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Lect. 29
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LECTURE 29

ORTHOGONAL MULTIRESOLUTION ANALYSIS WITH SPLINES

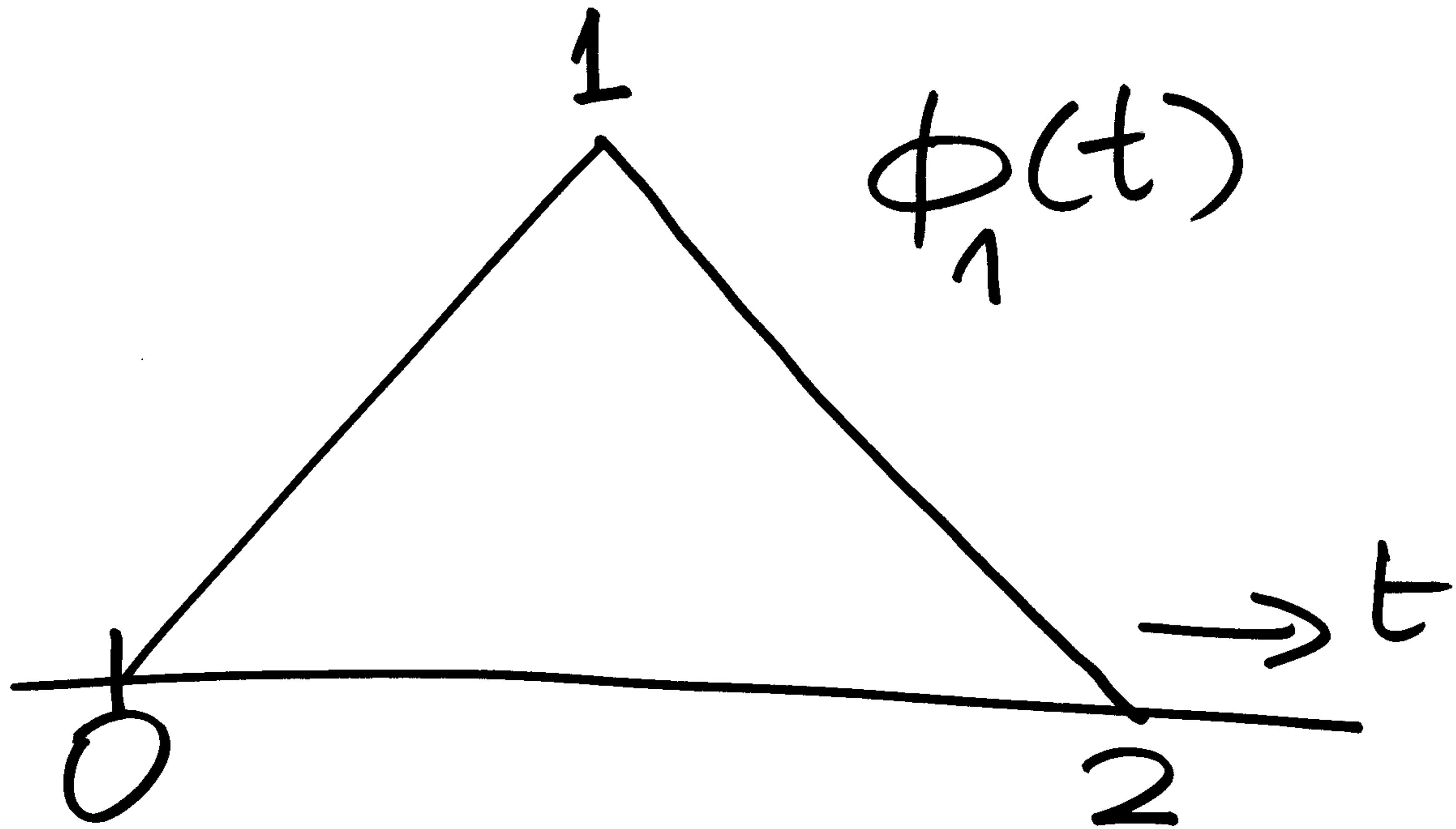
length-3 filter in the 5/3 filter bank:

$$\frac{1}{2}(1+z^{-1})^2$$

$$=\frac{1}{2}(1+2z^{-1}+z^{-2})$$

Corresponding scaling function:

$$\phi_1(t) = \frac{1}{2}\phi_1(2t) + \phi_1(2t-1) + \frac{1}{2}\phi_1(2t-2)$$


1
$\phi_1(t)$
0
2
t

$$\langle \phi_1(t), \phi_1(t-1) \rangle$$

$$\langle \phi_1(t), \phi_1(t+1) \rangle$$

These are <u>nonzero</u>.

