Prof. V. Gadar

Lec. No. 24

Date: 25/2/

LECTURE 24

LOGARITHMIC SCALE DISCRETIZATION

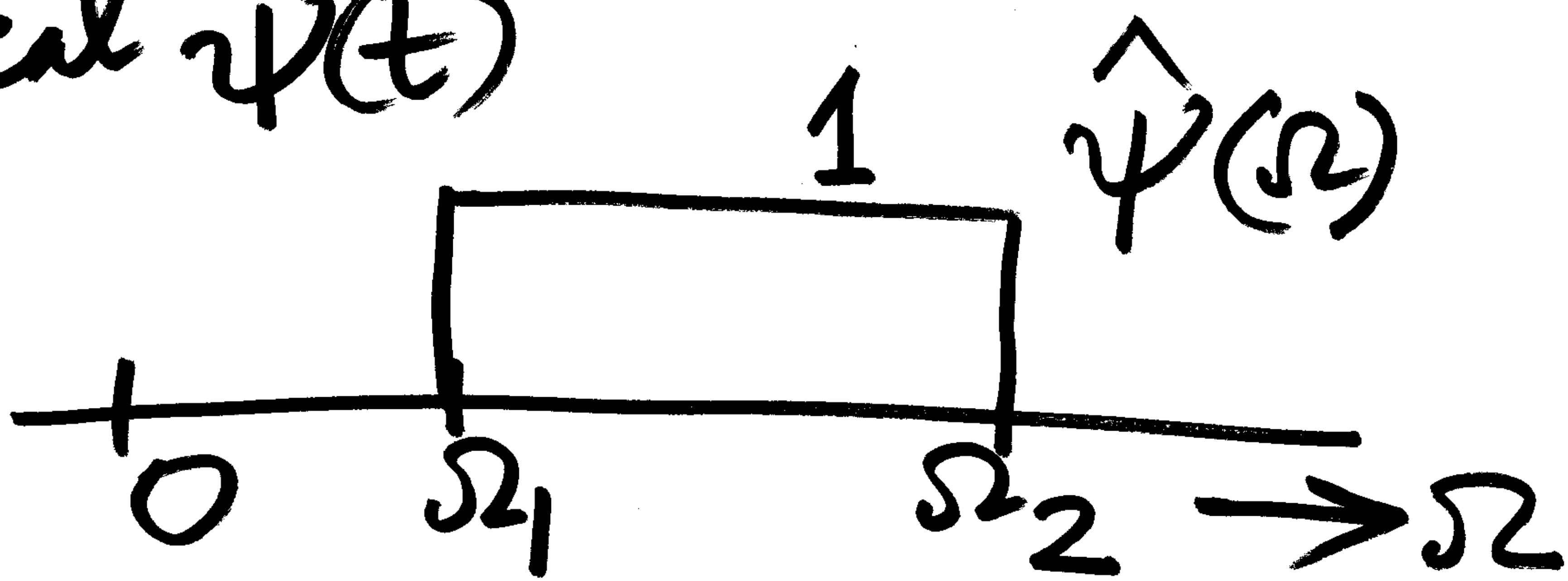
DYADIC DISCRETIZATION

Continuous Wavelet Transform at scale s

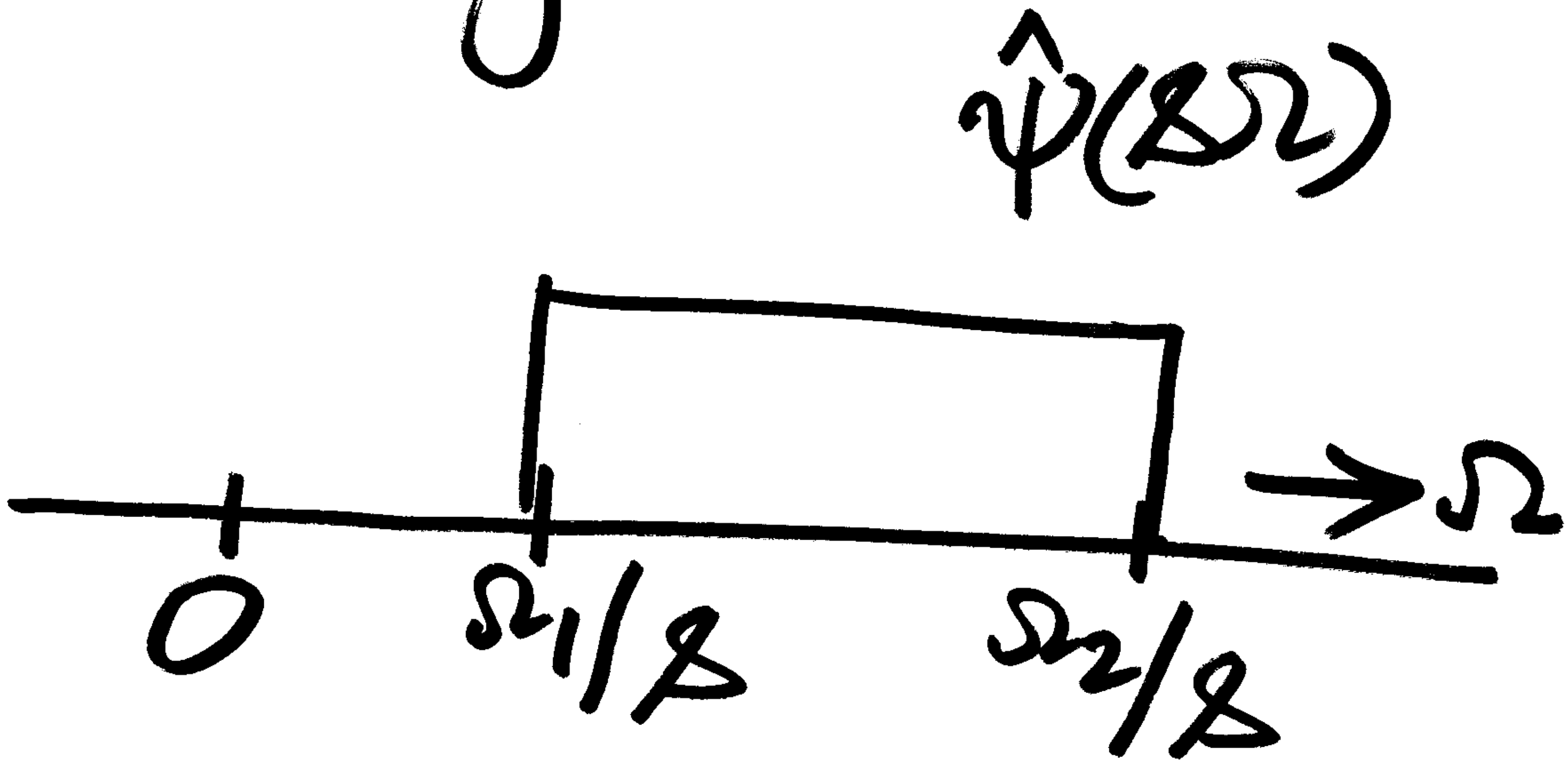
ignore constants $\rightarrow \boxed{\hat{\psi}(s\Omega)} \rightarrow$ translation τ = output independent variable

'Ideal' admissible 'wavelets':

real $\psi(t)$



For any $s > 0$

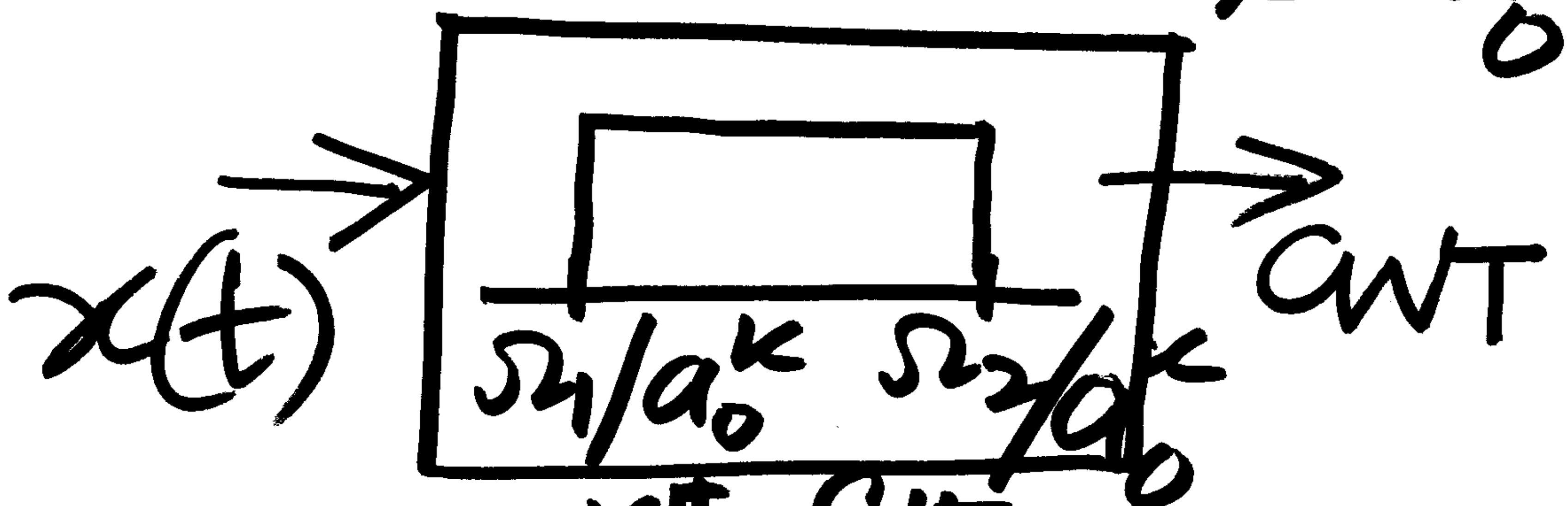


Logarithmic discretization

$\Rightarrow \delta = a_0^{k}, \ \underline{a_0 > 1}$

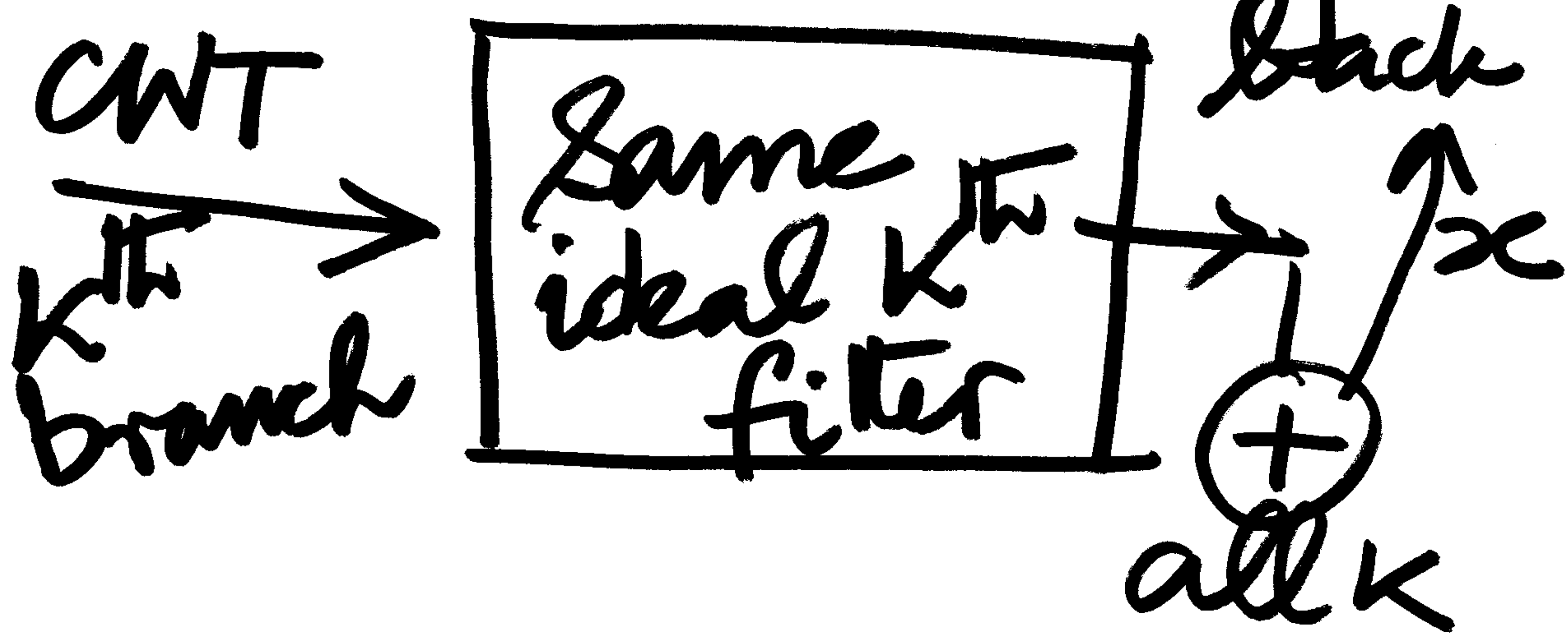
k : <u>all integer values</u>

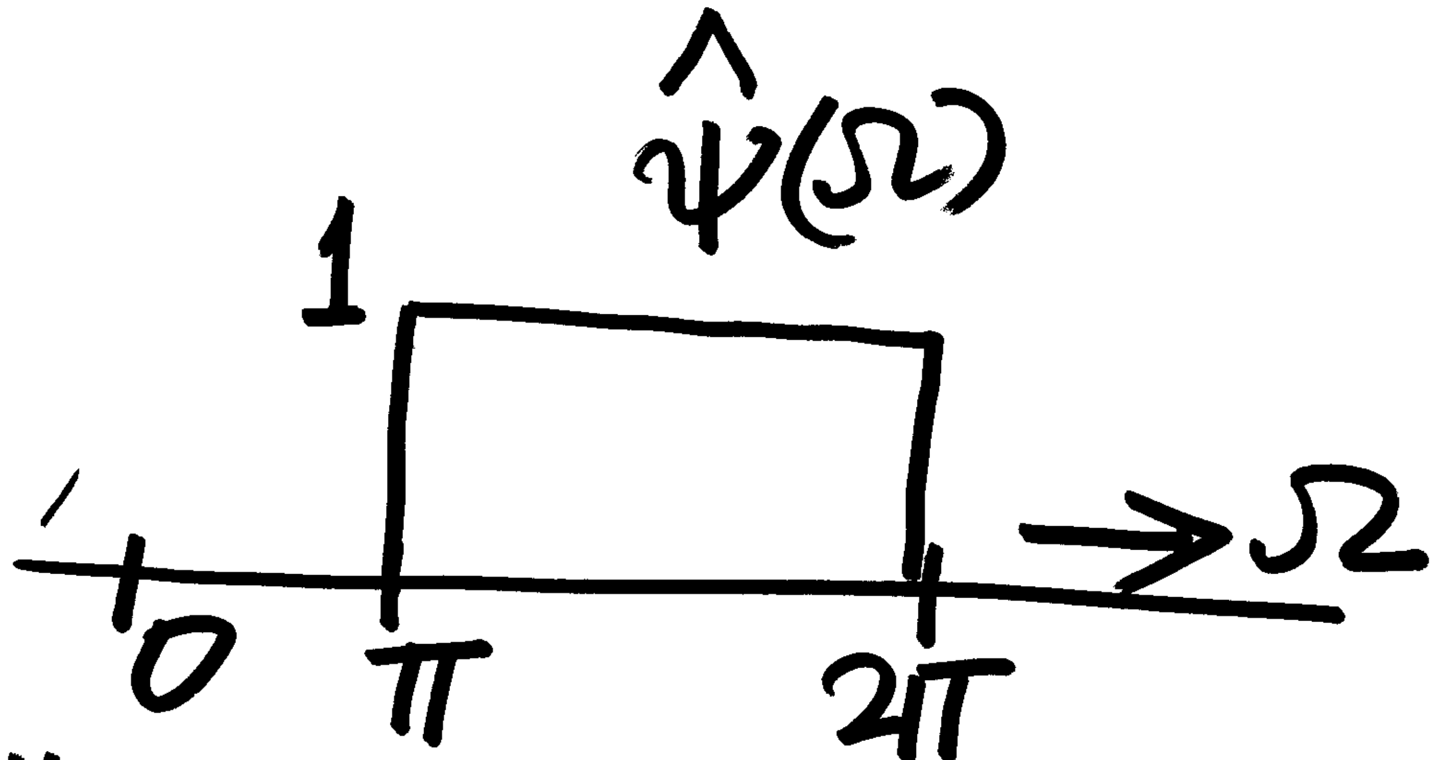
For each k we have a filter $s = a_0^k$



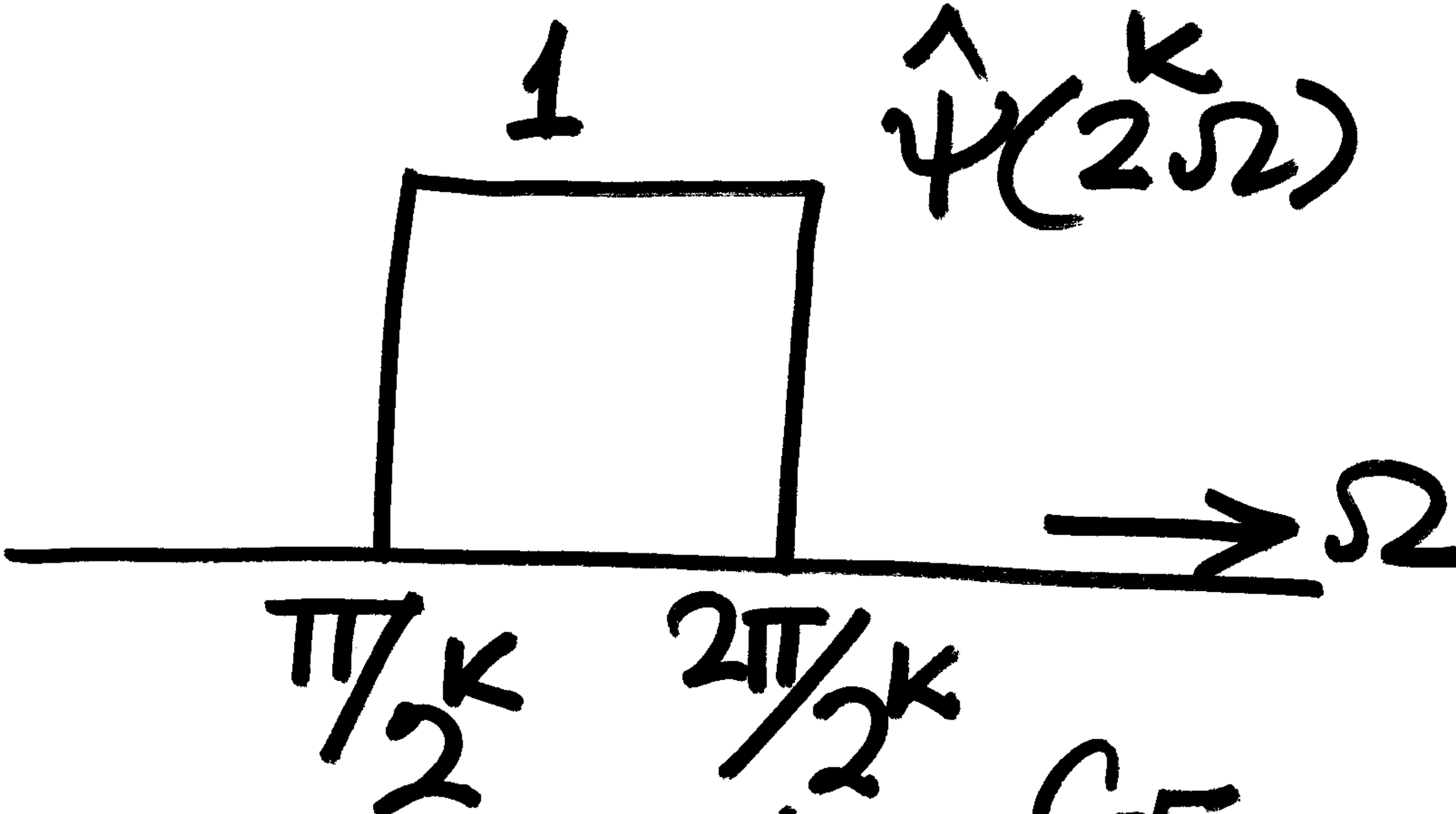
k^{th} filter

K^{th} branch of reconstruction





k^{th} branch $\rightarrow$ $a_0 = 2$



Nonoverlapping for integer k

Discretization of scale parameter:
equivalent to constructing filter bank

"Analysis" filters

$$\overline{\hat{\psi}(a_0^k \Omega)}$$

frequency resp

'Synthesis' filters

$$\hat{\psi}(a_0^k \Omega)$$

frequency response

On the k^{th} branch

$$x(t) \xrightarrow[\text{Fourier Transform}]{} \hat{x}(\Omega)$$

input Fourier Transform ANALYSIS

$$\hat{x}(\Omega)\,\hat{\psi}(a_0^k \Omega)$$

K^{th} synthesis branch:

$$\hat{x}(\Omega)\,\overline{\hat{\psi}(a_0^K\Omega)}\,\hat{\psi}(a_0^K\Omega)$$

$$= \hat{x}(\Omega)\,|\hat{\psi}(a_0^K\Omega)|^2$$

Overall output

$$= \hat{x}(\Omega)\underbrace{\left\{\sum_{k=-\infty}^{+\infty}\left|\hat{\psi}(a_0^k\Omega)\right|^2\right\}}$$

Ideally, this $= 1 \quad \forall\, \Omega$

To relax this requirement for 'designability':

$$c_1 \leq \sum_{k=-\infty}^{+\infty} \left| \hat{\psi}(a_0^k \Omega) \right|^2 \leq c_2$$

$$0 < c_1 \leq c_2 < +\infty$$

In that case, define another

$$\tilde{\psi}(t) \xrightarrow[\text{Fourier Transform}]{} \hat{\tilde{\psi}}(\Omega)$$

$$\hat{\tilde{\psi}}(\Omega) = \frac{\hat{\psi}(\Omega)}{\sum_{k=-\infty}^{+\infty} |\hat{\psi}(a_0^k \Omega)|^2}$$

Valid because of $\underline{G > 0}$.

We first show, (ψ real)

ψ is automatically admissible.

$$\frac{}{\int_0^\infty |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha}}\ ?.$$

$$\int_0^\infty |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha}$$

$$= \sum_{k=-\infty}^{+\infty} \int_{a_0^k}^{a_0^{k+1}} |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha}$$

Put $\alpha = a_0^k \beta$

$$\int_{a_0^k}^{a_0^{k+1}} |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha} \qquad d\alpha = a_0^k \, d\beta$$

$$\alpha = a_0^k \beta$$

$$= \int_1^{a_0} \hat{\psi}(a_0^k \beta) \frac{d\beta}{\beta}$$

$$\int_0^{\infty} |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha} = \int_1^{a_0} \left\{ \sum_{k=-\infty}^{+\infty} |\hat{\psi}(a_0^k \beta)|^2 \right\} \frac{d\beta}{\beta}$$

$$\sum_{k=-\infty}^{+\infty} |\hat{\psi}(a_0^k \beta)|^2$$

is upper bounded by c_2

'Admissibility integral' $\int_0^\infty |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha}$

is upper bounded by:

$$\int_1^{a_0} C_2 \frac{d\beta}{\beta}$$

$$= C_2 \ln \beta \Big|_1^{a_0}$$

$$= C_2 \ln a_0$$

$$\sum_{k=-\infty}^{+\infty} \left| \hat{\psi}(a_0^k \Omega) \right|^2$$

Sum of dilated spectra (SDS)

$$SDS(\mathcal{Y}, a_0)(\Omega)$$

Secondary arguments

Primary arguments

$\cancel{SDS(a_0}$

$$c_1 \le SDS(\psi, a_0)(\Omega) \le c_2$$

c_2 guarantees ADMISSILITY.

$$\hat{\tilde{\psi}}(\Omega) = \frac{\hat{\psi}(\Omega)}{SDS(\psi, a_0)(\Omega)}$$

We show that $\tilde{\psi}$ is ADMISSIBLE

For this purpose we need to consider

$$SDS(\tilde{\psi}, a_0)(\Omega)$$

$$|\hat{\psi}(a_0^k \Omega)|^2 = \frac{|\hat{\psi}(a_0^k \Omega)|^2}{\left\{\sum_{l=-\infty}^{+\infty} |\hat{\psi}(a_0^l a_0^k \Omega)|^2\right\}}$$

$$\left|\tilde{\hat{\psi}}(a_0^k \Omega)\right|^2 = \frac{\left|\hat{\psi}(a_0^k \Omega)\right|^2}{\left\{SDS(\psi, a_0)(\Omega)\right\}^2}$$

$$SDS(\tilde{\psi}, a_0)(\Omega) = \frac{\sum_{k=-\infty}^{+\infty} \left| \hat{\psi}(a_0^k \Omega) \right|^2}{\{SDS(\psi, a_0)(\Omega)\}^2}$$

$$SDS(\tilde{\psi}, a_0)(\Omega) = \frac{SDS(\psi, a_0)(\Omega)}{\{SDS(\psi, a_0)(\Omega)\}^2}$$

Cancellation from numerator and denominator is valid because of bounds c_1, c_2

$$SDS(\tilde{\psi}, a_0)(\Omega) = \frac{1}{SDS(\psi, a_0)(\Omega)}$$

$$0 < c_1 \le SDS(\psi, a_0)(\Omega) \le c_2 < \infty$$

$$\Downarrow$$

$$\infty > \frac{1}{c_1} \ge \frac{1}{SDS(\psi, a_0)(\Omega)} \ge \frac{1}{c_2} > 0$$

$$SDS(\psi, a_0)(\Omega)$$

$\tilde{\psi}$ is also an

ADMISSIBLE

WAVELET

Use $\tilde{\psi}$ on synthesis side: K^{th} branch:

$$\longrightarrow \boxed{\hat{\tilde{\psi}}(a_0^{k}\Omega) \text{ filter}} \longrightarrow$$

With this, analysis and synthesis together:

$$\hat{x}(\Omega) \sum_{k} \overline{\hat{\psi}(a_0^k \Omega)} \cdot \hat{\tilde{\psi}}(a_0^k \Omega)$$

$$\hat{x}(\Omega) \cdot \sum_{k=-\infty}^{+\infty} \cdots$$

$$\cdots \frac{}{\hat{\psi}(a_0^k \Omega)} \frac{\hat{\psi}(a_0^k \Omega)}{SDS(\psi, a_0)(\Omega)}$$

$$= \hat{x}(\Omega) \cdot \frac{SDS(\psi, a_0)(\Omega)}{SDS(\psi, a_0)(\Omega)}$$

Because of c_1, c_2

we can cancel $\frac{1}{\sqrt{\Omega}}$

$$= \hat{x}(\Omega) \times 1$$

Specific case of $a_0 = 2$

'Dyadic'

$SDS(\mathcal{V}, 2)(\Omega)$
must be upper
and lower
bounded.

Exercise: For the Haar wavelet $\psi(t)$ show that

$$0 < C_1 \leq SDS(\psi, 2)(\Omega) \leq C_2 < \infty$$

For the "Derivative of Gaussian" wavelets, a very wide range of a_0 is possible.