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Lec-26
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LECTURE 26

PROOF OF THE THEOREM OF (DYADIC) MULTIRESOLUTION ANALYSIS

Look at a typical function $f(t)$ in the incremental subspace W_0

Characteristics of $f(t)$

1. $f(t)$ orthogonal to every $\phi(t-m)$ $m \in \mathbb{Z}$

2. $f(t) \in V_1$

$f(t)$ can be expanded in terms of $\phi(2t-n)$
$n \in \mathbb{Z}$

Let

$$f(t) = \sum_{n=-\infty}^{+\infty} f[n]\,\phi(2t-n)$$

coefficients of expansion

$\phi(t) \in V_0 \subset V_1$

$\Rightarrow \phi(t)$ can be expanded in terms of $\phi(2t-n)$
$n \in \mathbb{Z}$

$$\phi(t) = \sum_{n=-\infty}^{+\infty} h[n]\,\phi(2t-n)$$

lowpass filter impulse response

$$\phi(t-m)$$

$$= \sum_{n=-\infty}^{+\infty} h[n]\,\phi(2t-2m-n)$$

$$\langle f(t), \phi(t-m) \rangle = 0$$

$$\Rightarrow \left\langle \sum_n f[n]\,\phi(2t-n),\ \sum_l h[l]\,\phi(2t-2m-l) \right\rangle = 0$$

$\forall \underline{m}$.

$$\langle \phi(2t-k_1), \phi(2t-k_2) \rangle$$

$$= \int_{-\infty}^{+\infty} \phi(2t-k_1)\overline{\phi(2t-k_2)}\, dt$$

$$2t = \lambda.$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \phi(\lambda - k_1)\overline{\phi(\lambda - k_2)}\, d\lambda$$

$$= \frac{1}{2}\,\delta[k_1 - k_2]$$

from orthogonality of $\phi(t-m)$

Thus $\langle f(t), \phi(t-m) \rangle$

$$= \sum_n \sum_\ell \cancel{4}\, f[n]\, \overline{h[\ell]} \cdots$$

$$\langle \phi(2t-n), \phi(2t-2m-\ell) \rangle$$

Only $n = 2m + \ell$

"Survives"

$$= \sum_{\ell} f[2m+\ell]\,\overline{h[\ell]} \cdot \frac{1}{2}$$

We are essentially looking at the cross-correlation of sequences $f[\cdot]$ and $h[\cdot]$

$$\Upsilon_{f,h}[p] = \sum_{\ell} f[p+\ell]\,\overline{h[\ell]}$$

z transform ?

Exercise: Prove that z-transform of $r_{fh}[p]$, $p \in \mathbb{Z}$ is $F(z)H(z^{-1})$ where ...

$F(z) =$ z-transform of $f[n]$

$H(z) =$ z-transform of $h[n]$ (also lowpass system function)

$$r_{fh}[p]\Big|_{p=2m} = 0$$

$$\forall\, m \in \mathbb{Z}$$

The cross-correlation of f and h evaluated at all even shifts $= 0$

What we are saying is:

$$r_{fh}[p] \rightarrow \boxed{\downarrow 2} \rightarrow \text{zero sequence}$$

In the z-domain:

$$
\begin{tikzcd}
r_{fh}[p] \arrow[r] & \downarrow 2 \arrow[r] & \uparrow 2 \arrow[d] \\
z\text{ transform} = 0. & & \text{all zero sequence} \arrow[ll]
\end{tikzcd}
$$

$$r_{fh}[p] \xrightarrow{z\text{-transf}} R_{fh}(z)$$

Then

$$r_{fh}[p] \longrightarrow \boxed{\downarrow 2} \longrightarrow \boxed{\uparrow 2} \longrightarrow$$

z-transf

$$\frac{1}{2}\left\{R_{fh}(z) + R_{fh}(-z)\right\}$$

$$R_{fh}(z) + R_{fh}(-z) = 0.$$

$$F(z)H(z^{-1}) + F(-z)H(-z^{-1}) = 0$$

$$\Rightarrow \underbrace{\frac{F(z)}{F(-z)} = \frac{-H(-z^{-1})}{H(z^{-1})}}$$

$F(z)$ must be of the form

$$-\underbrace{\Lambda(z)H(-\bar{z}')}_{\text{'cancelled' factor}}$$

The "citizen specific" information for $f(t)$ is contained in $\Lambda(z)$.

By equating denominators we also have

$$F(z) = \Lambda(z) H(\bar{z}')$$

$$F(z) = -\Lambda(z) H(-\bar{z}')$$

$$F(-z) = \Lambda(z) H(\bar{z}')$$

compare

$$z \leftarrow -z$$

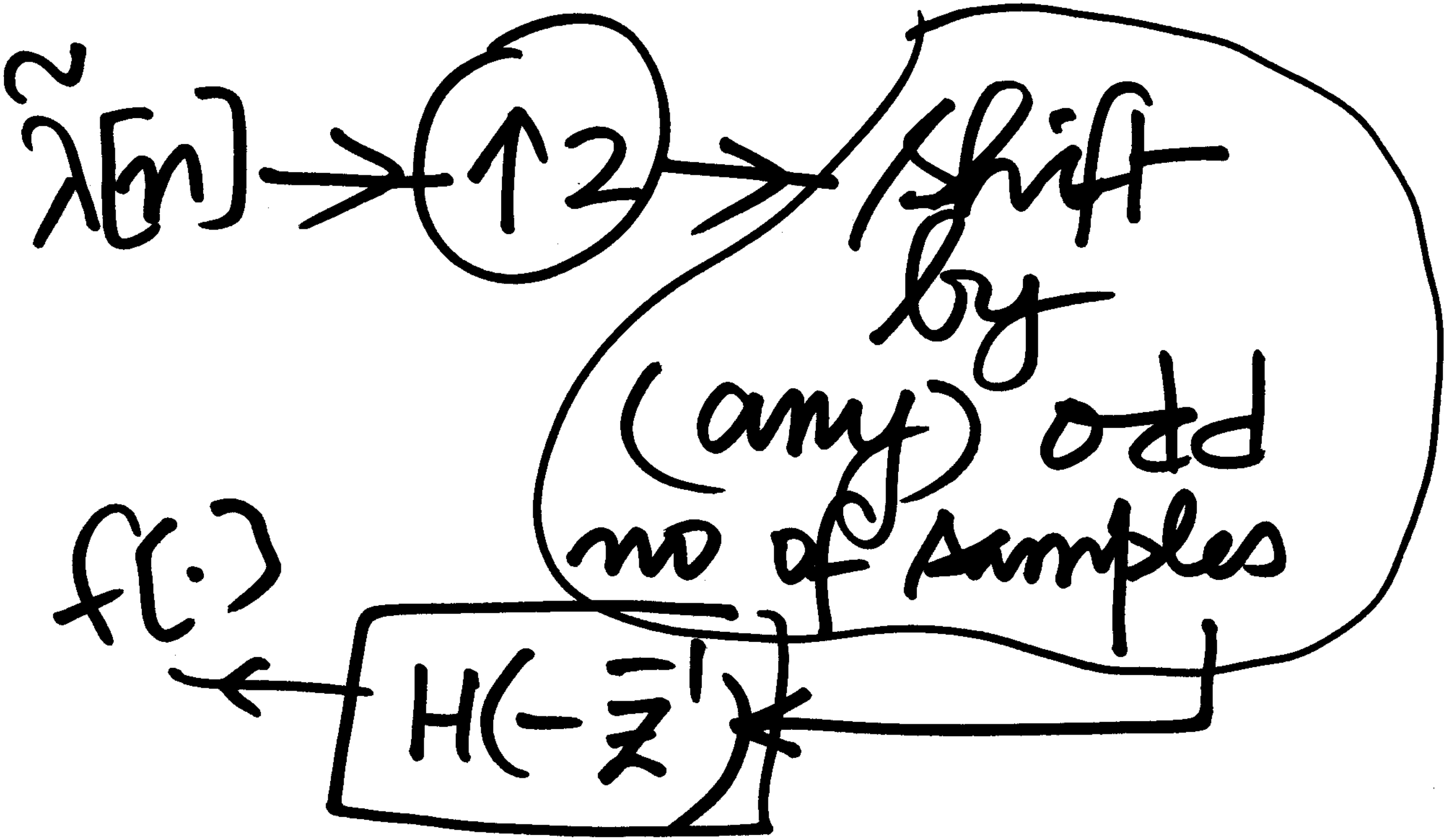
$$F(-z) = -\Lambda(-z) H(\bar{z}')$$

We must have on comparison

$$-\Lambda(Z) = -\Lambda(-Z)$$

$$\Lambda(z) + \Lambda(-z) = 0.$$

$$\tilde{\lambda}[n] \longrightarrow \uparrow 2 \longrightarrow \text{shift by odd samples} \xrightarrow{z\ \text{domain}} \Lambda(z)$$



λ̃[n]
↑2
Shift by (any) odd no of samples
H(−z⁻¹)
f[·]

We could in particular choose odd no of samples

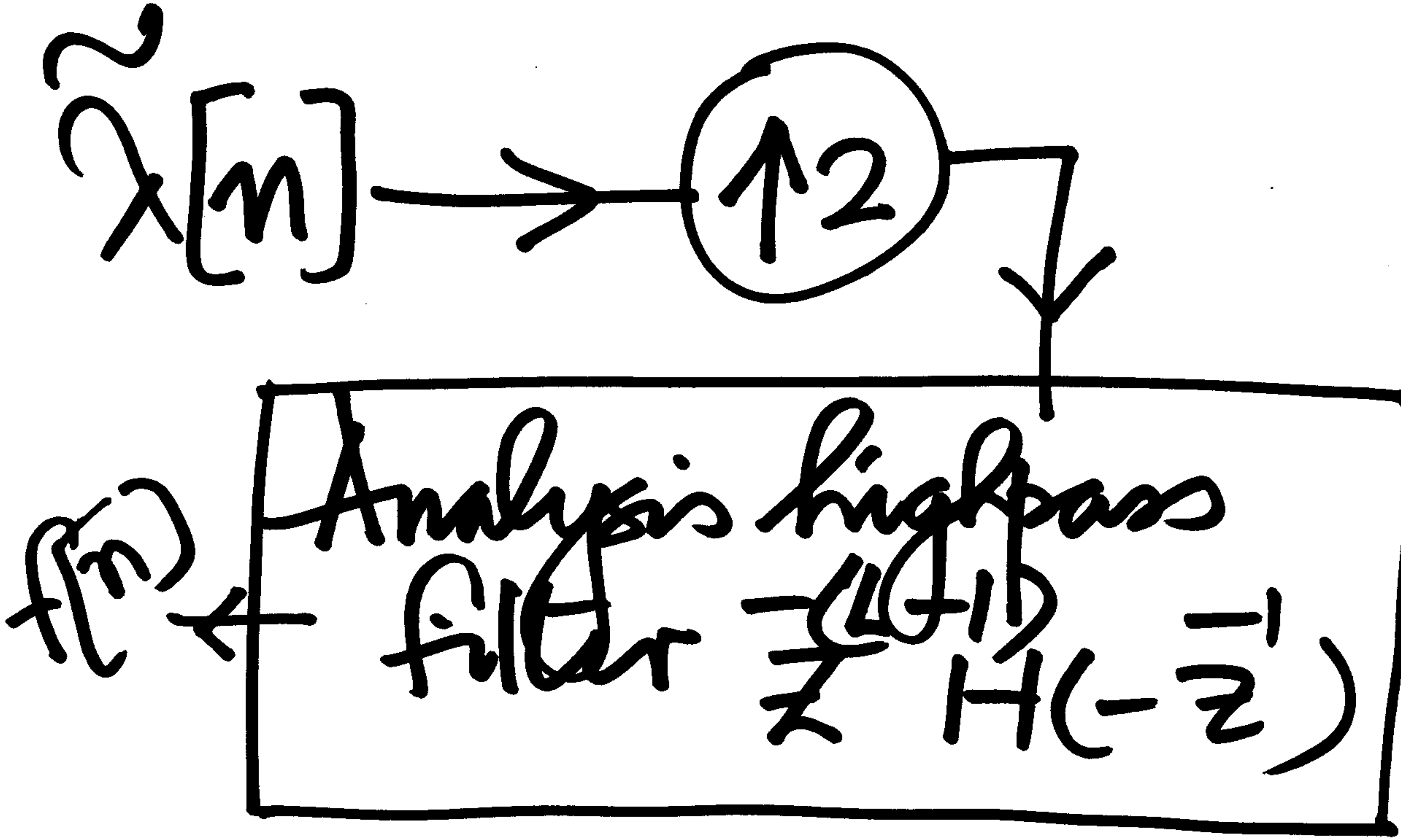
$$= L - 1$$

L = LOWPASS FILTER LENGTH

Recall that

$$z^{-(L-1)} H(-z^{-1})$$

Analysis highpass filter



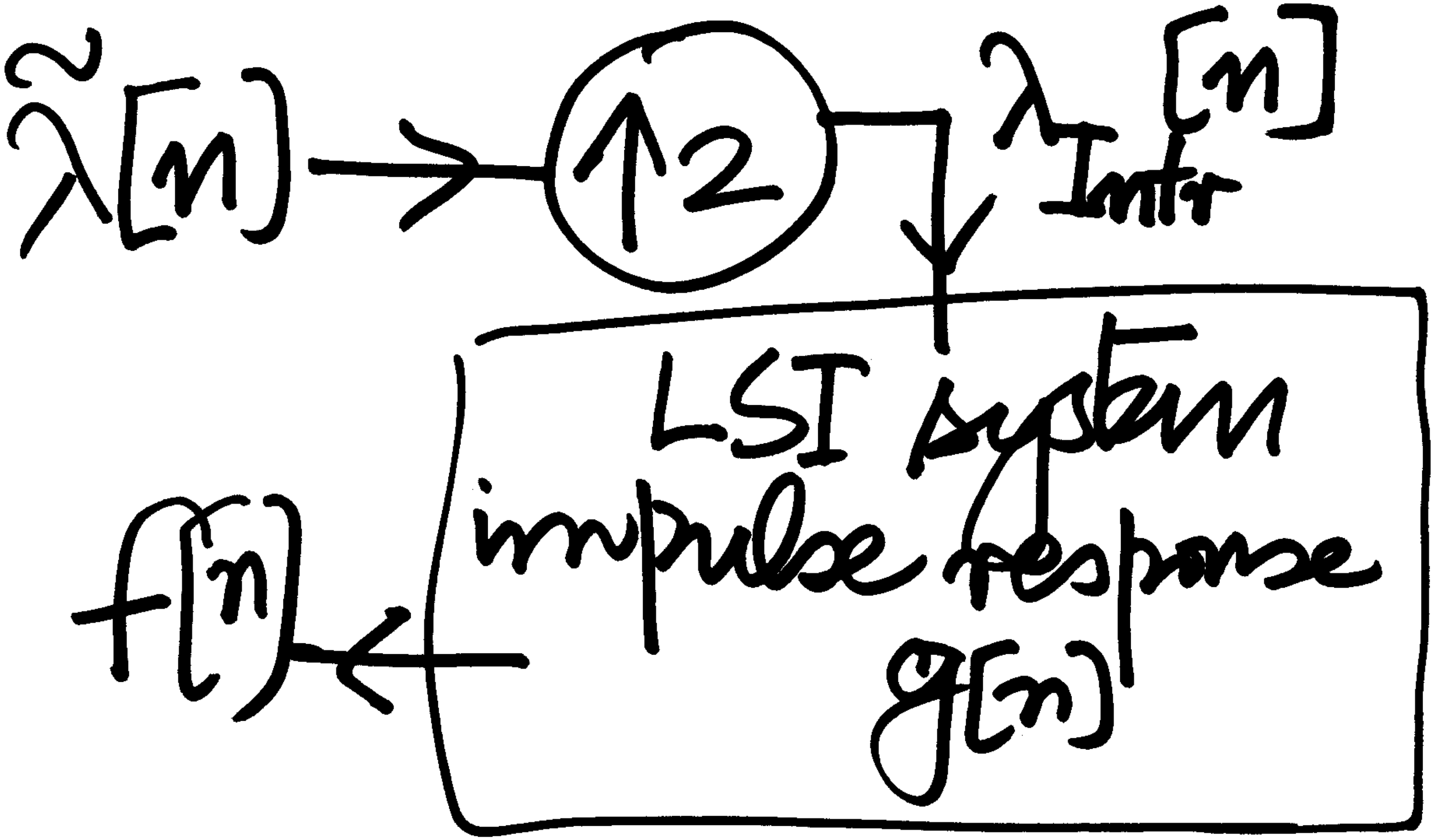
$\tilde{x}[n]$
$\uparrow 2$
Analysis highpass filter
$z^{-(L-1)} H(-z^{-1})$
$f[n]$

$$f(t) = \sum_{n} f[n]\,\phi(2t-n)$$

Let $g[n]$ be the inverse z-transform of $z^{-(L-1)} H(-z^{-1})$

Essentially

$g[n]$: impulse response of analysis highpass filter


$\tilde{\lambda}[n]$
$\uparrow 2$
$\lambda_{Intr}[n]$
LSI system
impulse response
$g[n]$
$f[n]$

$f[n] = \lambda_{\text{Intr}}[n]$ Convolved with $g[n]$

$$= \sum_{k=-\infty}^{+\infty} \lambda_{Intr}[k]\, g[n-k]$$

$\lambda_{Intr}[k]$ nonzero only at $2k$ (even)

$$= \sum_{k=-\infty}^{+\infty} \lambda_{\mathrm{Intr}}[2k]\, g[n-2k]$$

$$= \sum_{k=-\infty}^{+\infty} \tilde{\lambda}[k]\, g[n-2k]$$

$$f[n] = \sum_{k=-\infty}^{+\infty} \tilde{\lambda}[k]\, g[n-2k]$$

Write $f(t)$ ...

$$f(t) = \sum_{n=-\infty}^{+\infty} \cdots \underbrace{\sum_{k=-\infty}^{+\infty} \tilde{\lambda}[k] \cdot g[n-2k]}_{f[n]} \cdots \phi(2t-n)$$

Interchanging $\sum_n \sum_k$:

$$\sum_{k=-\infty}^{+\infty} \tilde{\lambda}[k] \left\{ \cdots \right\}$$

$$\{ \cdots \} = \sum_{n=-\infty}^{+\infty} g[n-2k]\, \phi(2t-n).$$

$$n-2k = q$$

$$= \sum_{q=-\infty}^{+\infty} g[q]\, \phi(2t - q - 2k)$$

$$= \sum_{q} \underbrace{g[q]\, \phi(2(t-k)-q)}$$

Define $\psi(\cdot) \in V_1$.

$$\psi(t) = \sum_{q \in \mathbb{Z}} g[q]\,\phi(2t-q)$$

We effectively have:

$$f(t) = \sum_{k=-\infty}^{+\infty} \tilde{\lambda}[k]\,\psi(t-k)$$

We have shown that $\{\psi(t-k)\}_{k \in \mathbb{Z}}$ spans W_0.

The proof is almost
Complete, except
to
demonstrate:

(i) $\{\psi(t-k)\}_{k \in \mathbb{Z}}$ form an <u>orthogonal</u> set

(iii) $\langle \psi(t-k), \phi(t-m) \rangle = 0 \quad \forall k, m \in \mathbb{Z}$