Prof. Gadre
Lec - 32
Date - 15/3/10

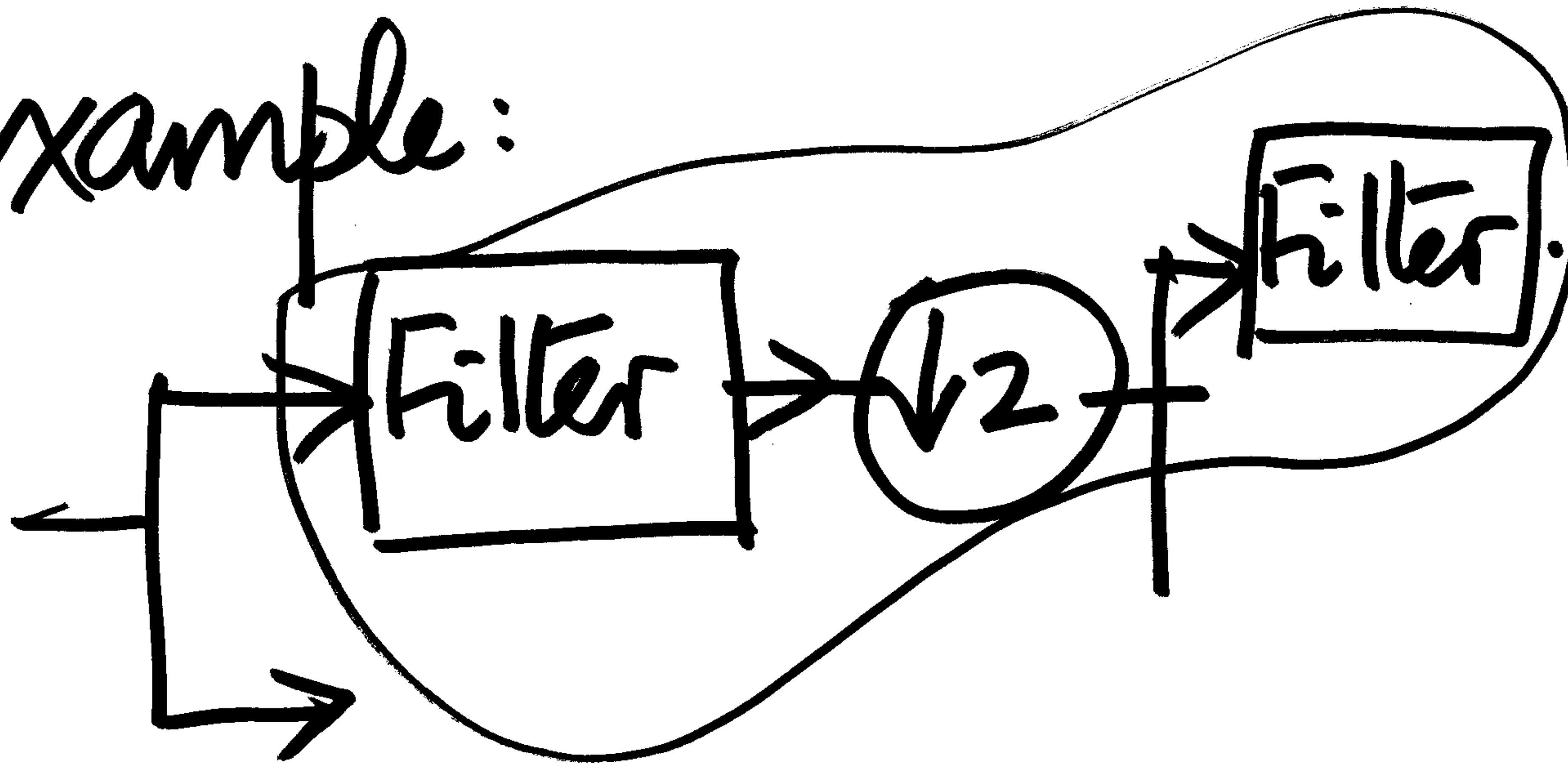
LECTURE 32

'NOBLE' IDENTITIES AND THE HAAR WAVEPACKET TRANSFORM

'NOBLE' IDENTITIES

Theme: To deal with cascades of sampling rate changes and filters.

Example:



'Noble' Identity for Downsampling

$$\longrightarrow \left(\downarrow 2\right) \longrightarrow \boxed{H(z)} \longrightarrow$$

How to interchange

$$x[n] \longrightarrow \boxed{\downarrow 2} \xrightarrow{x_1} \boxed{H(z)} \longrightarrow y$$

impulse response $h[n]$

$$x_1[n] = x[2n]$$

$$\forall n \in \mathbb{Z}$$

$$y[n] = (x_1 * h)[n]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x_1[k]\, h[n-k]$$

$$= \sum_{\ell=-\infty}^{+\infty} h[\ell]\, x_1[n-\ell]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[2k]\, h[n-k]$$

How does this fit into $\rightarrow \boxed{\text{Filter}} \rightarrow (\downarrow 2) \rightarrow$?

Let us use :

$$y[n] = \sum_{l=-\infty}^{+\infty} h[l]\, x_1[n-l]$$

$$x_1[n-l] = x[2n-2l]$$

$$y[n] = \sum_{l=-\infty}^{+\infty} h[l]\,x[2n-2l]$$

Just before the 'equivalent' downsampler

here, - - -

We have:

$$\sum_{l=-\infty}^{+\infty} h[l]\,x[n-2l]$$

This looks like a convolution....

the $h[l]$ seems to be located at the $2l^{th}$ place;

All $(2l+1)^{th}$ places are 0.

