LECTURE 25

THE THEOREM OF (DYADIC) MULTIRESOLUTION ANALYSIS

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Date: 26/2/10

Filter banks with <u>different</u> analysis and synthesis wavelets & scaling functions

: BIORTHOGONAL FILTER BANKS

We are <u>always</u> talking about filter banks with <u>PERFECT RECONSTRUCTION.</u>

<u>Same</u> analysis and synthesis wavelets / scaling functions: <u>ORTHOGONAL</u> FILTER BANKS

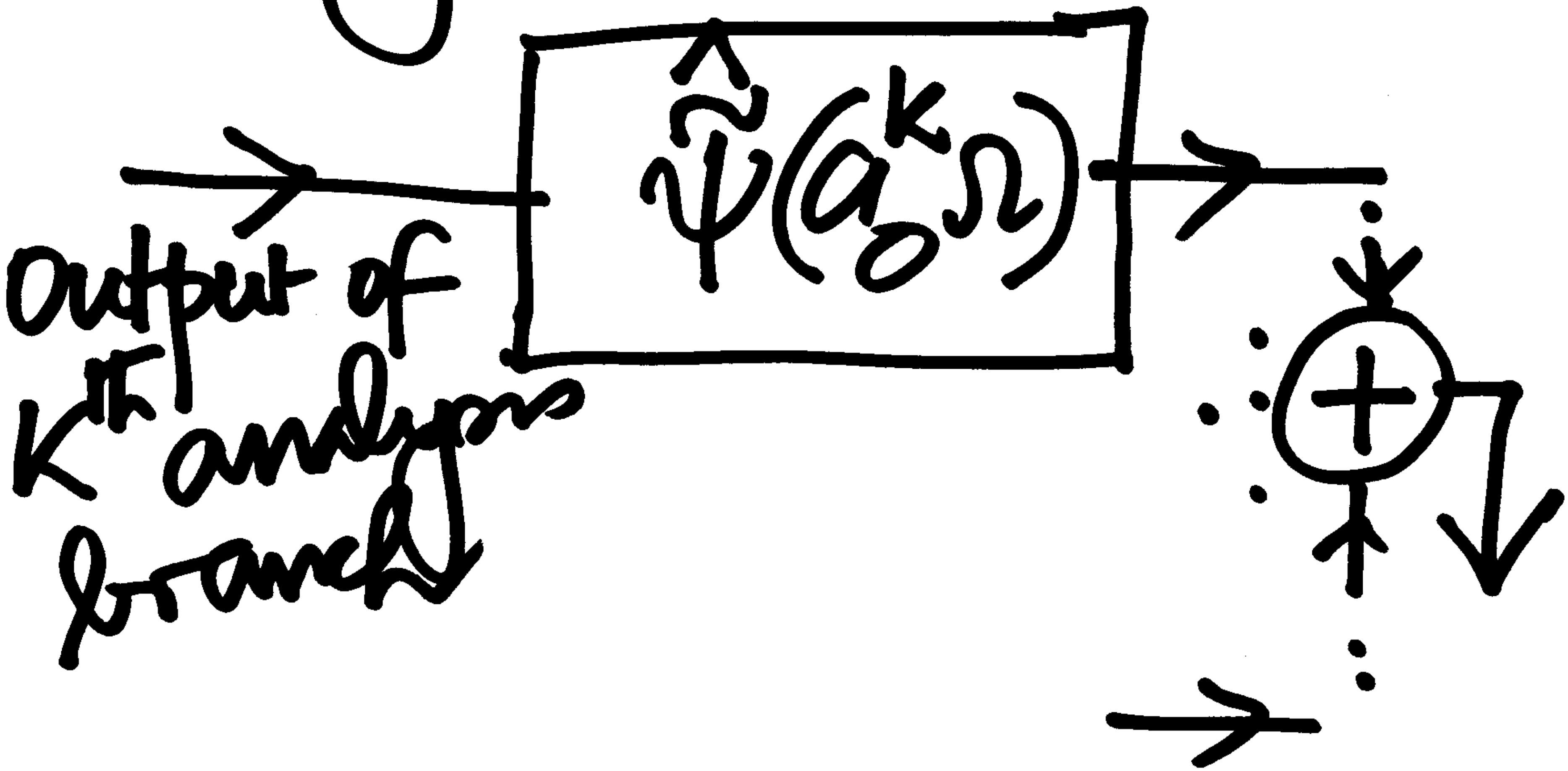
In general

k^{th} analysis branch

Input $x \longrightarrow \boxed{\overline{\hat{\Psi}(a_0^k \Omega)}} \longrightarrow$