Can we build a multiresolution Analysis with piecewise linear functions for Φ and ψ?

Scaling function

$\phi(t)$

is orthogonal to its

integer translates

$\phi(t-m),\ m \in \mathbb{Z}$

--- means

Autocorrelation of

$\phi(t)$

$= R_{\phi\phi}(\tau)$, when

sampled at $\tau = m$,

$m \in \mathbb{Z}$ ---

--- gives an impulse sequence!.

The sequence $R_{\phi\phi}(\tau)\Big|_{\tau=m}$ is an <u>impulse</u>

$m \in \mathbb{Z}$

$$\phi(t) \xrightarrow[\text{Transform}]{\text{Fourier}} \hat{\phi}(\Omega)$$

$$R_{\phi\phi}(\tau) \longrightarrow |\hat{\phi}(\Omega)|^2$$

exercise: review proof.

Sampling $R_{\phi\phi}(\tau)$ at $\tau = m$, $m \in \mathbb{Z}$ (sampling rate $= 1$) means ---

Summing all aliases of $|\hat{\Phi}(\Omega)|^2$

$$\kappa_0 \sum_{k=-\infty}^{+\infty} \left|\hat{\Phi}\left(\Omega + \frac{2\pi k}{1}\right)\right|^2$$

Constant of Amps

If this sequence

$$R_{\varphi\varphi}(\tau)\Big|_{\tau=m}$$

$m \in \mathbb{Z}$ is an impulse, its DTFT

— —

essentially

$$\sum_{k=-\infty}^{+\infty} \left| \hat{\Phi}(\Omega + 2\pi k) \right|^2$$

must be <u>Constant</u>.

$\{\phi(t-m)\}_{m \in \mathbb{Z}}$ forms an orthogonal set ___ ___ _____ is equivalent to

$$\sum_{k=-\infty}^{+\infty} \left| \hat{\Phi}(\Omega + 2\pi k) \right|^2 = \text{Constant}.$$