Define

$$h_1[n] = 0 \quad n \text{ odd}$$

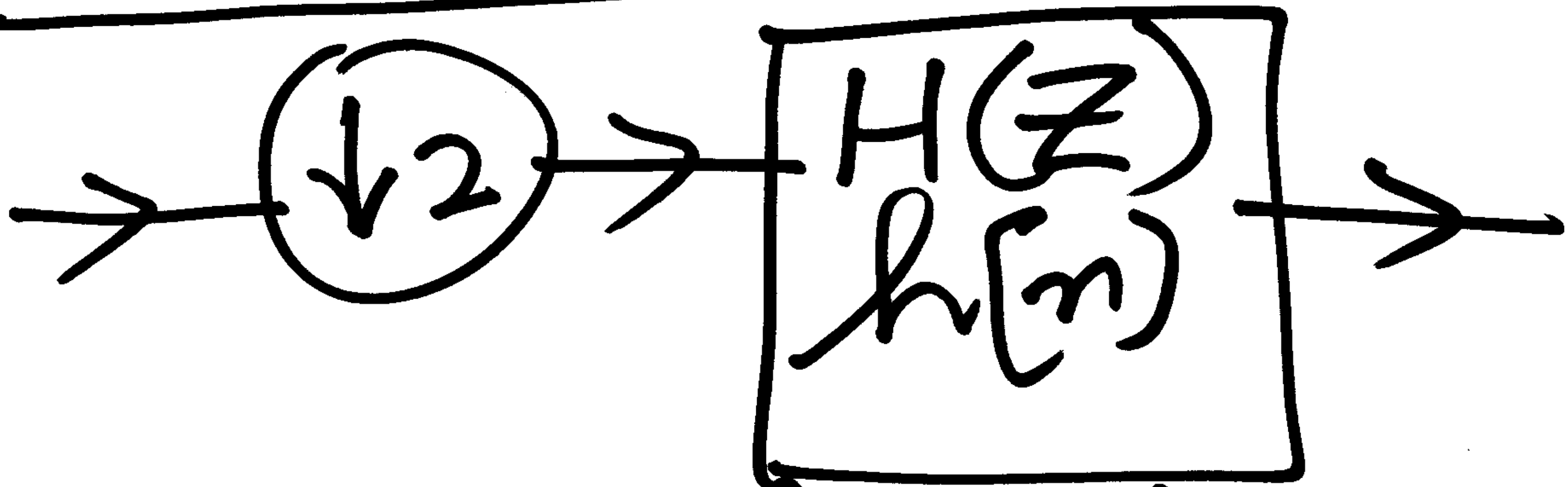
(n not multiple of 2)

$$= h[n/2], \quad n \text{ even}$$

$$y[n] = \sum_{l=-\infty}^{+\infty} h_1[l]\, x[2n - l]$$

'n' before $\downarrow 2$

Conclusion:



is equivalent to ---

$$\longrightarrow \boxed{H(z^2)} \longrightarrow \textcircled{\downarrow 2} \longrightarrow$$

Impulse response of filter preceding downsampler

— — —

-- is the original impulse response upsampled by 2

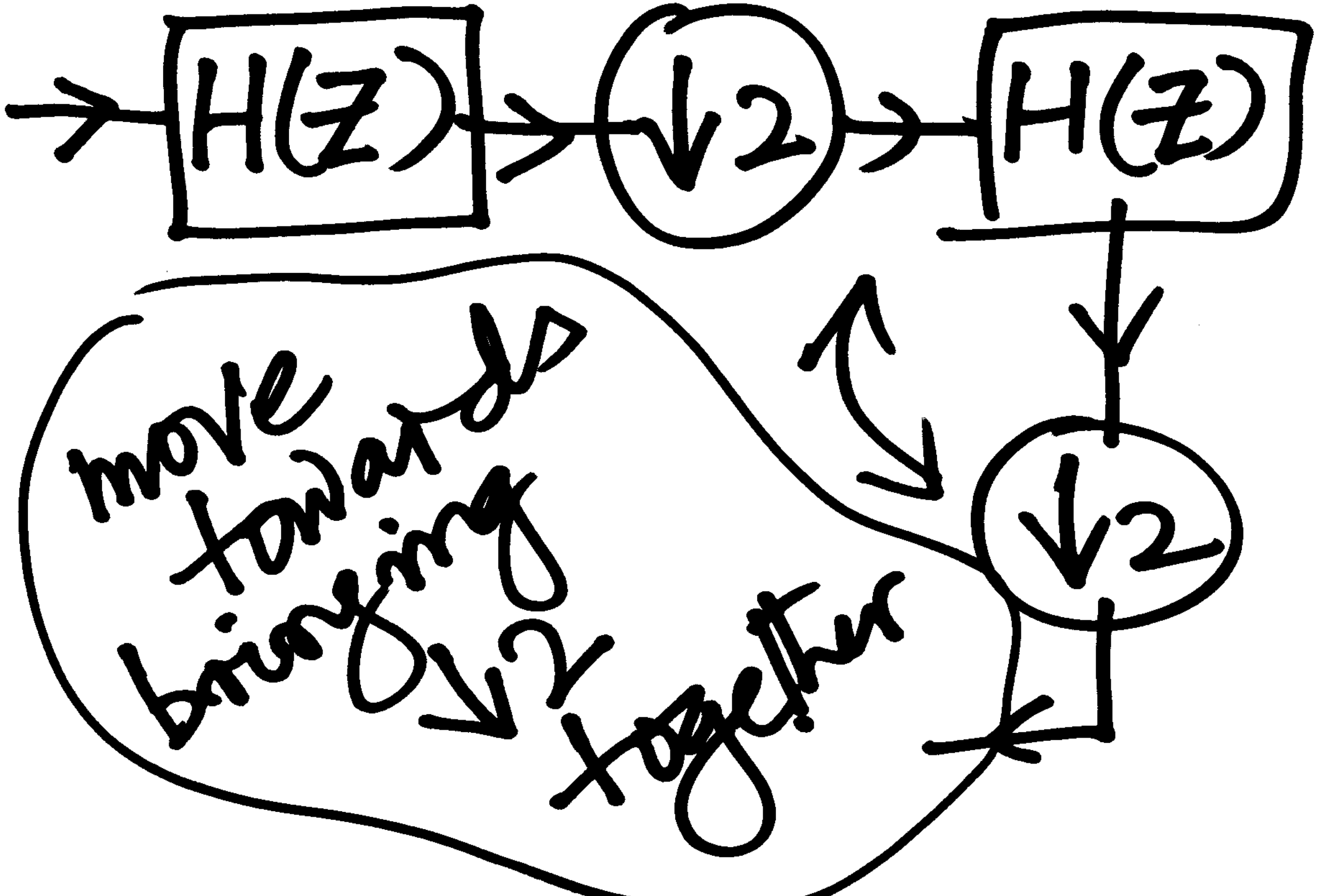
$$h[n] \rightarrow \boxed{\uparrow 2} \rightarrow h_{\uparrow}[n]$$

$$\longrightarrow \downarrow 2 \longrightarrow \boxed{H(z)} \longrightarrow$$

is efficient computationally.

$$\longrightarrow \boxed{H(z^2)} \longrightarrow \downarrow 2 \longrightarrow$$

is <u>inefficient</u> computationally!.