$a_0 > 1$, k: all integers

K^{th} synthesis branch:



$$\hat{\tilde{\psi}}(\Omega) = \frac{\hat{\psi}(\Omega)}{SDS(\psi, a_0)(\Omega)}$$

SDS = sum of dilated spectra

$$SDS(\psi, a_0)(\Omega) = \sum_{k=-\infty}^{+\infty} \left| \hat{\psi}(a_0^k \Omega) \right|^2$$

Provided, $\exists\ C_1, C_2$

there exist

$$0 < C_1 \leq SDS(\psi, q_0)(\Omega) \leq C_2 < \infty$$

$\psi(\cdot)$ is admissible

Because of c_1,

$\hat{\psi}$ was meaningful!

$\tilde{\psi}$ was admissible,

Bounds on SDS:

$\frac{1}{c_2}, \frac{1}{c_1}$.

Challenge/exercise:
Come up with examples of $\hat{\psi}$ which satisfy the requirement with $c_1 = c_2$!.

Construction of an orthogonal filter bank:

Define:

$$\hat{\tilde{\psi}}(\Omega) = \frac{\hat{\psi}(\Omega)}{+\sqrt{SDS(\psi, a_0)(\Omega)}}$$

$$0 < C_1^{1/2} \leq \sqrt[+]{SDS(\psi, a_0)(\Omega)} \leq C_2^{1/2} < \infty$$

+ve square root

$\tilde{\Psi}$ is admissible.

Consider

$SDS(\tilde{\Psi}, a_0)(\Omega)$

$$SDS(\vec{\psi}, a_0)(\Omega) = \sum_{k=-\infty}^{+\infty} \frac{|\hat{\psi}(a_0^k \Omega)|^2}{SDS(\psi, a_0)(\Omega)}$$

$$\mathrm{SDS}(\psi, a_0)(\Omega)$$

$$= \mathrm{SDS}(\psi, a_0)(a_0^m \Omega)$$

Proof in general:

$$SDS(\psi, a_0)(a_0^m \Omega)$$

$$= \sum_{k=-\infty}^{+\infty} \left| \hat{\psi}(a_0^k a_0^m \Omega) \right|^2$$

$$\equiv \sum_{k=-\infty}^{+\infty} \left| \hat{\psi}(a_0^k \Omega) \right|^2$$

Proved.

$$SDS(\hat{\tilde{\psi}}, a_0)(\Omega) = \frac{\sum_{k=-\infty}^{+\infty} |\hat{\psi}(a_0^k \Omega)|^2}{SDS(\psi, a_0)(\Omega)} = 1$$

for all Ω

$\tilde{\psi}$ is admissible on account of satisfying $\frac{\text{upper}}{\text{bound on SDS}} = \text{lower} = 1.$

$\tilde{\psi}$ can be used both on the analysis and synthesis side.

Dyadic Multiresolution Analysis (MRA):
examples: Haar MRA
Daubechies' MRAs
and so on

Special case:

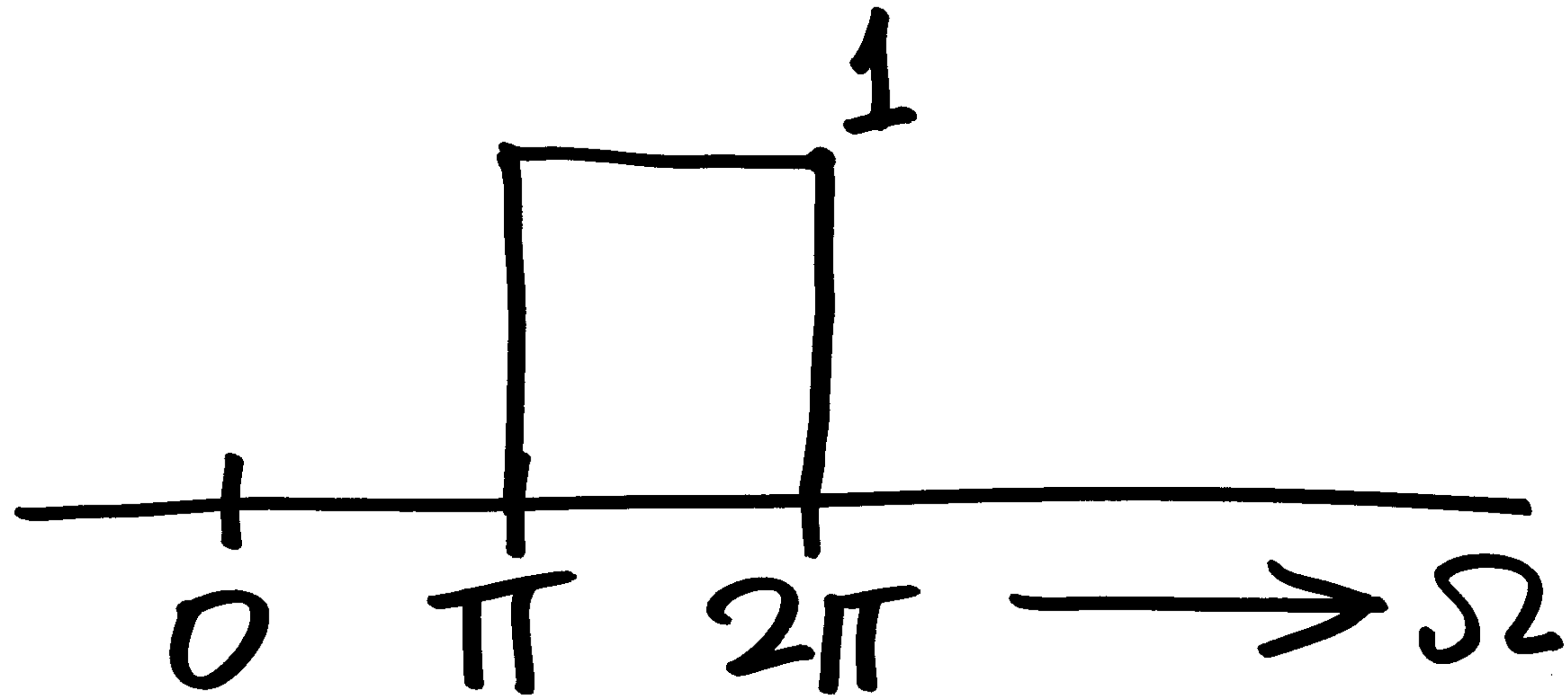
$$a_0 = 2$$

wavelet obeys the requirement ($a_0 = 2$)

$$0 < C_1 \leq SDS \leq C_2 < \infty \quad \forall \Omega$$

and the wavelet admits discretizing the translation parameter

On the k^{th} analysis branch: output is broadly a BANDPASS FUNCTION.


1
0
π
2π
Ω

Band occupancy

$= \pi$

We could use a

sampling rate $= \frac{2\pi}{\pi} = 2$

If you simply use the Nyquist criterion,

f_s = sampling freq

$2\pi f_s = 4\pi, \; f_s = 2$

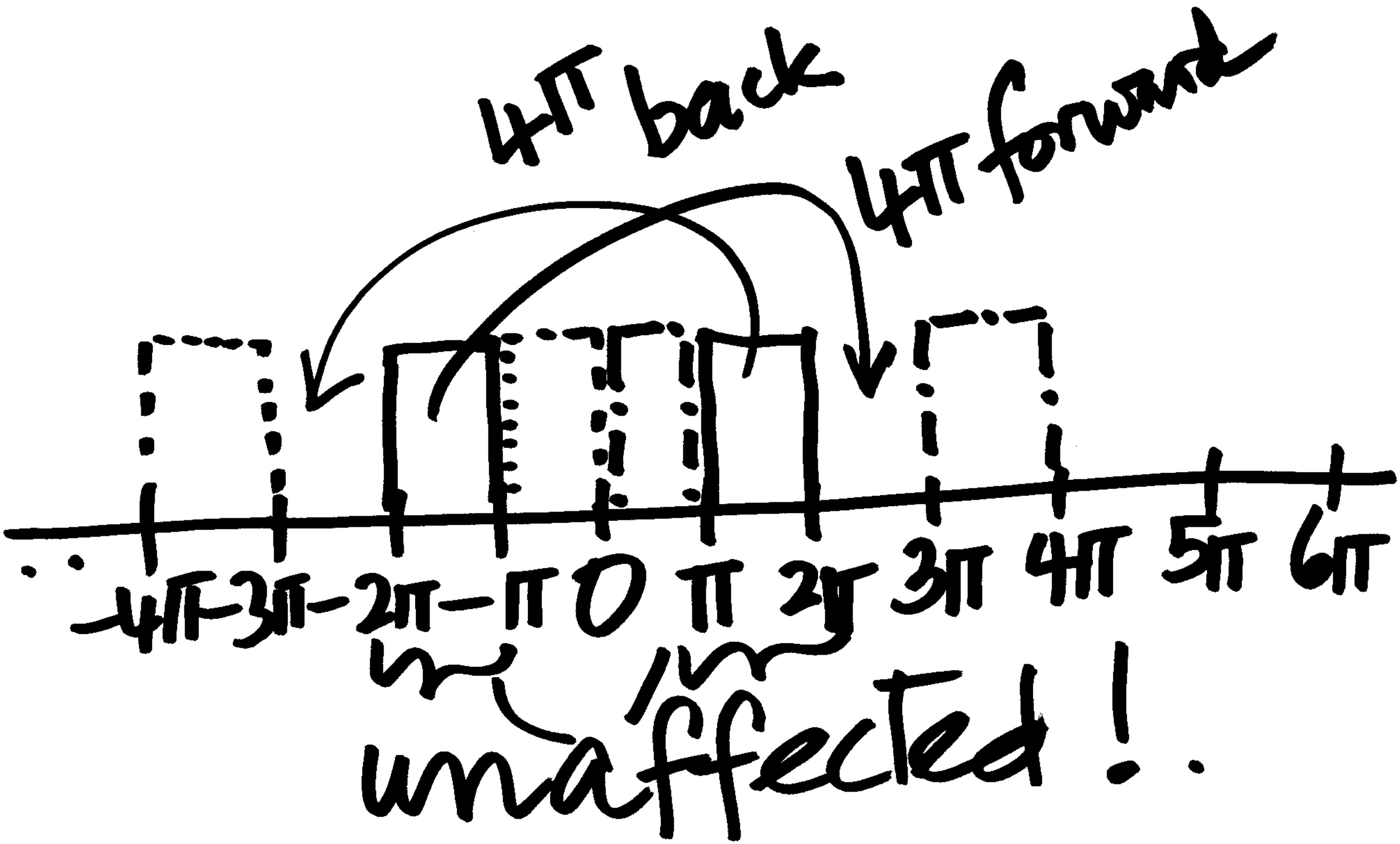
But we can also do with a sampling rate

$$2\pi f_s = 2 \times \underline{\underline{\pi}} = 2\pi$$

$$\Rightarrow \underline{\underline{f_s = 1}}.$$

Suppose we do use a sampling rate $= 1$

Aliases: shifts of original spectrum by $2\pi \times 1 \times k$ $\forall$ integer k



4π back
4π forward
... −4π −3π −2π −π 0 π 2π 3π 4π 5π 6π
unaffected!!

Let us now focus on $\underline{a_0 = 2}$.

We need to use a logarithmic (2^k) change of sampling.

On the k^{th} branch sampling rate relates to 2^k.

Axioms of a (dyadic) MRA:

(i) 'Ladder' axiom

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots$$

(ii) $\overline{\bigcup_{m \in \mathbb{Z}} V_m} = \mathcal{L}_2(\mathbb{R})$

(iii) $\bigcap_{m \in \mathbb{Z}} V_m = \{0\}$

(iv) If $x(t) \in V_0$

$$x(2^m t) \in V_m$$

(Implicitly this provides for logarithmic sampling).

(V) Translation

$$x(t) \in V_0$$

$$x(t-n) \in V_0$$

$$\forall n \in \mathbb{Z}$$

(vi) orthogonal basis

$\exists$ $\phi(t)$ so that

$\{\phi(t-n)\}_{n \in \mathbb{Z}}$ basis for V_0.

Theorem of (Dyadic) MRA:

Given axioms (i) $\rightarrow$ (vi), $\exists$ a $\psi(\cdot) \in V_1$; $\psi(t) \in \mathcal{L}_2(\mathbb{R})$, such that

$$\{\psi(2^m t - n)\}$$ forms

$m \in \mathbb{Z},\ n \in \mathbb{Z}$

an ORTHOGONAL basis for $\mathcal{L}_2(\mathbb{R})$