We shall call

$$\sum_{k=-\infty}^{+\infty} \left| \hat{\Phi}(\Omega + 2\pi k) \right|^2$$

as Sum of Translated Spectra of $\Phi(\cdot)$

Sum of translated spectra

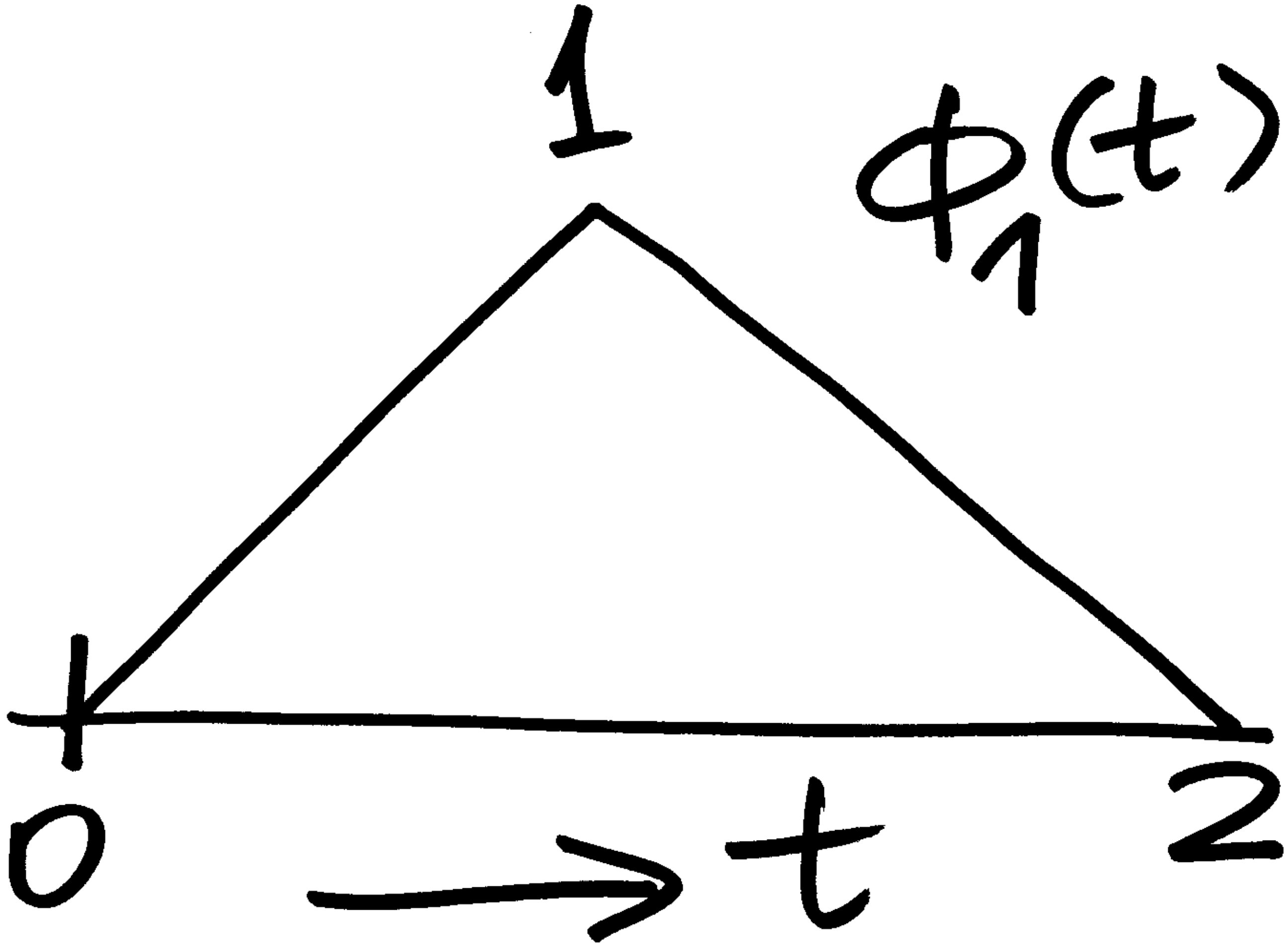
$$\underbrace{STS(\Phi, 2\pi)}_{\text{secondary}}(\Omega)$$