H(Z)
↓2
H(Z)
↓2
move towards bringing ↓2 together

Noble Identity for upsampling:

<u>Transposition</u> in multirate systems: means the following: - -

1. Reverse direction of signal flow

2. When reversing an (up-) (resp. down-) sampler ---

put there a corresponding (down-) resp. (up-) sampler of same factor.

Noble Identity for Downsampling

$$\longrightarrow \downarrow 2 \longrightarrow \boxed{H(z)} \longrightarrow$$

$$\equiv \longrightarrow \boxed{H(z^2)} \longrightarrow \downarrow 2 \longrightarrow$$

Corresponding transpose:

$$\longrightarrow \boxed{H(z)} \longrightarrow \uparrow 2 \longrightarrow$$

$$\equiv \longrightarrow \uparrow 2 \longrightarrow \boxed{H(z^2)} \longrightarrow$$

This is the 'noble' identity for upsampling!

Exercise:

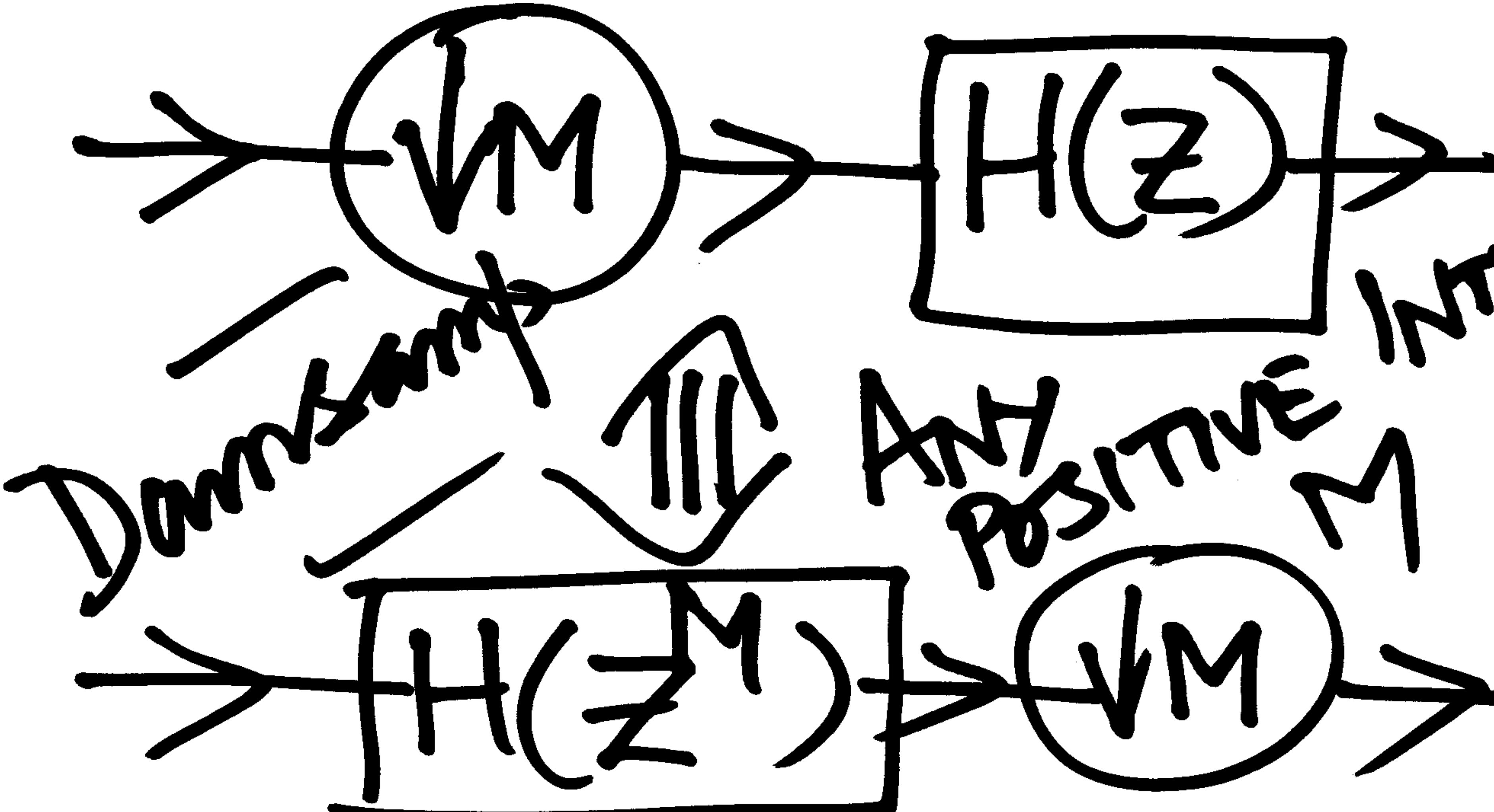
Prove the noble identity for upsampling $\uparrow 2$

Challenge!

