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Lect. No. 30
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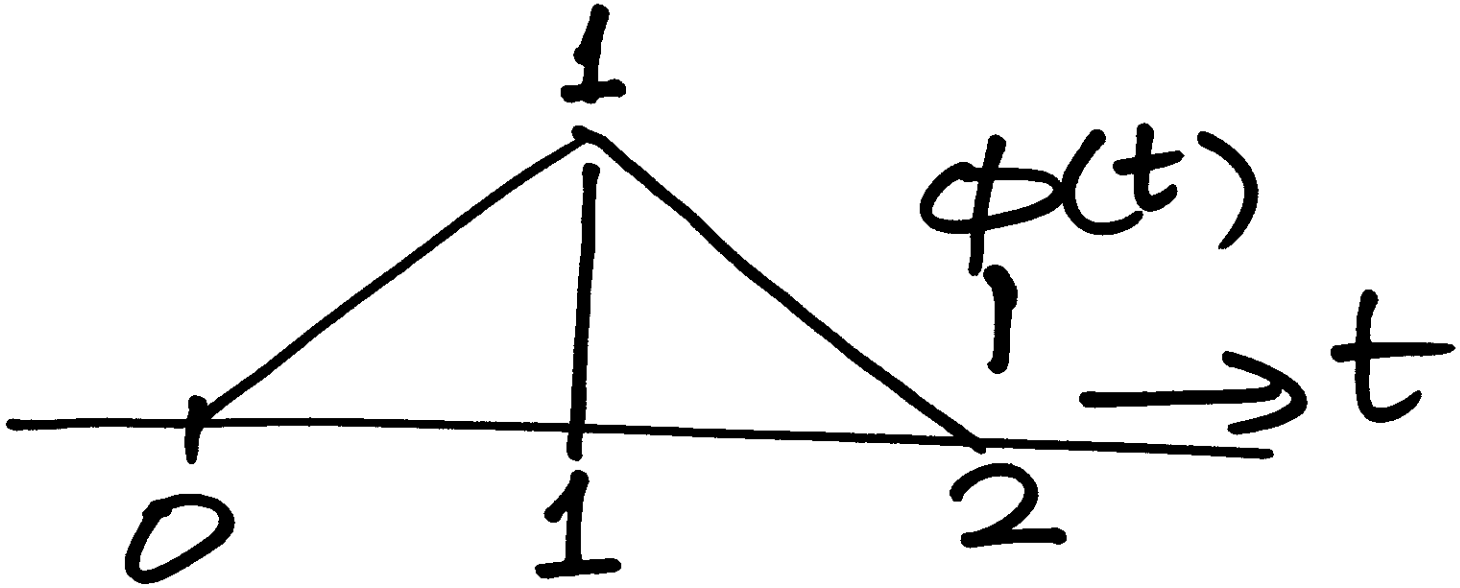
LECTURE 30

BUILDING PIECEWISE LINEAR SCALING FUNCTION, WAVELET

$$STS(\Phi_1, 2\pi)(\Omega)$$

$$= \sum_{k=-\infty}^{+\infty} |\Phi_1(\Omega + 2\pi k)|^2$$

<u>Not</u> Constant as required



$$0 < A \le STS(\Phi_1, 2\pi)(\Omega) \le B$$

$$B < \infty$$

$$\frac{B}{A} = 3.$$

$$\hat{\tilde{\phi}}_1(\Omega) = \frac{\hat{\phi}_1(\Omega)}{+N\sqrt{STS(\phi_1, 2\pi)(\Omega)}}$$

$$\mathrm{STS}(\tilde{\Phi}_1, 2\pi)(\Omega)$$

$$\equiv \frac{\mathrm{STS}(\Phi_1, 2\pi)(\Omega)}{\mathrm{STS}(\Phi_1, 2\pi)(\Omega)}$$

$$= 1$$

Therefore $\tilde{\Phi}_1(t)$ is orthogonal to all its integer translates $\forall m \in \mathbb{Z}$ $\tilde{\Phi}_1(t-m)$

$$\hat{\tilde{\phi}}_1(\Omega) = \frac{\hat{\phi}_1(\Omega)}{+\sqrt{\frac{2}{3}\left(1+\frac{1}{2}\cos\Omega\right)}}$$

