)^2$$

$$t = \frac{1}{2}(k_i + k_j)h$$

$$i \neq j$$



B

$f(hk) = 0$

$t \in B$