Ω: primary

arguments

In general

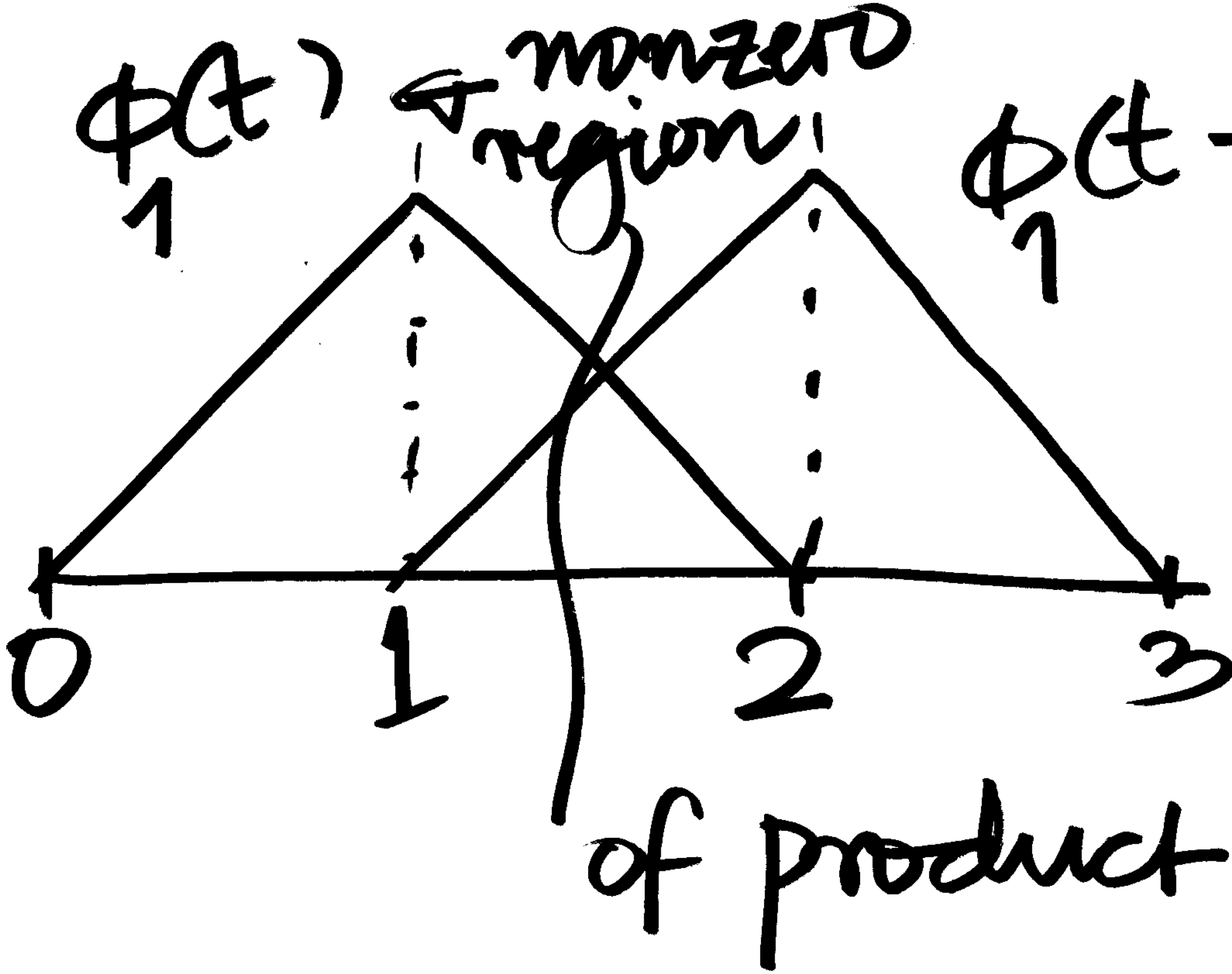
$$STS(\Phi, T)(\Omega) = \sum_{K=-\infty}^{+\infty} \left| \hat{\Phi}(\Omega + TK) \right|^2$$


1
$\phi_1(t)$
0
t
2

For $\phi_1(t)$ consider

$$R_{\phi_1\phi_1}(1) = R_{\phi_1\phi_1}(-1)$$

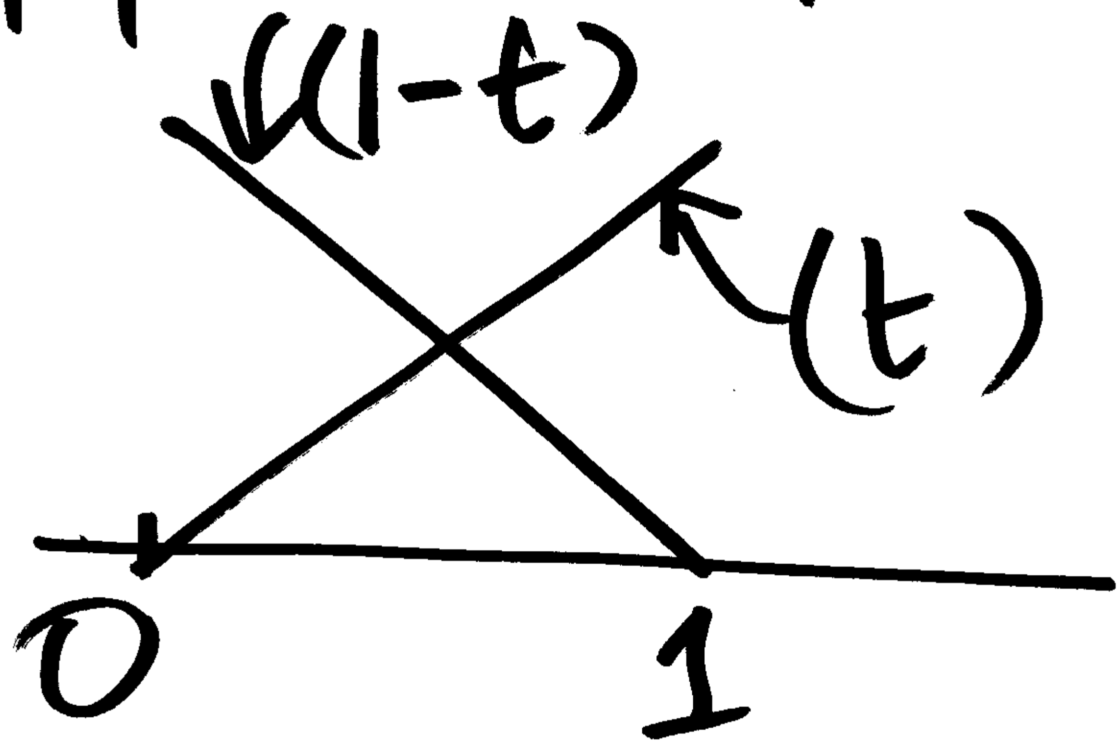
$$= \int_{-\infty}^{+\infty} \phi_1(t)\phi_1(t-1)\,dt$$



Φ(t)
nonzero region
of product
Φ(t-1)
1
1
0
1
2
3

$$R_{\phi_1\phi_1}(1) = R_{\phi_1\phi_1}(-1)$$

$=$



$$= \int_0^1 t(1-t)\,dt$$

$$= \int_0^1 (t - t^2)\,dt$$

$$= \left. \frac{t^2}{2} - \frac{t^3}{3} \right|_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6}$$

$$= 1/6$$

$$R_{\phi_1 \phi_1}(0) = \int_0^2 \phi_1^2(t)\, dt$$

$$= 2\int_0^1 \phi_1^2(t)\, dt$$

from symmetry of $\phi_1(t)$ about $t = 1$.

$$= 2\int_0^1 t^2\,dt$$

$$= \left.\frac{2t^3}{3}\right|_0^1 = \frac{2}{3}$$

Discrete Time Fourier Transform (DTFT) of autocorrelation sequence $R_{\phi_1 \phi_1}(\tau)\Big|_{\tau = m}$

= DTFT of sequence

$$-\;0\;\Big)\;\frac{1}{6}\qquad \underset{\substack{\uparrow\\ 0}}{\frac{2}{3}}\qquad \frac{1}{6}\;\Big(\;0\;-$$

$$= \frac{1}{6} e^{j\omega} + \frac{2}{3} + \frac{1}{6} e^{-j\omega}.$$

$$\sum_{k=-\infty}^{+\infty} \left| \hat{\Phi}(\Omega + 2\pi k) \right|^2$$

$$= \left(\frac{2}{3} + \frac{1}{6} e^{j\Omega} + \frac{1}{6} e^{-j\Omega} \right)$$

$\times$ constant ~~ignore~~

$$= \frac{2}{3} + \frac{1}{6}\left(e^{j\Omega} + e^{-j\Omega}\right)$$

$$= \frac{2}{3} + \frac{1}{6} \cdot 2\cos\Omega$$

$$= \frac{2}{3} + \frac{1}{3}\cos\Omega$$

$$= \frac{2}{3}\left\{1 + \frac{1}{2}\cos\Omega\right\}.$$

$$\frac{1}{3} \leqslant \frac{2}{3}\left(1+\frac{1}{2}\cos\Omega\right) \leqslant 1$$

$(\cos\Omega = -1)$ $\quad$ $(\cos\Omega = +1)$.

Let us define:

$\tilde{\phi}_1(t)$ in terms of its Fourier transform

$$\hat{\tilde{\Phi}}_1(\Omega) = \frac{\hat{\Phi}_1(\Omega)}{+\sqrt{STS(\Phi_1, 2\pi)(\Omega)}}$$

What is

$$STS(\tilde{\Phi}_1, 2\pi)(\Omega)\ ?$$

$$= \sum_{k=-\infty}^{+\infty} \left| \hat{\tilde{\Phi}}_1(\Omega + 2\pi k) \right|^2$$

Periodicity of STS:

$$\mathrm{STS}(\Phi, 2\pi)(\Omega) = \mathrm{STS}(\Phi, 2\pi)(\Omega + 2\pi) \quad \forall\, \Omega$$

$$STS(\phi, 2\pi)(\Omega + 2\pi)$$

$$= \sum_{k=-\infty}^{+\infty} \left|\hat{\phi}(\Omega + 2\pi + 2\pi k)\right|^2$$

$$= \quad \cdots\cdots$$

$$= \sum_{k=-\infty}^{+\infty} \left| \hat{\Phi}\left(\Omega + 2\pi(k+1)\right) \right|^2$$

Proved

We could as well write k here!

$$STS(\tilde{\Phi}_1, 2\pi)(\Omega)$$

$$= \sum_{k=-\infty}^{+\infty} \frac{|\hat{\Phi}_1(\Omega + 2\pi k)|^2}{STS(\Phi_1, 2\pi)(\Omega + 2\pi k)}$$

invoke periodicity

$$= \frac{1}{STS(\Phi_1, 2\pi)(\Omega)} \sum_{k=-\infty}^{+\infty} |\hat{\Phi}_1(\Omega+2\pi k)|^2$$

essentially STS

$$= \frac{STS(\Phi_1, 2\pi)(\Omega)}{STS(\Phi_1, 2\pi)(\Omega)} = 1$$

We have shown:

$$\mathrm{STS}\left(\tilde{\Phi}_1, 2\pi\right)(\Omega)$$

$$= 1 \quad \left(\underline{\text{Constant}}.\right)$$

$\tilde{\Phi}_1$ is orthogonal to its integer translates (all $m \in \mathbb{Z}$).

What does $\tilde{\Phi}_1$ look like?

$$\hat{\tilde{\Phi}}_1(\Omega) = \frac{\hat{\Phi}_1(\Omega)}{\sqrt{STS(\Phi_1, 2\pi)(\Omega)}}$$

$$\hat{\tilde{\Phi}}_1(\Omega) = \frac{\hat{\Phi}_1(\Omega)}{+\sqrt{\frac{2}{3}\left(1+\frac{1}{2}\cos\Omega\right)}}$$

$$= \hat{\Phi}_1(\Omega)\left(\frac{2}{3}\right)^{-1/2}\left(1+\frac{1}{2}\cos\Omega\right)^{-1/2}$$

$$(1+\gamma)^{-1/2}$$

$(1+x)^{-1/2}$

given $\underline{\underline{|x|<1}}$

can be expanded
....

$(1+x)^R$ Real R

Generalization of Binomial Theorem ...

$$1 + R \cdot x + \frac{R(R-1)}{2!} x^2 + \frac{R(R-1)(R-2)}{3!} x^3 + \cdots$$

A typical term is:

$$K_p \cdot \gamma^p = K_p (\cos\Omega)^p \left(\frac{1}{2}\right)^p$$

p term

$p > 0$ positive integer

$(\cos \Omega)^p$ can be expanded in terms of $e^{j\Omega}$

$$(\cos\Omega)^p$$

$$= \left(\frac{e^{j\Omega} + e^{-j\Omega}}{2}\right)^p$$

Essentially we could aggregate coefficients of e^{j2k}, k integer

$$\hat{\tilde{\Phi}}_1(\Omega) = \sum_{k=-\infty}^{+\infty} \tilde{c}_k \, e^{-j\Omega k} \hat{\Phi}_1(\Omega)$$

Taking inverse FT
on both sides
essentially ___ ___ ___

$$\tilde{\phi}_1(t) = \sum_{k=-\infty}^{+\infty} \tilde{c}_k \phi_1(t+k)$$