$(1+\gamma)^R$

R real

$|\gamma| < 1$

Generalized binomial expansion

Also a Taylor series expansion

$$(1+r)^R =$$

$$1 + R \cdot r + \frac{R(R-1)}{2!} r^2 + \frac{R(R-1)(R-2)}{3!} r^3 \ldots$$

$$\hat{\tilde{\Phi}}_1(\Omega) =$$

$$\hat{\Phi}_1(\Omega)\sqrt{\left(\frac{2}{3}\right)^{-1}}\left(1+\frac{1}{2}\cos\Omega\right)^{-1/2}$$

$$\cos^N \Omega$$

$$= \left(\frac{e^{j\Omega} + e^{-j\Omega}}{2}\right)^N$$

$$\hat{\Phi}_1(\Omega) = \sum_{l=-\infty}^{+\infty} \tilde{c}_l e^{j\Omega l} \hat{\Phi}_1(\Omega)$$

$$\tilde{C}_l = \tilde{C}_{-l}$$

from symmetry in expanding $(\cos\Omega)^l$

Taking inverse Fourier Transform

$$\tilde{\phi}_1(t) = \sum_{l=-\infty}^{+\infty} \tilde{c}_l \phi_1(t+l)$$

This shows that $\Phi_1(t)$ is piecewise <u>linear</u>.

One can calculate a few $\tilde{c}_\ell$:

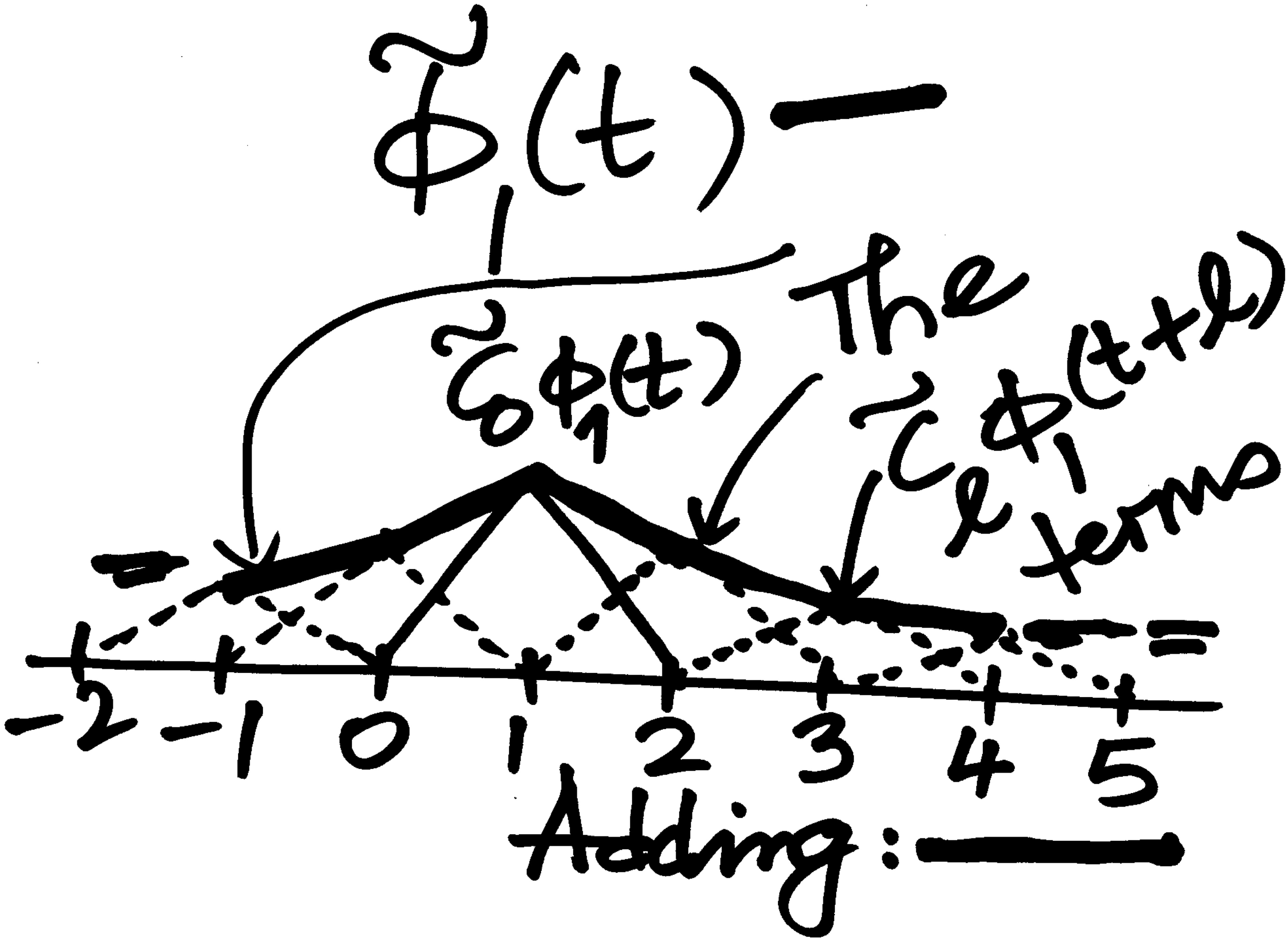
maybe $\ell = 0$

$\ell = \pm 1$, $\ell = \pm 2$

As $\ell \to +\infty$

$\tilde{\zeta}_{\ell}$ and $\tilde{\zeta}_{-\ell}$

decay $\longrightarrow 0$.



$\tilde{\Phi}_1(t)$
$\tilde{c}_0\phi_1(t)$
The $\tilde{c}_\ell\phi_1(t+\ell)$ terms
-2
-1
0
1
2
3
4
5
Adding:

$\tilde{\Phi}_1(t)$ is :

(i) piecewise linear,

a sum of piecewise linear functions

(ii) orthogonal to all integer translates

$$\langle \phi_1(t-m), \phi_1(t-n) \rangle = 0 \, ; \ \forall m, n \in \mathbb{Z}, \ m \neq n$$

(iii) $\tilde{\Phi}_1(\cdot)$ obeys a dilation equation

To be shown

critical for MRA!

We shall use the frequency domain ~~Time domain~~ dyadic equation :

For a general scaling function $\phi(\cdot)$

$$\phi(t) = \sum_{k=-\infty}^{+\infty} h[k]\,\phi(2t-k)$$

Taking Fourier Transf

$$\hat{\Phi}(\Omega) = \frac{1}{2} H\left(\frac{\Omega}{2}\right) \hat{\Phi}\left(\frac{\Omega}{2}\right)$$

where $H(\cdot)$ = DTFT of $h[k]$.

To establish a dyadic dilation equation on $\tilde{\Phi}_1(\cdot)$ - - -

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We essentially need to consider

$$\hat{\tilde{\Phi}}_1(\Omega) \Big/ \hat{\tilde{\Phi}}_1\left(\frac{\Omega}{2}\right)$$

— — —

$\bar{a}$nd show that this is of the form $\frac{1}{2} H\left(\frac{\Omega}{2}\right)$

where

$$H(\Omega) = \sum_{k=-\infty}^{+\infty} h[k] e^{-jk\Omega} \quad (\text{DTFT})$$

We need to establish, essentially, that:

— —

$$\frac{\hat{\tilde{\Phi}}_1(\Omega)}{\hat{\tilde{\Phi}}_1\left(\frac{\Omega}{2}\right)}$$ is a DTFT ___ ___ ___

That is, it is

(i) periodic with period 2π

(ii) bounded on any interval of 2π

Boundedness is needed:

$$\frac{1}{2\pi}\int f(\Omega)\, e^{j\Omega n}\, d\Omega$$

must converge

where

$$f(\Omega) = \frac{\hat{\tilde{\Phi}}_1(\Omega)}{\hat{\tilde{\Phi}}_1\left(\frac{\Omega}{2}\right)}$$

$$\frac{\hat{\tilde{\Phi}}_1(\Omega)}{\hat{\Phi}_1\left(\frac{\Omega}{2}\right)} =$$

$$= \frac{\hat{\Phi}_1(\Omega)\left\{\frac{2}{3}\left(1+\frac{1}{2}\cos\frac{\Omega}{2}\right)\right\}^{1/2}}{\left\{\frac{2}{3}\left(1+\frac{1}{2}\cos\Omega\right)\right\}^{1/2}\hat{\Phi}_1\left(\frac{\Omega}{2}\right)}$$

Focus on

$$\left\{ \frac{\frac{2}{3}\left(1+\frac{1}{2}\cos\frac{\Omega}{2}\right)}{\frac{2}{3}\left(1+\frac{1}{2}\cos\Omega\right)} \right\}^{1/2}$$

This is definitely of the form

$$\frac{1}{2} H\left(\frac{\Omega}{2}\right)$$

with properties of $H(\cdot)$ desired !!

To demonstrate this,
replace

$$\frac{\Omega}{2} \leftarrow \Omega$$

or $\Omega \leftarrow 2\Omega$

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to get

$$\left\{ \frac{\frac{2}{3}\left(1+\frac{1}{2}\cos\Omega\right)}{\frac{2}{3}\left(1+\frac{1}{2}\cos 2\Omega\right)} \right\}^{1/2}$$

$0 < A \leq \text{Numerator} \leq B < \infty$

$0 < A \leq \text{Denominator} \leq B < \infty$

We have already proved!

$$0 < \frac{A}{B} \leq \frac{\text{Numerator}}{\text{Denominator}} \leq \frac{B}{A} < \infty$$

Fraction is also bet 2 positive finite bounds.

$$\frac{\hat{\Phi}_1(\Omega)}{\hat{\Phi}_1\left(\frac{\Omega}{2}\right)} = \left(\frac{\hat{\Phi}_1(\Omega)}{\hat{\Phi}_1\left(\frac{\Omega}{2}\right)}\right)(\Omega)$$

Q already obeys
periodicity and
boundedness
requirements.

The ratio $\dfrac{\hat{\Phi}_1(\Omega)}{\hat{\Phi}_1\left(\frac{\Omega}{2}\right)}$ obeys requirem.

$\tilde{\Phi}_1(t)$ must obey a dyadic dilation equation

$$\tilde{\phi}_1(t) = \sum_{k=-\infty}^{+\infty} \tilde{h}[k]\,\tilde{\phi}_1(2t-k)$$

$$\tilde{h}[k] \xrightarrow[\text{Discrete Time Fourier Trans}]{} \tilde{H}(\Omega)$$

$$\tilde{H}(\Omega) = \sum_{k=-\infty}^{+\infty} h[k] e^{-jk\Omega}$$

$$\frac{1}{2}\tilde{H}\left(\frac{\Omega}{2}\right) = \frac{\hat{\tilde{\Phi}}_1(\Omega)}{\hat{\tilde{\Phi}}_1\left(\frac{\Omega}{2}\right)}$$

$$= \frac{\hat{\Phi}_1(\Omega)}{\hat{\Phi}_1\left(\frac{\Omega}{2}\right)} \cdot \hat{Q}$$

which we had written

$$= \frac{1}{2} c_0 \left(1 + 2e^{-j\frac{\Omega}{2}} + e^{-j2\frac{\Omega}{2}}\right) \cdots$$

$$\cdots \left[\frac{\frac{2}{3}\left(1 + \frac{1}{2}\cos\frac{\Omega}{2}\right)}{\frac{2}{3}\left(1 + \frac{1}{2}\cos\Omega\right)} \right]^{1/2}$$