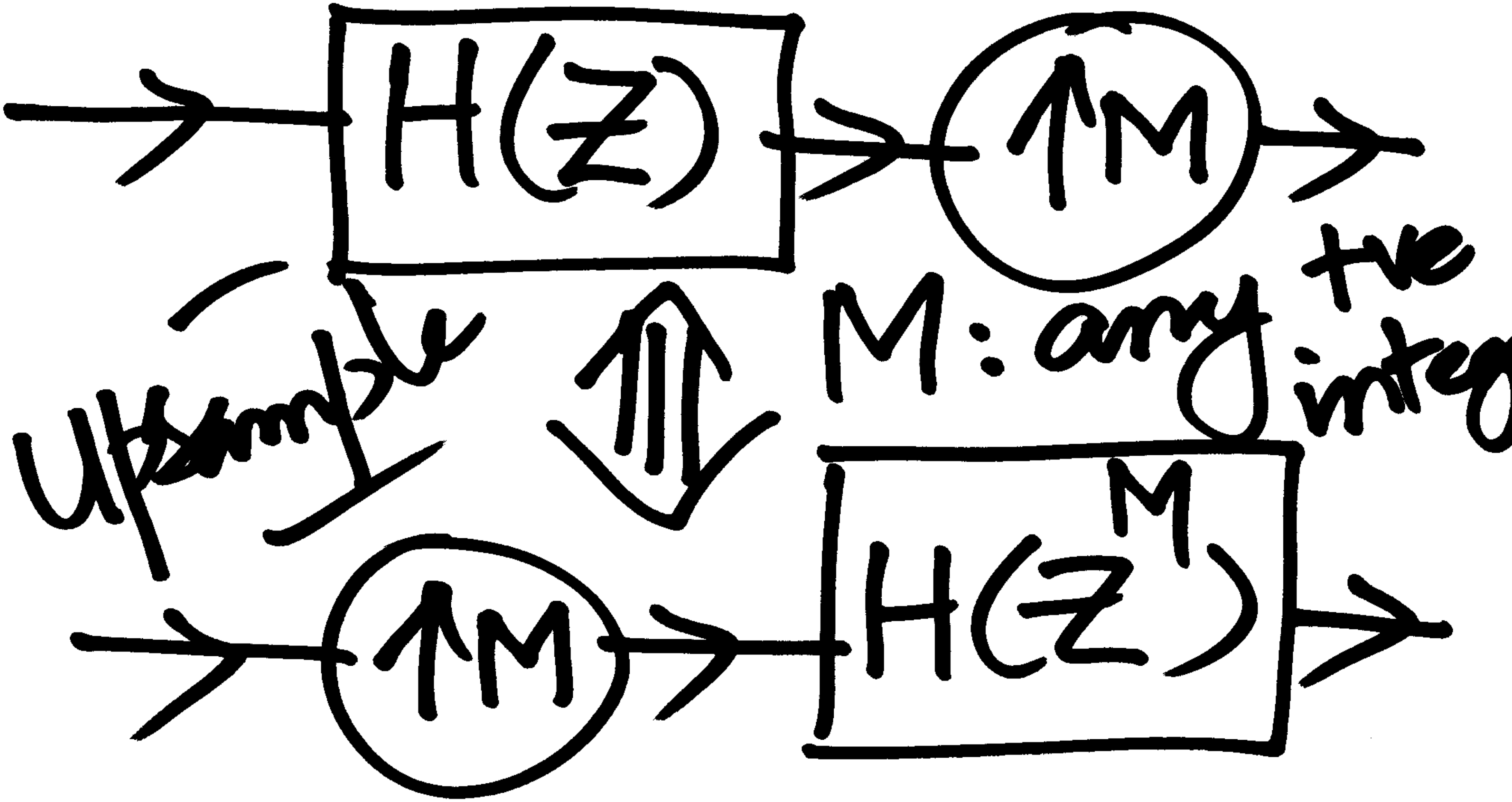
Prove that transposition leads to a valid alternate structure

Exercise:

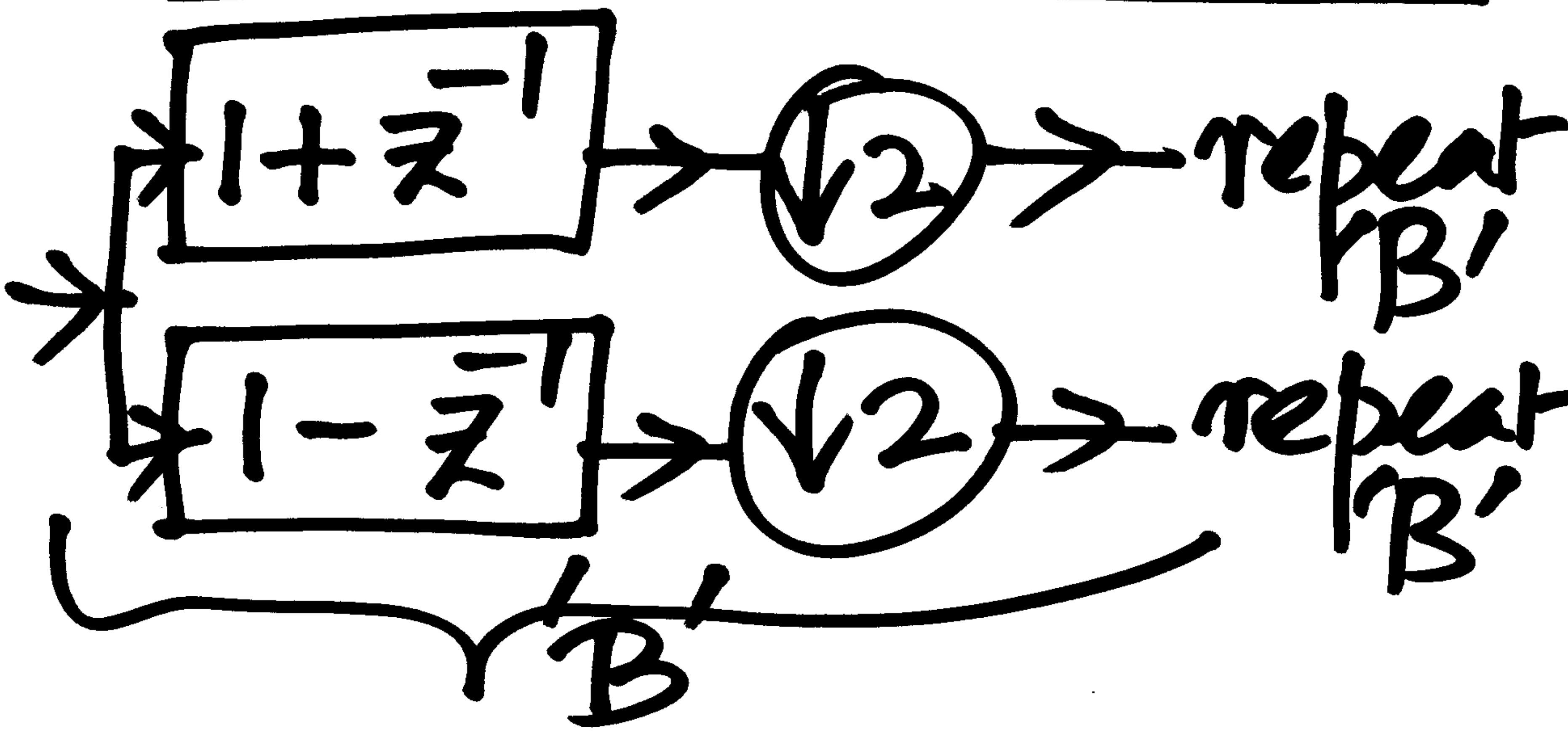
Prove the more general 'noble' identities:



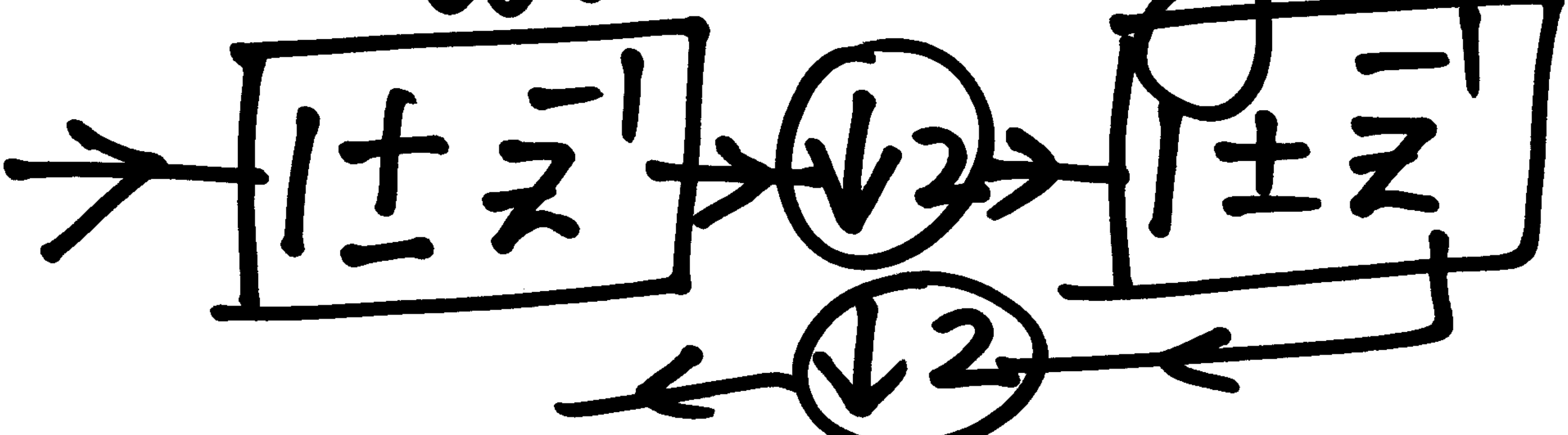
↓M
H(Z)
Downsamp
≡
ANY POSITIVE INTE
M
H(Z^M)
↓M


H(Z)
↑M
Upsample
M: any +ve integer
↑M
H(Z^M)

