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Lec - 27
Date :- 04/3/

LECTURE 27

INTRODUCING VARIANTS OF THE MULTIRESOLUTION ANALYSIS CONCEPT

To complete a few details of the proof of theorem of MRA:

We had already shown:

$\psi(t) \in V_1$

and can be expanded:

$$\psi(t) = \sum_{n \in \mathbb{Z}} g[n] \, \phi(2t-n)$$

$g[n]$ = impulse response of the Analysis HIGHPASS

Haar case:-

Highpass impulse response

$$\underset{\substack{\uparrow\\0}}{1} \quad -1$$

Accordingly

$$\psi_{Haar}(t) = 1 \cdot \phi(2t) - 1\,\phi(2t-1)$$

(example)

$g[n]$ corresponds to the inverse z-transform of

$$z^{-(L-1)} H(-z^{-1})$$

. . . where . .

$H(z)$ = analysis lowpass filter.

$\langle \psi(t), \phi(t-m) \rangle$?

any integer m ?

$$\psi(t) = \sum_{n} g[n]\,\phi(2t-n)$$

$$\phi(t-m) = \sum_{n_1} h[n_1]\,\phi(2t-2m-n_1)$$

$$\langle \psi(t), \phi(t-m) \rangle$$

$$= \sum_{n} \sum_{n_1} g[n] \overline{h[n_1]} \cdots$$

$$\cdots \langle \phi(2t-n), \phi(2t-2m-n_1) \rangle$$

$$\left\langle \phi(2t-n),\ \phi(2t-2m-n_1) \right\rangle = \frac{1}{2}\delta[n-(2m+n_1)]$$

Drop $\sum_m$ and leave:

$$\frac{1}{2}\sum_{n_1} g[2m+n_1]\,\overline{h[n_1]}$$

= cross correlation of $g[\cdot]$ and $h[\cdot]$ evaluated at $2m$

Z-transform of the cross correlation of $g[\cdot]$ and $h[\cdot]$

$$= G(z)\,H(z^{-1})$$

$$G(z) = z^{-(L-1)} H(-z^{-1})$$

Cross correlation

z - transform

$= \quad - \quad -$

$$z^{-(L-1)} H(-z^{-1}) H(z^{-1})$$

(simplify matters: assume impulse responses <u>real</u>)

Consider that:

$$\bar{z}^{(L-1)} H(-\bar{z}^{-1}) \cdot H(\bar{z}^{-1})$$

$$+ \left(\cdot\cdot \text{same} \cdot\cdot \right)_{z \leftarrow -z}$$

$$= \cdots\cdot$$

$$z^{-(L-1)} H(-z^{-1}) H(z^{-1})$$

$$+(-1)^{L-1} z^{-(L-1)} H(z^{-1}) H(-z^{-1})$$

L even (remember)

$= 0$ (identically)

The cross correlation of g and h is zero $\forall\, 2m$

$$\Rightarrow \langle \psi(t), \phi(t-m) \rangle = 0 \quad \forall m \in \mathbb{Z}$$

Consider

$$\langle \phi(t), \phi(t-m) \rangle$$

We can similarly show ....

$$= \frac{1}{2} \sum_n h[2m+n] \overline{h[n]}$$

(Autocorrelation of impulse response of lowpass analy. filt.)

$$H(z)H(\bar{z}^{-1})$$

We ask:

$$H(z)H(\bar{z}^{-1}) + H(-z)H(-\bar{z}^{-1}) \quad ?$$

The design equations for an orthogonal filter bank ensure:

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = \text{const}$$

Autocorrelation sequence $= 0$ at all even locations $(2m)$ except $\underline{m = 0}$.

1. Must we have essentially the same analysis and synthesis filters?

JPEG 2000

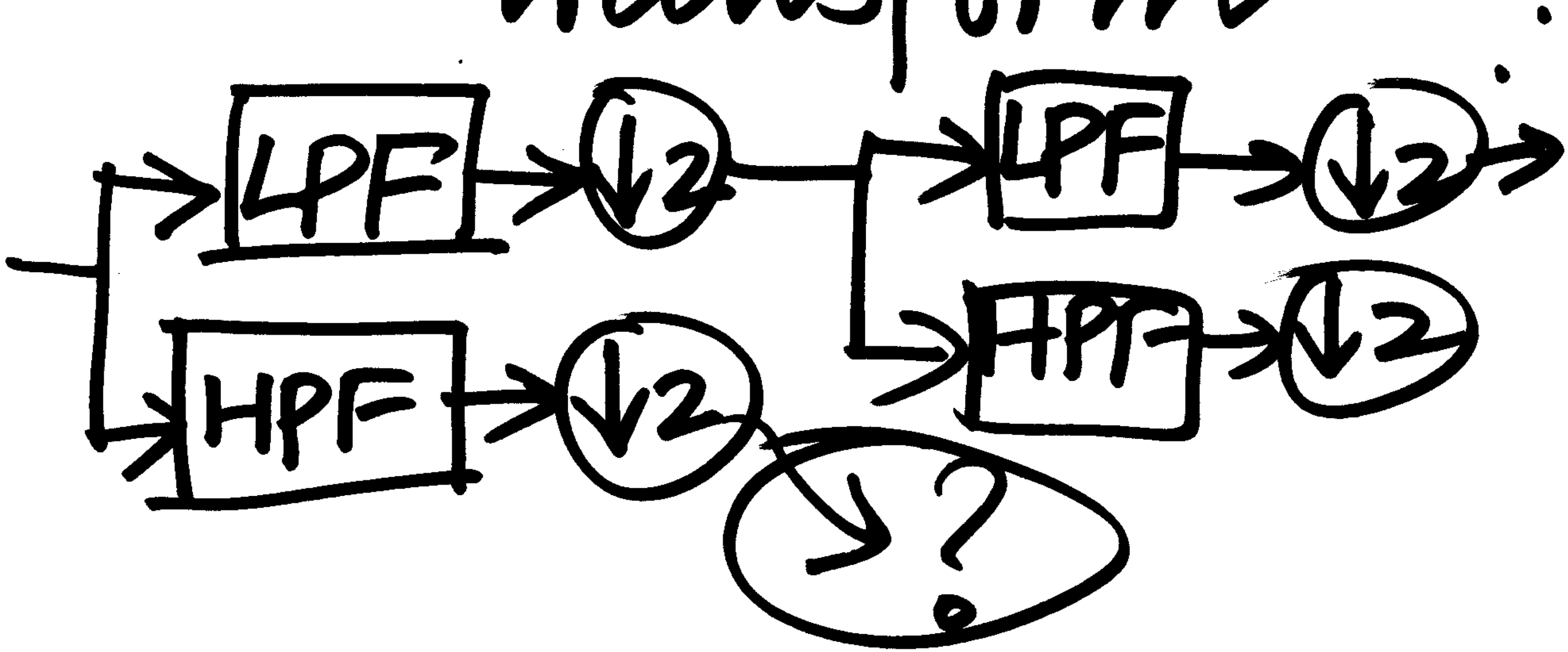
Standards for Data Compression

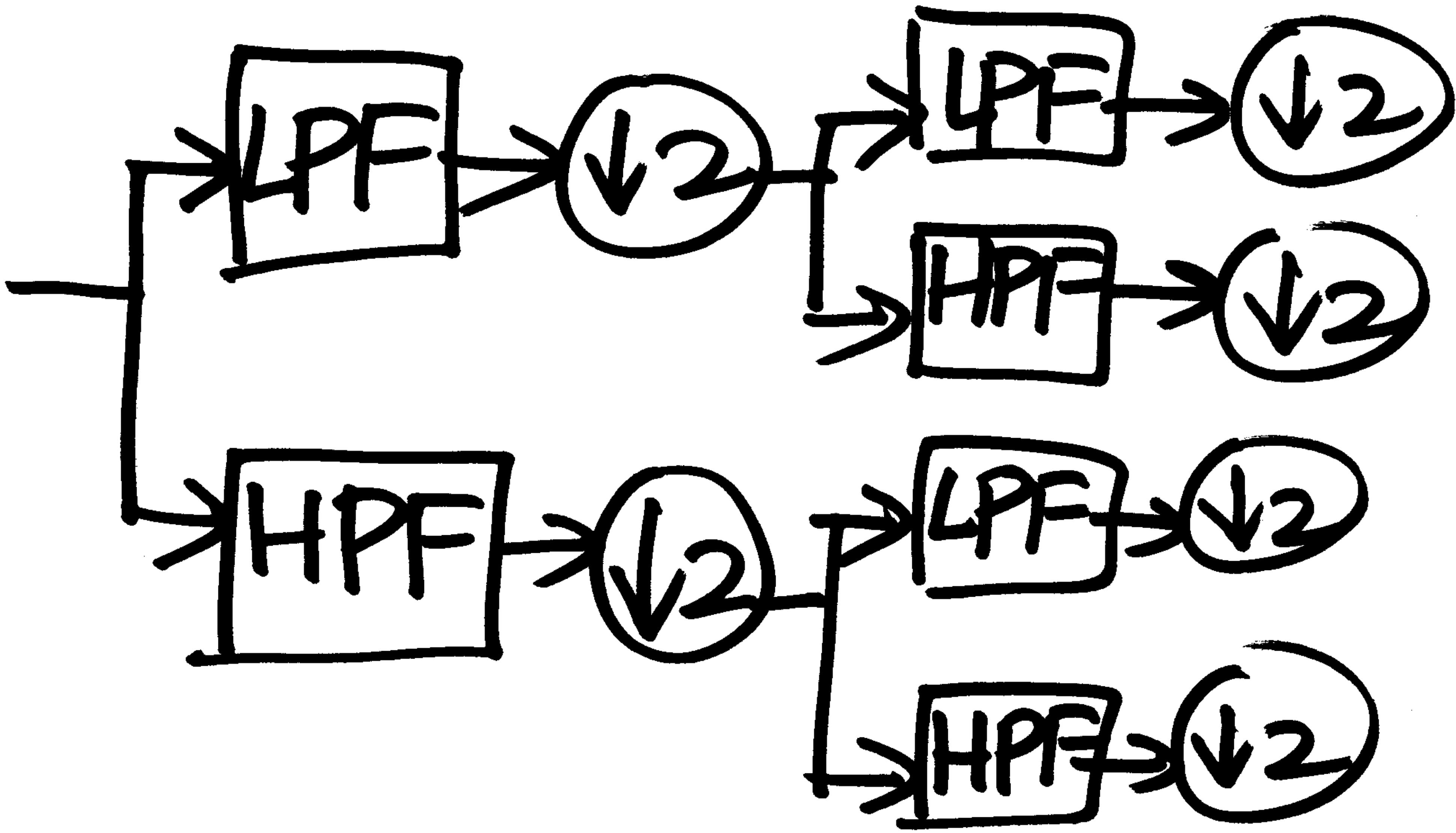
JPEG 2000
employs
Biorthogonal
filter banks.

2. Do filters in filter bank <u>have</u> to be finite impulse response?

3. Must we always iterate on the lowpass branch ?

Discrete wavelet transform



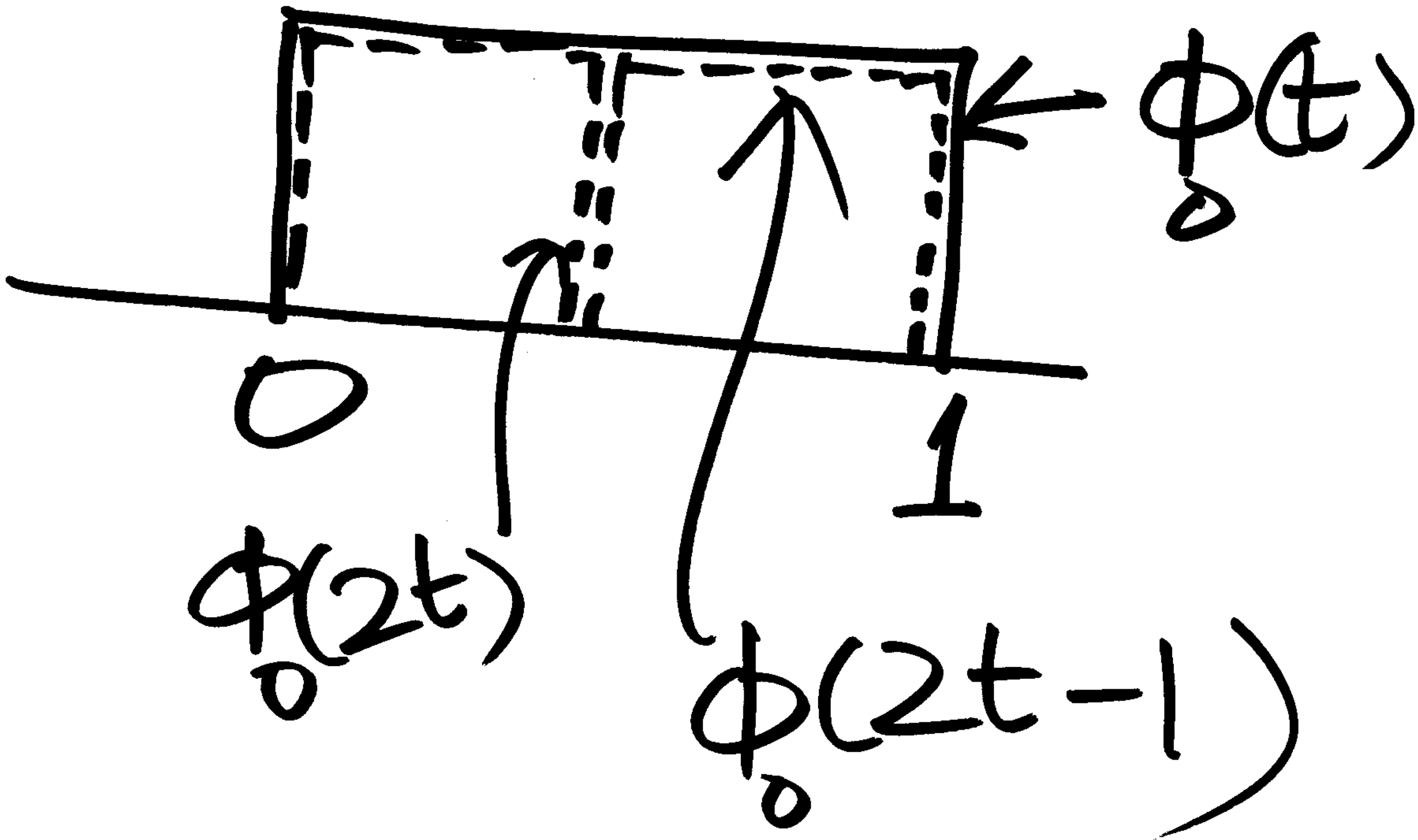

LPF
↓2
HPF
↓2
LPF
↓2
HPF
↓2
LPF
↓2
HPF
↓2

Wave packet transform: iterating on highpass branch as well.

Haar case:

Dilation equation for $\phi_0(t)$:

$$\phi_0(t) = \phi_0(2t) + \phi_0(2t-1)$$



$\phi_0(t)$
0
1
$\phi_0(2t)$
$\phi_0(2t-1)$

$$\phi_0(t) = \phi_0(2t) + \phi_0(2t-1)$$

$$*$$

$$\phi_0(t) = \phi_0(2t) + \phi_0(2t-1)$$

Given

$$h(t) * g(t) = r(t)$$

what is

$$h(a \cdot t + b) * g(a \cdot t + c) ?$$

Same scaling

$a \in \mathbb{R}$ (real)

note

Indeed

$$h(at+b) * g(at+c)$$

$$= \int_{-\infty}^{+\infty} h(a\lambda+b) \cdot g(a(t-\lambda)+c)\, d\lambda$$

Let $a\lambda + b = \gamma$

$a \neq 0, \; a \in \mathbb{R}$

$a > 0$

$$d\gamma = a\, d\lambda$$

$\lambda : -\infty \to +\infty \Rightarrow \gamma : -\infty \to +\infty.$

When $a < 0$

$$d\gamma = a\, d\lambda$$

$$\lambda : -\infty \longrightarrow +\infty$$

$$\Rightarrow \gamma : +\infty \longrightarrow -\infty$$

In general

$$\int_{-\infty}^{+\infty} h(a\lambda+b)\, g(a(t-\lambda)+c)\, d\lambda$$

$$= \ldots\ldots$$

$$= \frac{1}{|a|} \int_{-\infty}^{+\infty} h(\gamma)\, g(\gamma + at + b + c)\, d\gamma$$

$$at - a\lambda + c$$

$$= at - \gamma + b + c$$

$$= \frac{1}{|a|} \int_{-\infty}^{+\infty} \cdots$$

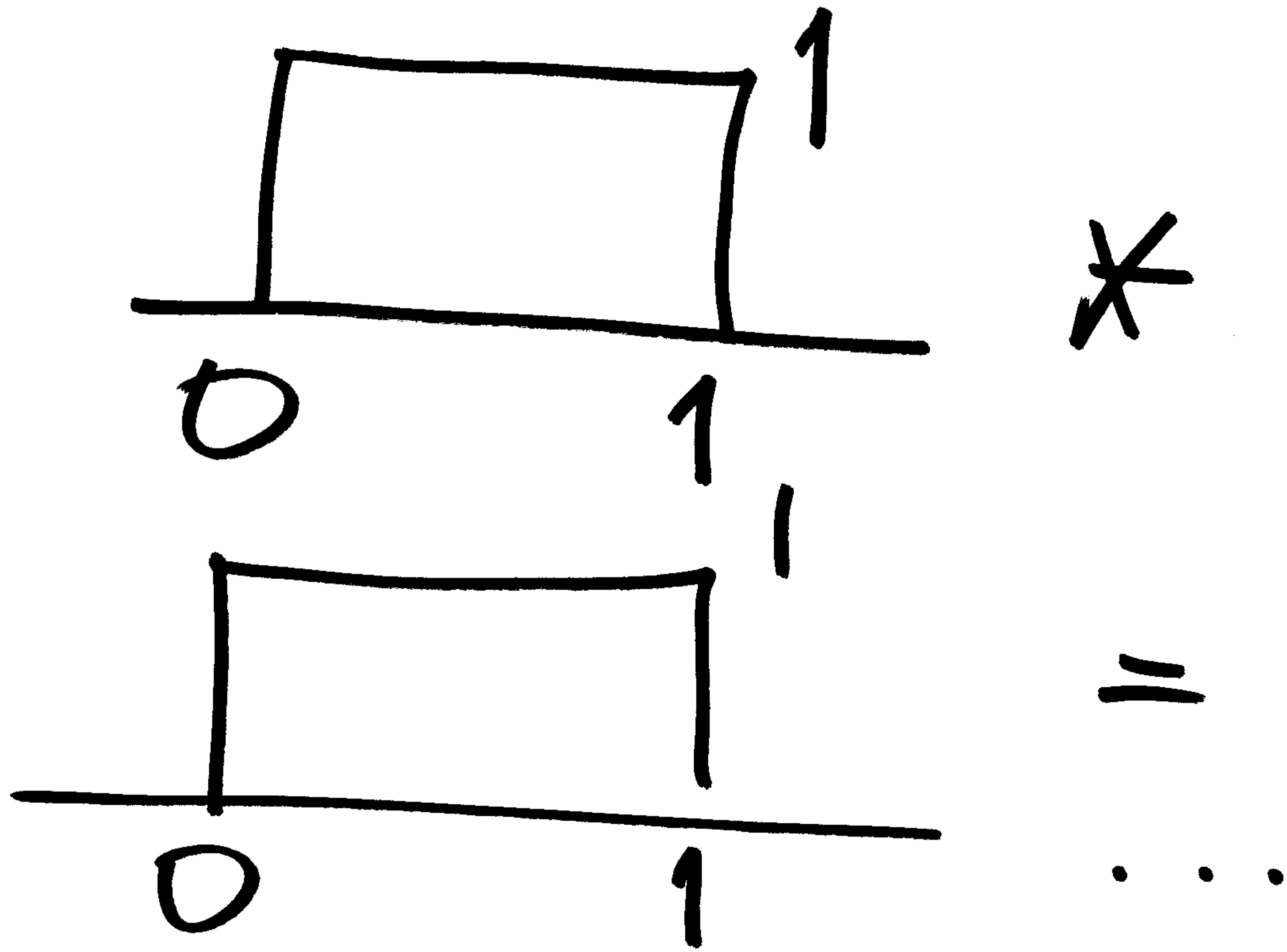
$$\cdots h(\gamma)\, g\!\left(at + b + c - \gamma\right) d\gamma$$

$$h(at+b) * g(at+c) = \frac{1}{|a|} h*g\Big|_{at+b+c}$$

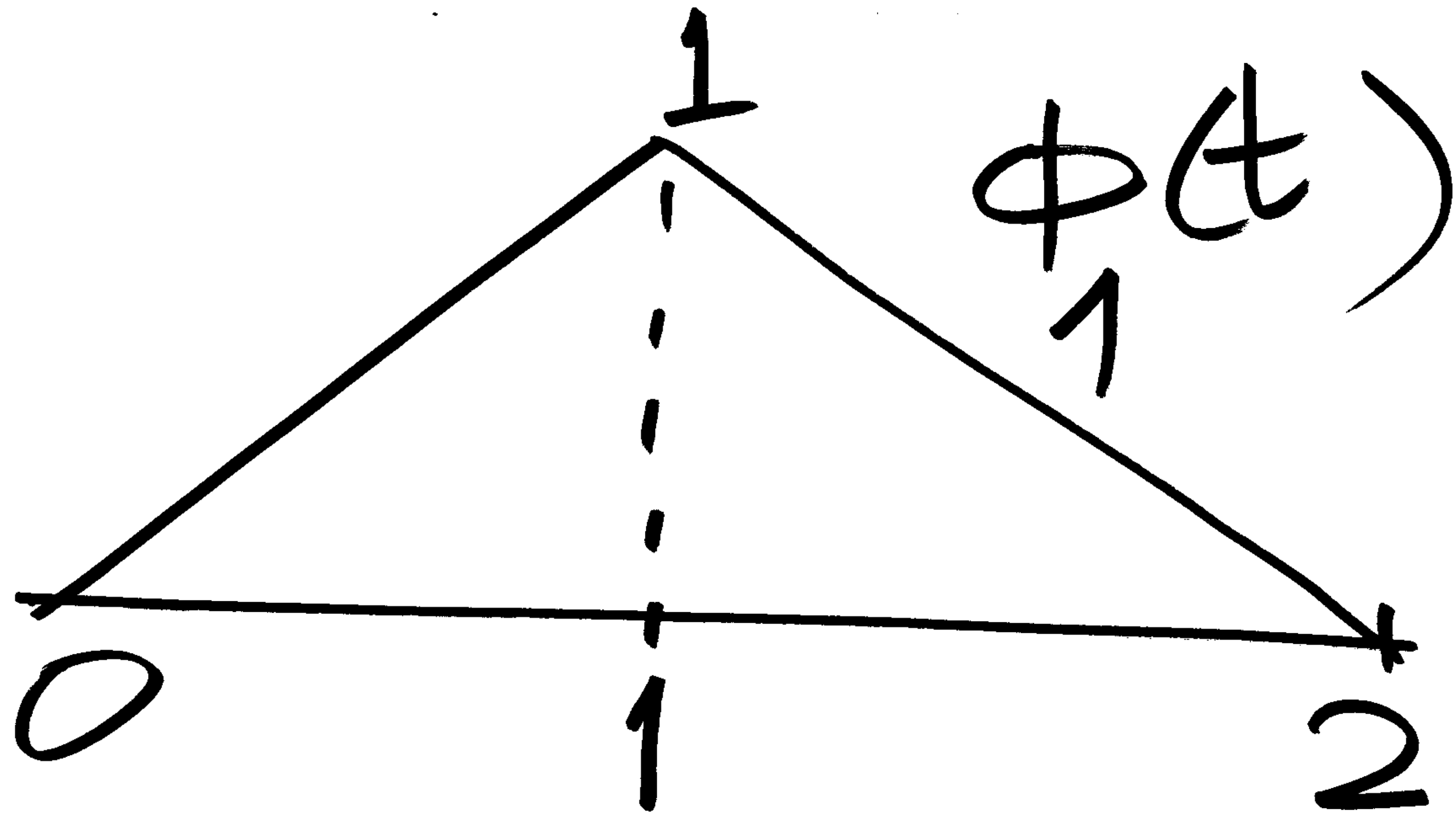
Using this :

Denoting

$$\phi_0(t) * \phi_0(t) = \phi_1(t)$$



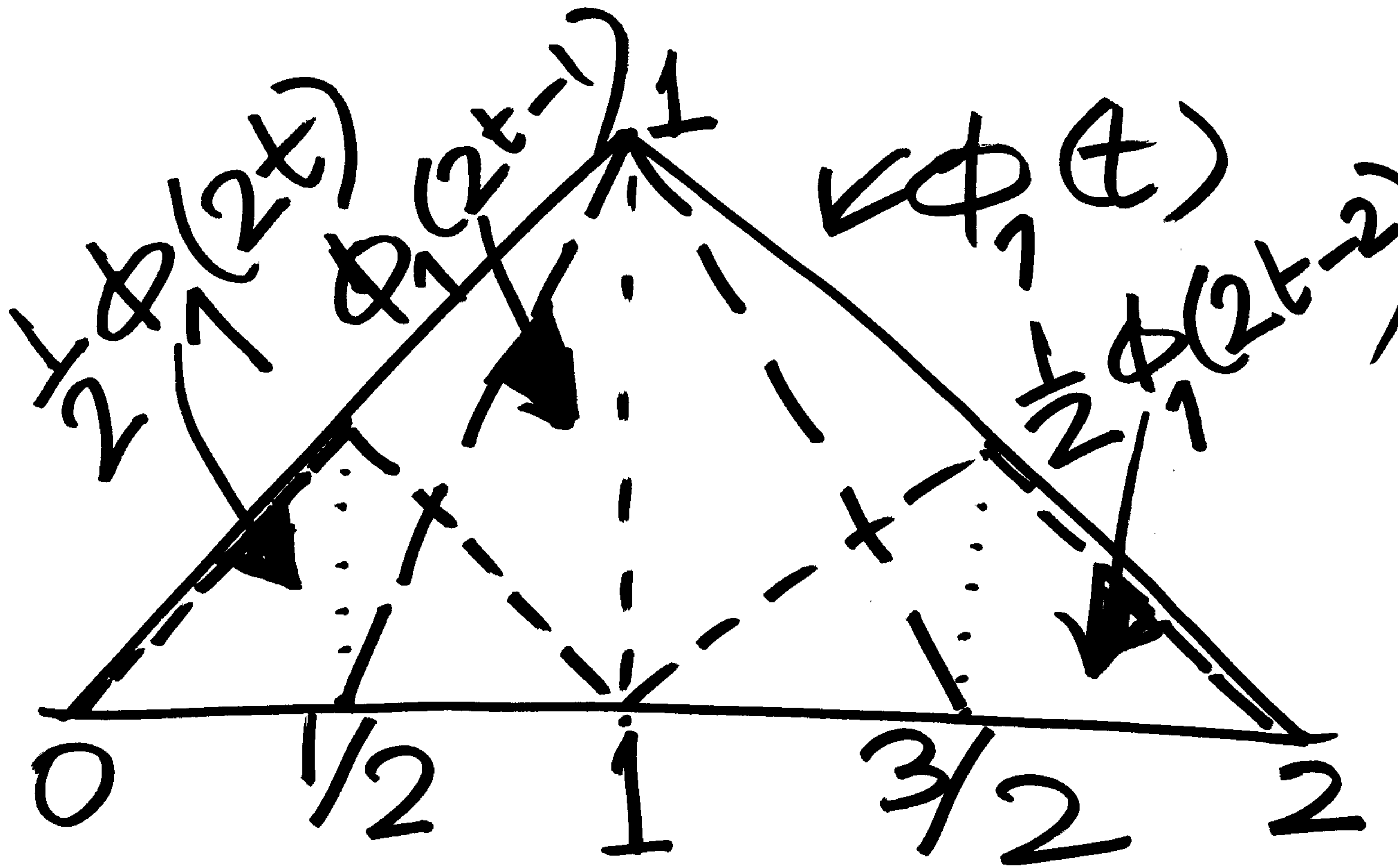
1
0
1
*
1
0
1
=
...



1
$\phi_1(t)$
0
1
2

$$\phi_1(t) = \frac{1}{2}\left\{\phi_1(t) + 2\phi_1(2t-1) + \phi_1(2t-2)\right\}$$

$$\phi_0(2t) * \phi_0(2t-1)$$
$$= \phi_0(2t-1) * \phi_0(2t)$$
$$= \frac{1}{2}\phi_1(2t-1).$$



$\frac{1}{2}\phi_1(2t)$
$\phi_1(2t-1)$
1
$\phi_1(t)$
$\frac{1}{2}\phi_1(2t-2)$
0
$1/2$
1
$3/2$
2

Coefficients in dilation equation

$$\phi_1(t) = \frac{1}{2}\phi_1(2t) + \phi_1(2t-1) + \frac{1}{2}\phi_1(2t-2)$$

$\frac{1}{2}$ $\quad 1 \quad$ $\frac{1}{2}$

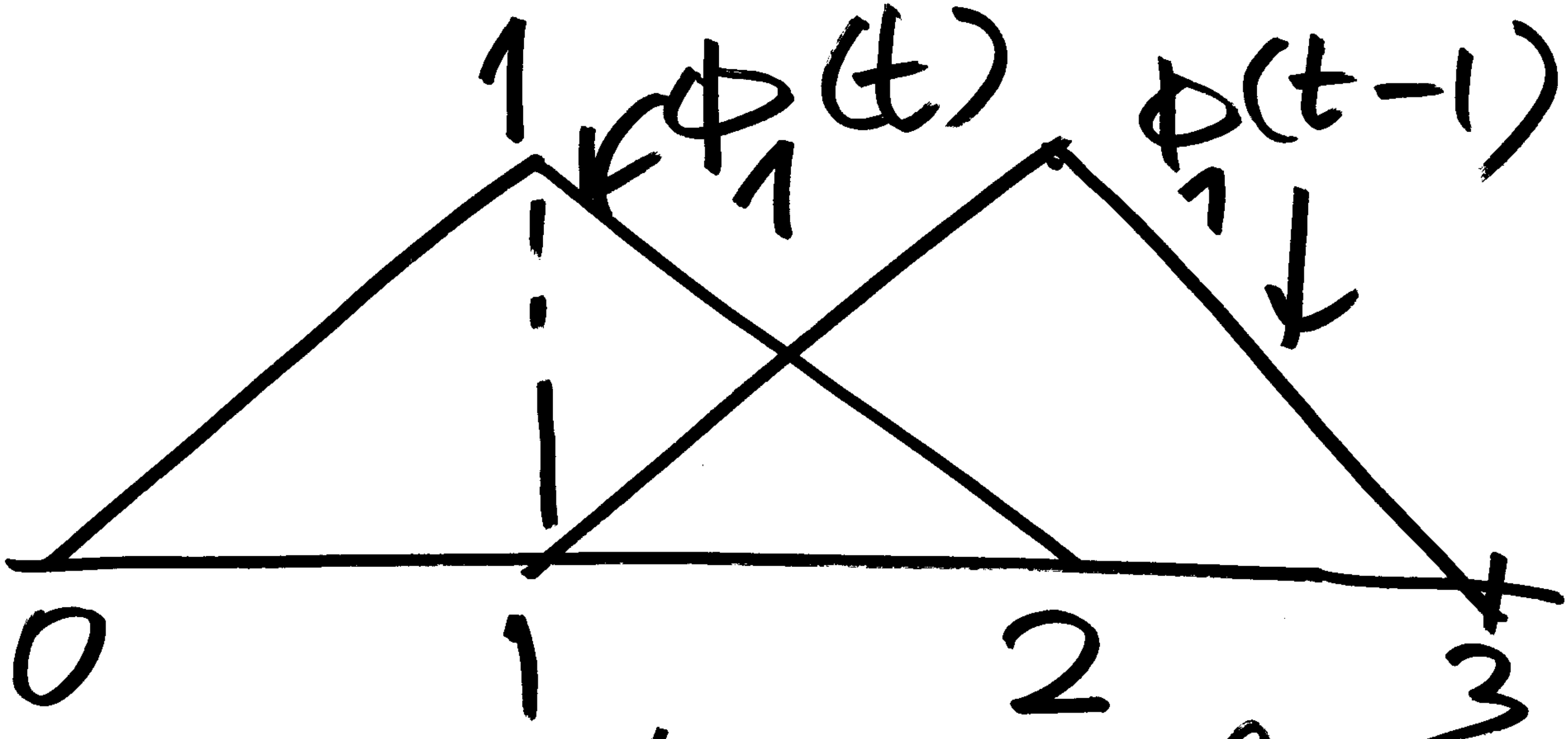
$\uparrow$

0

Corresponding filter

$$= \frac{1}{2} + 1 \cdot z^{-1} + \frac{1}{2} z^{-2}$$

$$= \frac{1}{2}\left(1+\bar{z}'\right)^{2} -$$



<u>Not</u> orthogonal.