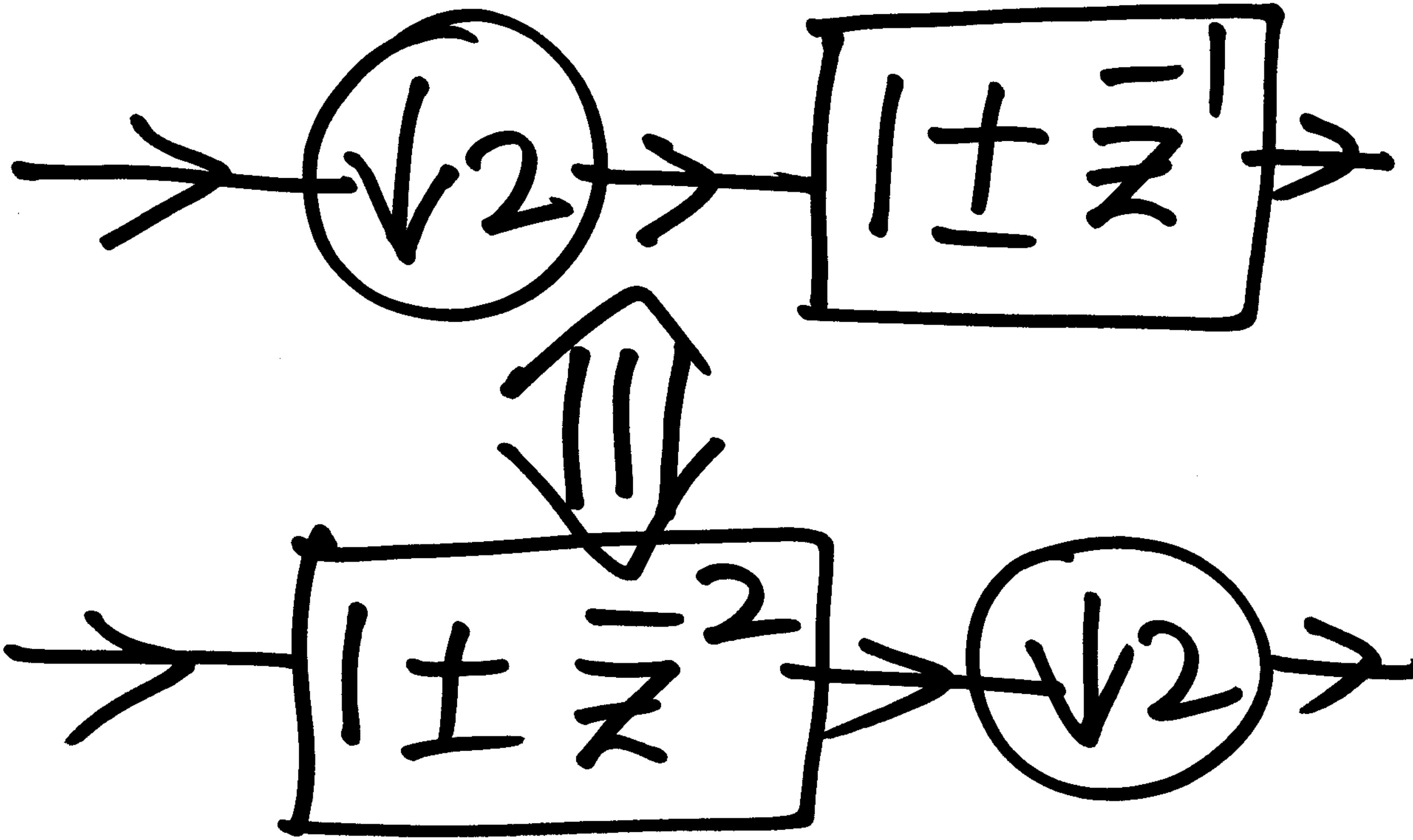
Haar filter bank



There are 4 Cascade structures distinctly:

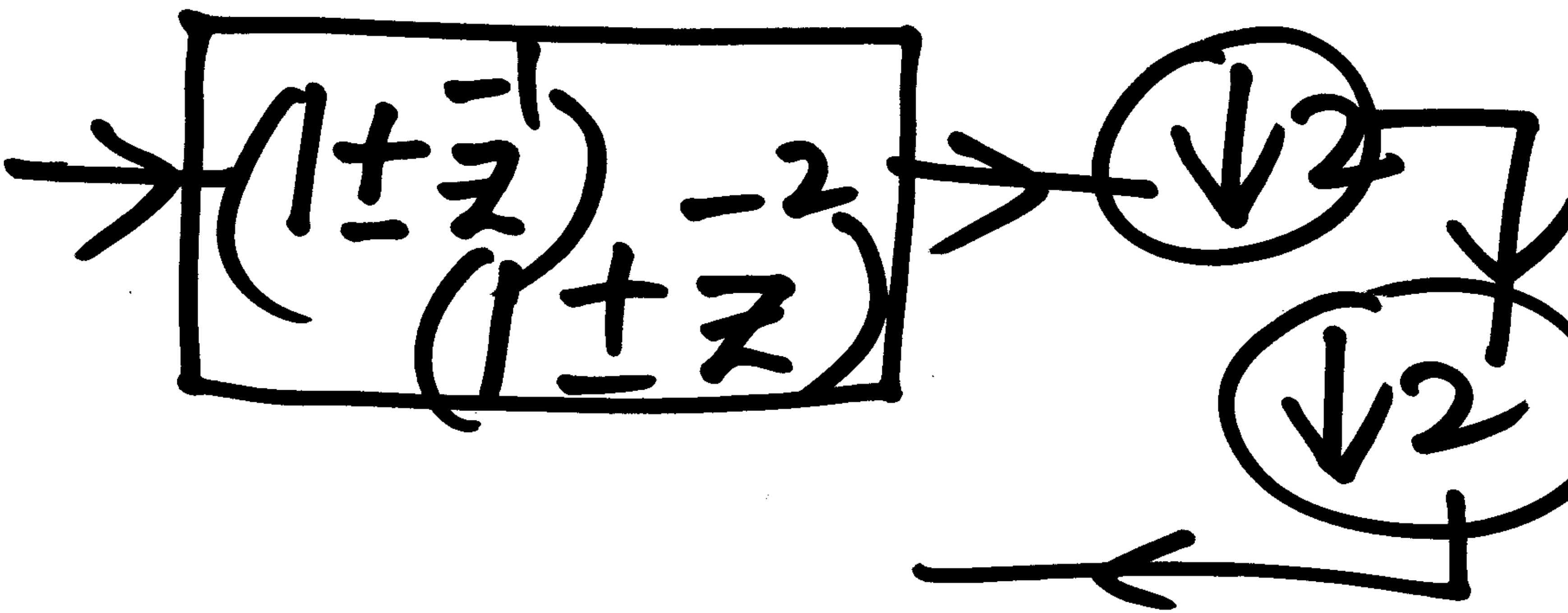


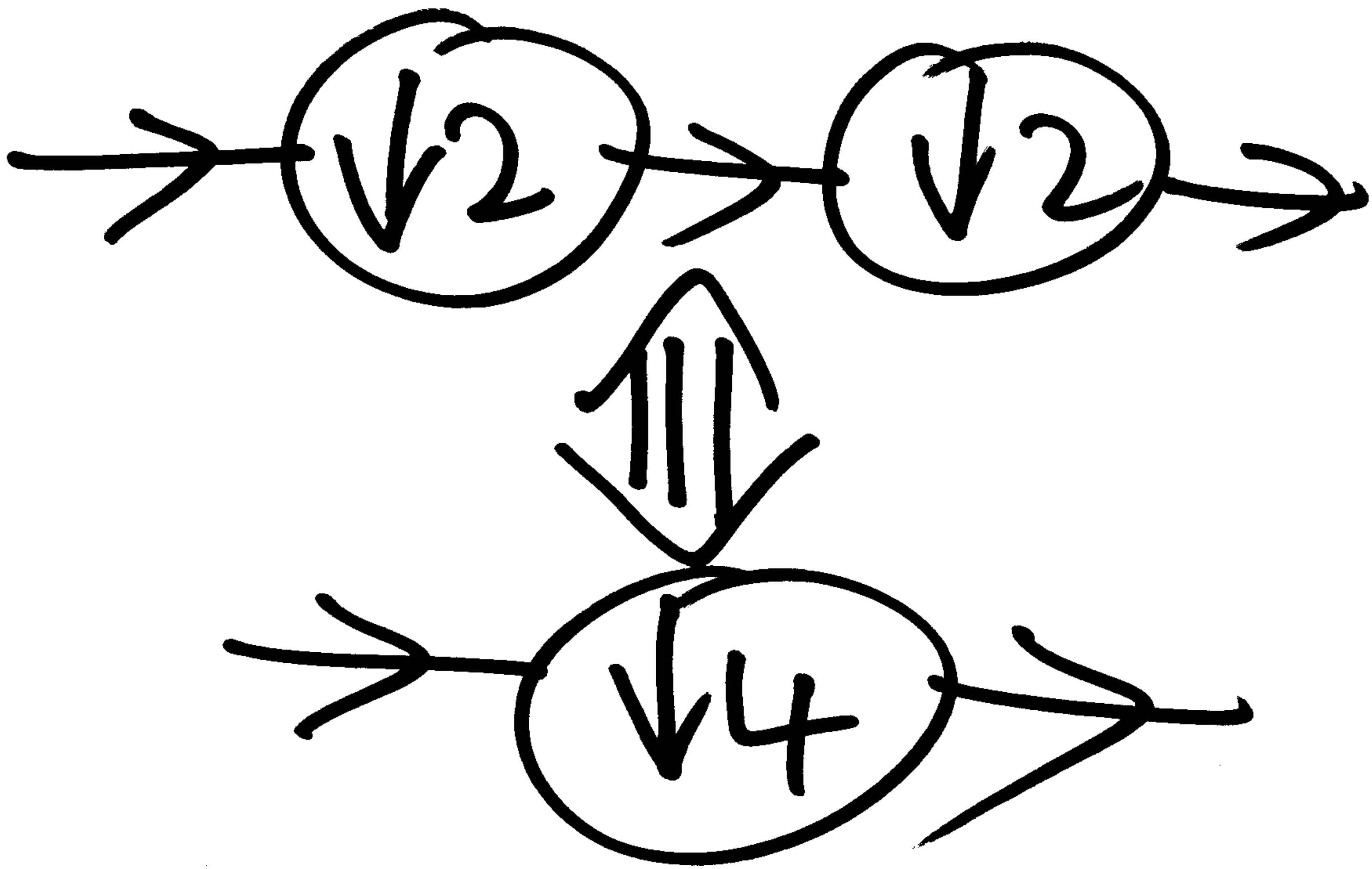
using 'noble' identity for 2



$\downarrow 2$
$1 \pm z^{-1}$
$1 \pm z^{-2}$
$\downarrow 2$

4 cascade structures become:





↓2
↓2
↓4

$$
\begin{tikzcd}
 & & V_0 \\
 & V_1 \arrow[ur, "\frac{1+z}{2}"] \arrow[r, "\frac{1-z}{2}"'] & W_0 \\
V_2 \arrow[ur, "\frac{1+z}{2}"] \arrow[dr, "\frac{1-z}{2}"'] & & \\
 & W_1 \arrow[r, "\frac{1+z}{2}"] \arrow[dr, "\frac{1-z}{2}"'] & W_{10} \\
 & & W_{11}
\end{tikzcd}
$$

$$V_2 \xrightarrow{1+z^{-1}} V_1 \xrightarrow{1+z^{-2}} V_0$$

$$(1+z^{-1})(1+z^{-2})$$

$$= 1+z^{-1}+z^{-2}+z^{-3}$$

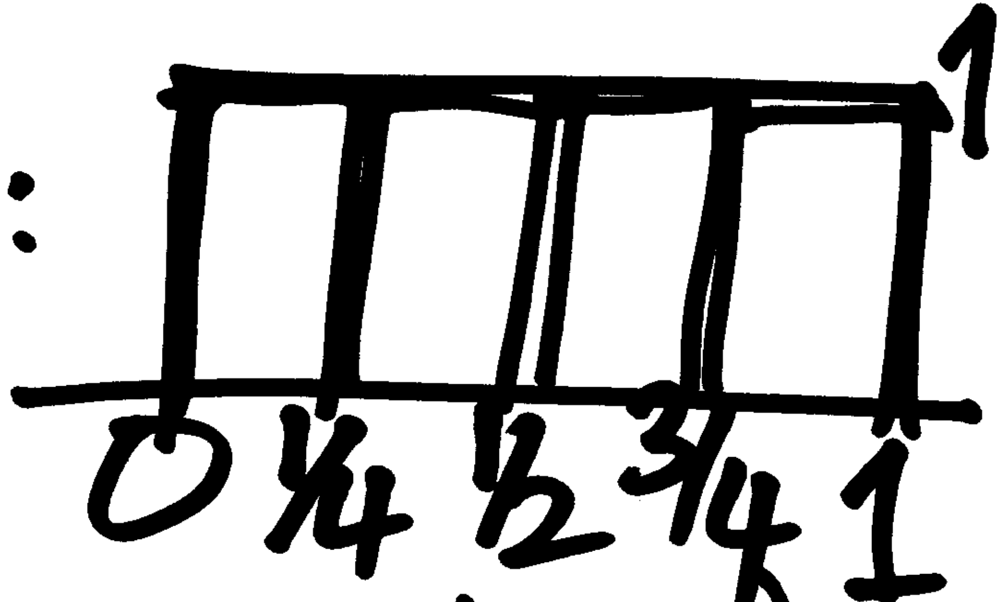
The sequence

$1 \quad 1 \quad 1 \quad 1$

0

expands $\phi(t)$ in terms of $\phi(4t-k)$

Indeed:



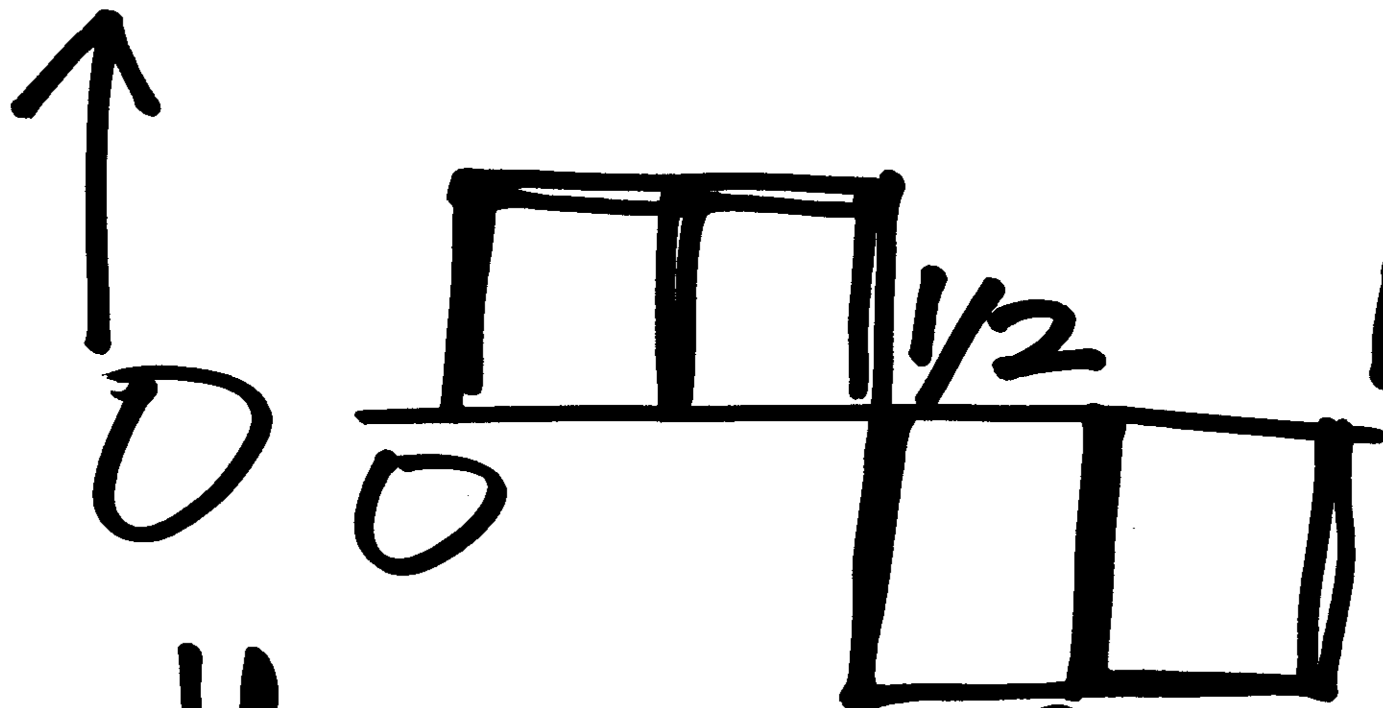
$$\phi(t) = \phi(4t) + \phi(4t-1) + \phi(4t-2) + \phi(4t-3)$$

$$V_2 \to V_1 \to W_0$$

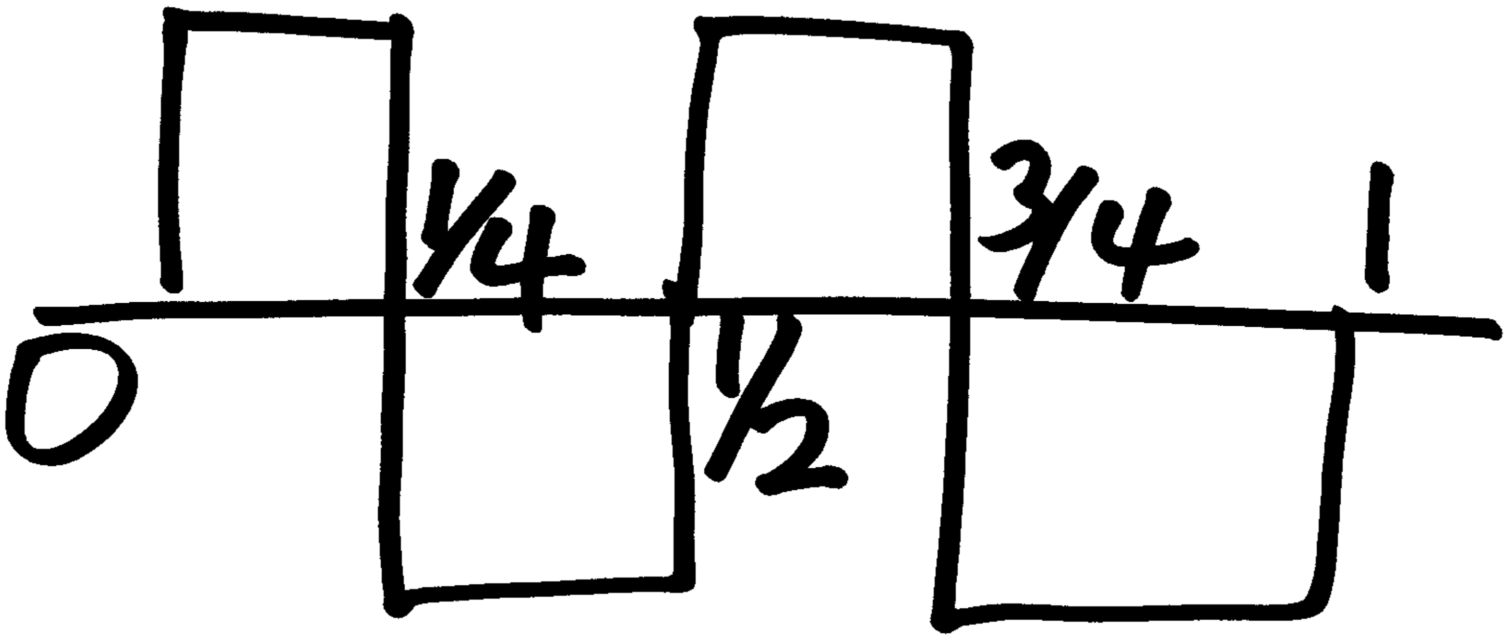
$$(1+z^{-1})(1-z^{-2})$$

$$(1+z^{-1}) - z^{-2} - z^{-3}$$

$1 \quad 1 \quad -1 \quad -1$



Haar wavelet (expected)



and its integer translates

$V_2 \to W_1 \to W_{10}$

$(1 - z^{-1})(1 + z^{-2})$

$1 \quad -1 \quad 1 \quad -1$

$\uparrow$
0

Finally, for W_{11}: