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Lec-23
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LECTURE 23

ADMISSIBILITY IN DETAIL

DISCRETIZATION OF SCALE

Important steps in reconstructing $x(t)$ from $CWT(x,\psi)(\tau, s)$

Triple integral involved

$$\underbrace{\frac{1}{2\pi}\int \hat{x}(\Omega)\, s^{1/2}\, \overline{\hat{\psi}(s\Omega)}\, e^{j\Omega\tau}\, d\Omega}_{CWT(x,\psi)(\tau,s)}$$

Reconstruct:

$$\int_{0}^{\infty}\int_{-\infty}^{+\infty} CWT(x,\psi)(\tau,s)\,\frac{1}{s^{1/2}}\,\psi\!\left(\frac{t-\tau}{s}\right) f(s)\,d\tau\,ds$$

Writing all together

Overall integral:

$$\int_0^{\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{x}(\Omega)\cdot s^{1/2}\cdot \overline{\hat{\psi}(s\Omega)}\, e^{j\Omega\tau}\cdot \frac{1}{s^{1/2}}\cdot \psi\left(\frac{t-\tau}{s}\right) f(s)\, \underset{\text{take last}}{d\Omega}\, \underset{\text{first}}{d\tau}\, \underset{\text{second}}{ds}$$

After taking care of

$$\int_{-\infty}^{+\infty} \qquad dz$$

what was left, by choosing $f(\delta) = \frac{1}{\delta^2}$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{x}(\Omega)\cdot\left\{\int_{0}^{\infty}\left|\hat{\psi}(s\Omega)\right|^{2}\frac{ds}{s}\right\}\cdot e^{j\Omega t}\,d\Omega$$

If we could make

$$\int_0^\infty |\hat{\psi}(s\Omega)|^2 \frac{ds}{s}$$

independent of Ω

(constant): C_ψ

What is left is:

$$\frac{1}{2\pi} \cdot C_\psi \int_{-\infty}^{+\infty} \hat{x}(\Omega) e^{j\Omega t} \, d\Omega$$

$$= C_\psi \cdot x(t)$$

Consider: $\displaystyle\int_0^\infty |\hat{\psi}(s\Omega)|^2 \frac{ds}{s}$

Put $s\Omega = \alpha$

$s : 0 \longrightarrow +\infty$

$\Omega = 0$ is a problem

Except for $\Omega = 0$,

$\Omega > 0$: $\alpha = \delta\Omega$

$\alpha : 0 \longrightarrow +\infty$

when $\delta : 0 \longrightarrow +\infty$

$$d\alpha = \Omega \, d\gamma$$

$$\alpha = \Omega \gamma \qquad \Omega \neq 0$$

$$\frac{d\alpha}{\alpha} = \frac{d\gamma}{\gamma}$$

$$\int_0^\infty |\hat{\psi}(s\Omega)|^2 \frac{ds}{s}$$

$$= \int_0^\infty |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha}$$

If $\psi(t)$ is a complex wavelet, we need separately to take care of the positive and negative spectrum

$$\int_0^\infty |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha} = \int_0^\infty |\hat{\psi}(-\beta)|^2 \frac{d\beta}{\beta} < \infty.$$

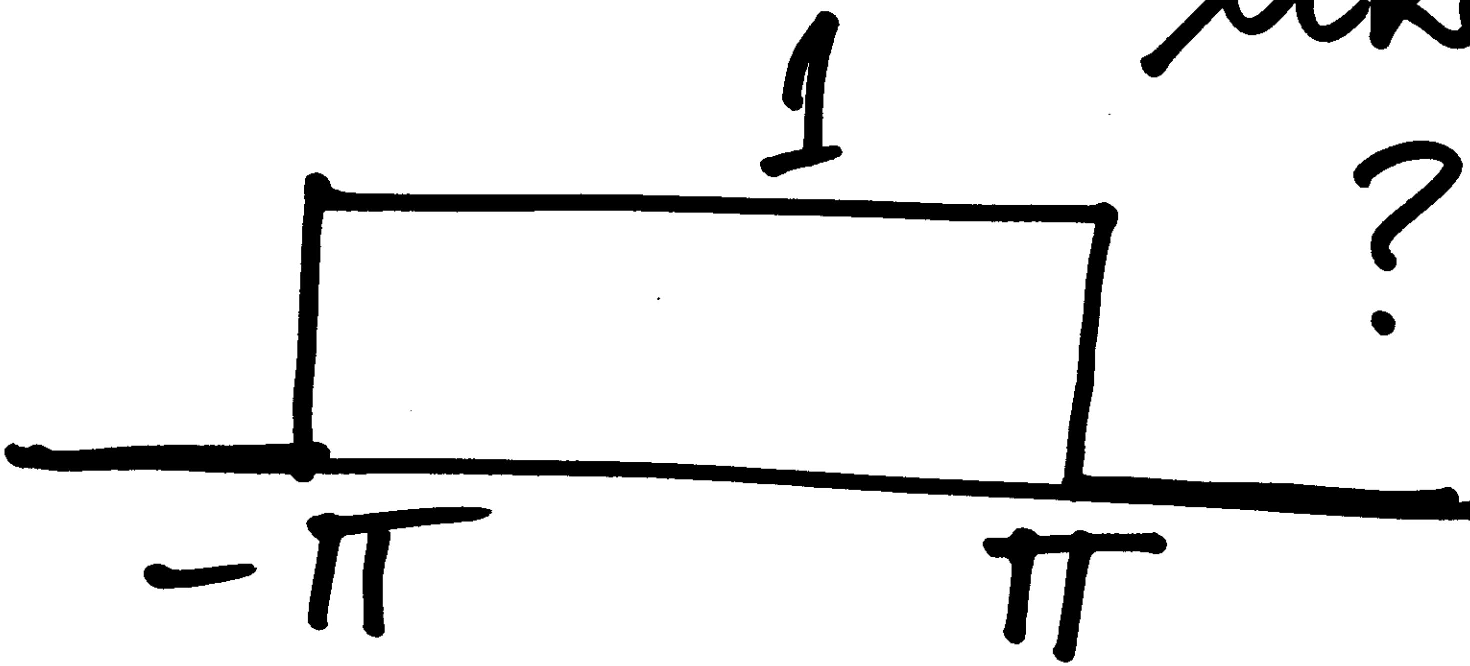
ADMISSIBILITY

'Admissibility' is a Condition required for RECONSTRUCTION.

$\Omega > 0$ Condition
(For real ψ this is enough)

$$\int_0^\infty |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha} \text{ finite}$$

Can $|\hat{\varphi}(\alpha)|^2$ look like:



?

$\Omega < 0$: $\alpha = \delta\Omega$

$\delta : 0 \longrightarrow +\infty$

$\alpha : 0 \longrightarrow -\infty$

$$\int_0^{-\infty} |2\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha}$$

Substitute $\beta = -\alpha$

$\alpha : 0 \to \cancel{+\infty}\ -\infty$; $\beta : 0 \to +\infty$

$$\int_0^\infty |\hat{M}(-\beta)|^2 \frac{d\beta}{\beta} \quad \text{for } \underline{\Omega < 0}$$

$$\frac{d\beta}{\beta} = \frac{-d\alpha}{-\alpha}$$

$$\Rightarrow d\beta/\beta = d\alpha/\alpha$$

Making this integral (on S) independent of Ω, means:

If $\psi(t)$ is real

$$\hat{\psi}(-\beta) = \overline{\hat{\psi}(\beta)}$$

$$|\hat{\psi}(-\beta)|^2 = |\hat{\psi}(\beta)|^2$$

$$\int_0^{\infty} |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha}$$

$$= \int_0^{\pi} 1 \cdot \frac{d\alpha}{\alpha}$$

Divergent!

$$\int \frac{dx}{x} \qquad \ln x \Big|_{0}^{\pi}$$

divergent

Here the trouble
comes as $\alpha \to 0$
The spectrum
should have vanished
as $\alpha \to 0$

What about

$$\longrightarrow \Omega$$

$$|\hat{\psi}(\alpha)|^2 \to 0 \text{ as } \alpha \to +\infty$$

$$\int_L^{\infty} |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha}$$

$L \leftarrow$ large enough

$$\simeq \int_L^{\infty} c_0 \cdot \frac{d\alpha}{\alpha}$$

$\ln \alpha \; +\infty$

$\downarrow$

divergent again!

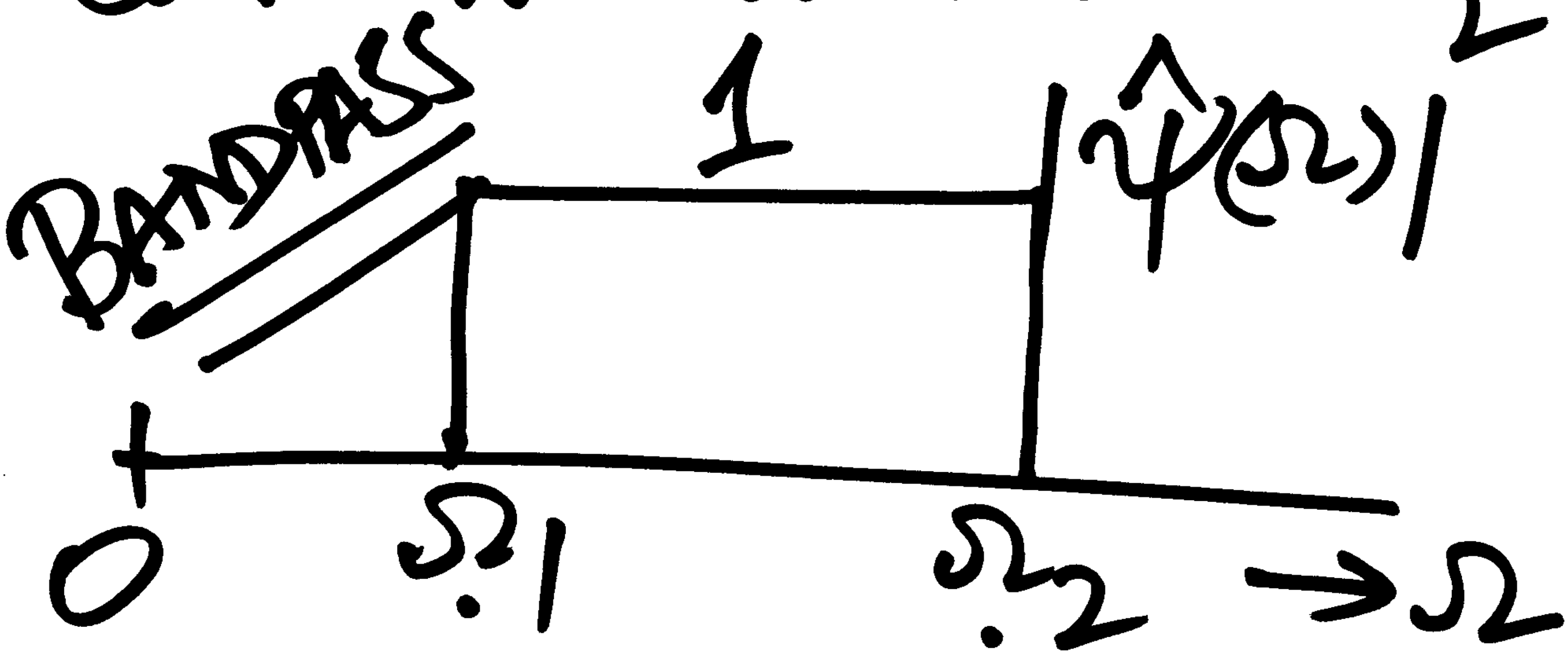
The spectrum $|\hat{\psi}(\alpha)|^2$

should have

vanished

as $\alpha \to +\infty$

Can we allow:



$$\int_0^\infty |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha} = \int_{\Omega_1}^{\Omega_2} 1 \cdot \frac{d\alpha}{\alpha}$$

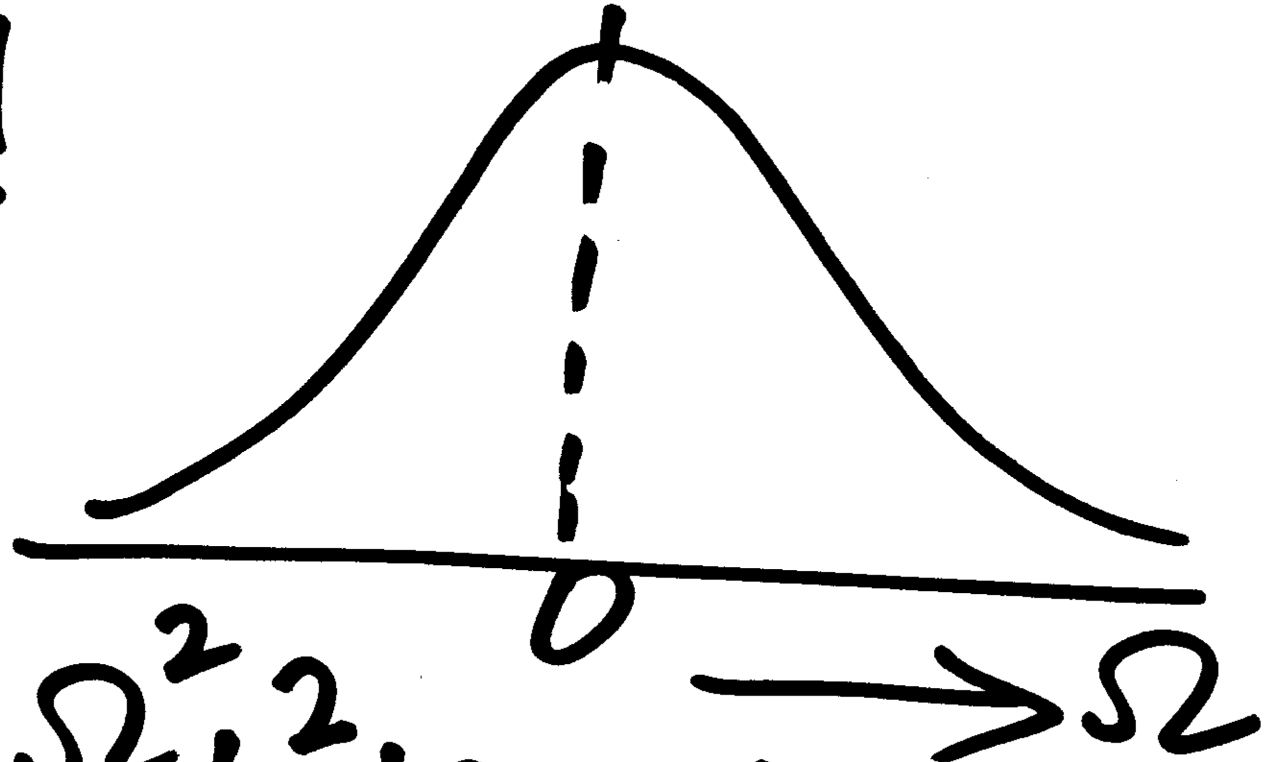
$$= \cancel{\ln \Omega_2}$$

$$\frac{\ln \Omega_2 - \ln \Omega_1}{\text{finite}}$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \longrightarrow e^{-\Omega^2}$$

Is this admissible?

No!

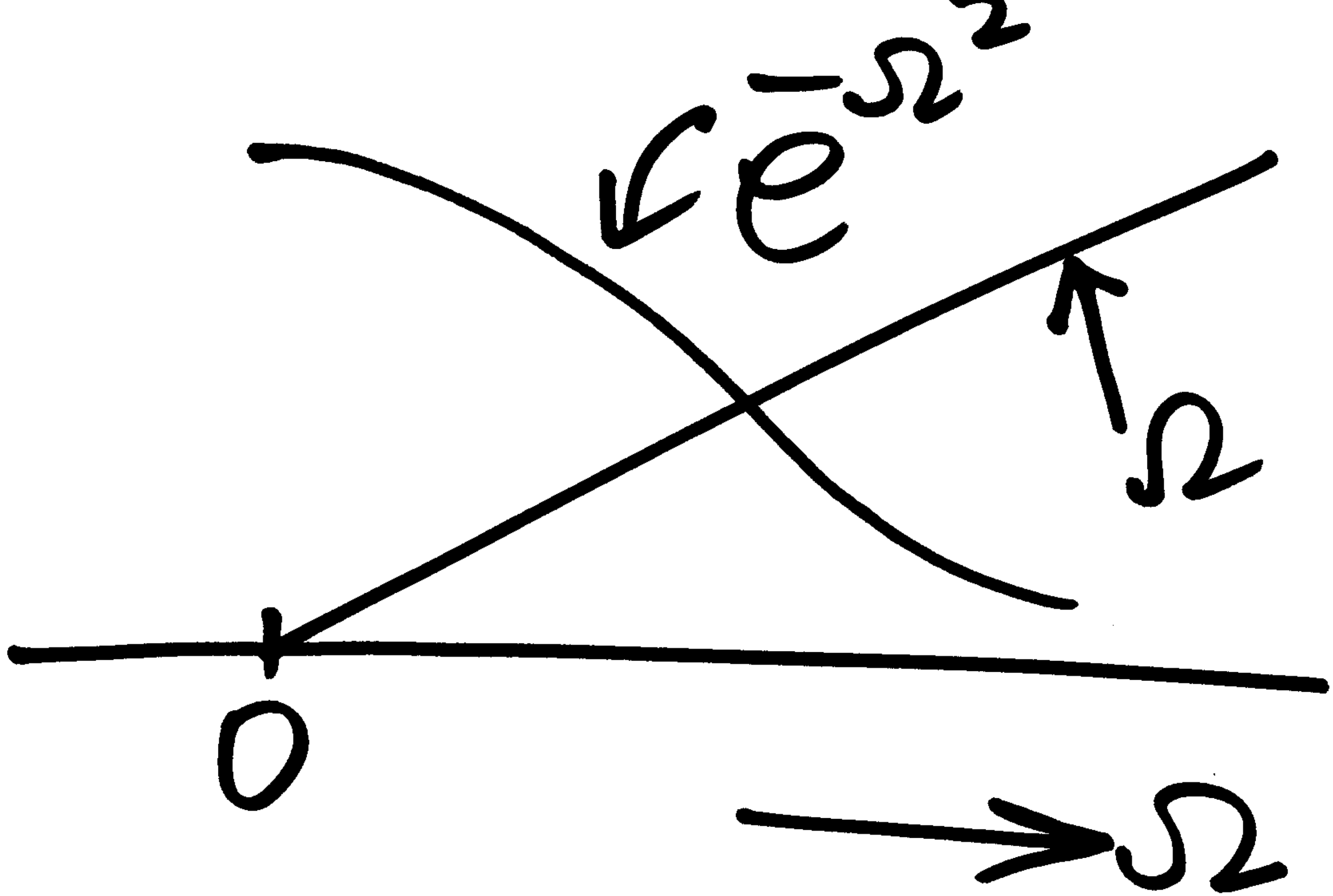


$$\int_0^1 \left|e^{-\Omega^2}\right|^2 \frac{d\Omega}{\Omega}$$ **diverges**

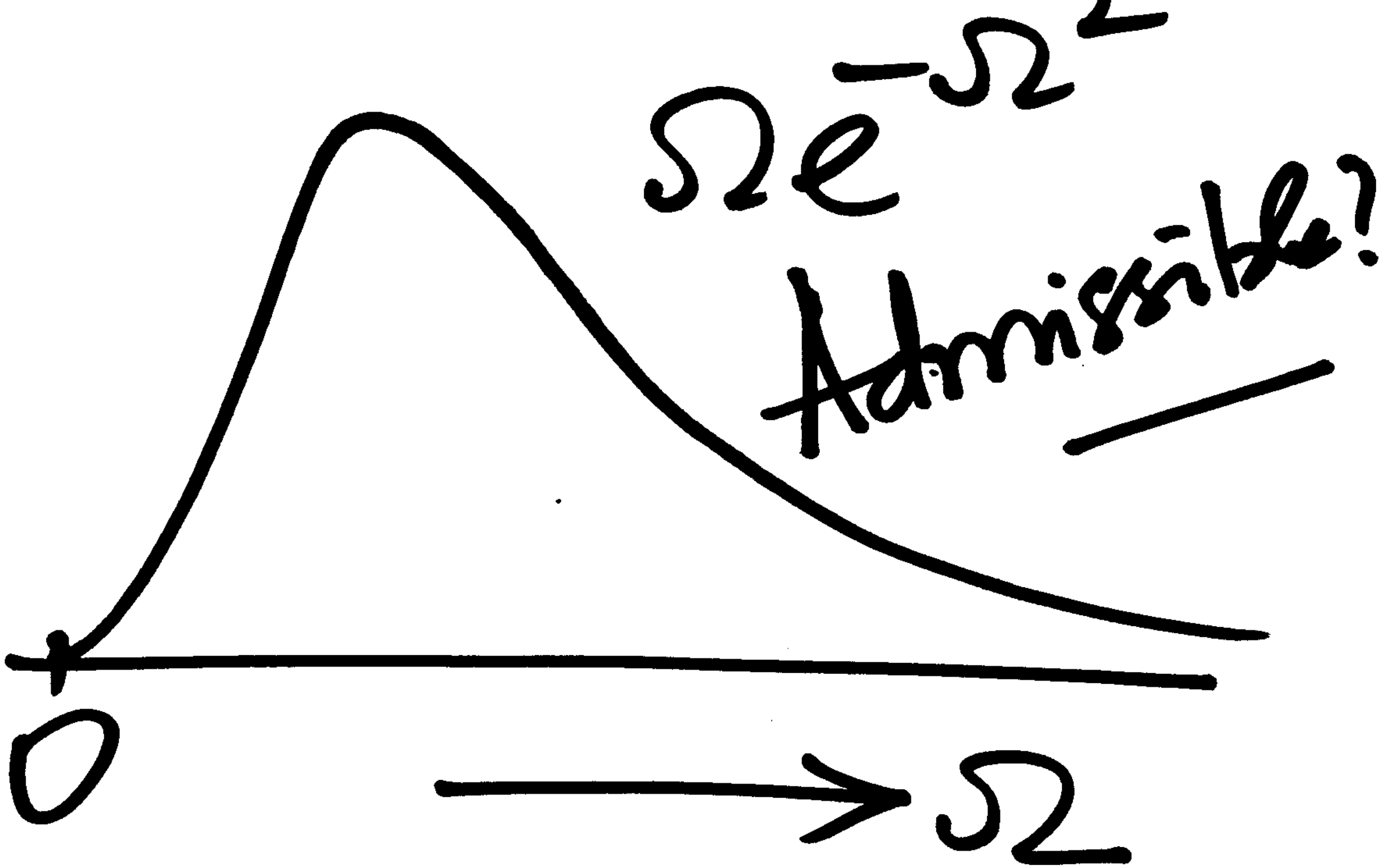
Derivative of Gaussian:

$$\frac{d}{d\Omega} e^{-\Omega^2} = -2\Omega e^{-\Omega^2}$$

$$\Rightarrow \Omega e^{-\Omega^2}$$



$e^{-\Omega^2}$
Ω
0
Ω


$\Omega e^{-\Omega^2}$
Admissible?
0
Ω

$$\int_0^\infty |\omega e^{-\omega^2}|^2 \frac{d\omega}{\omega}$$

$$= \int_0^\infty \omega^2 e^{-2\omega^2} \frac{d\omega}{\omega}$$

$$= \int_0^\infty e^{-2\Omega^2} \cdot \Omega \, d\Omega$$

$$\Omega^2 = \lambda$$

$$d\lambda = 2\Omega \, d\Omega$$

$$= \int_0^{\infty} e^{-2\lambda} \cdot \frac{1}{2} \, d\lambda$$

Converges

Exercises:

1. Calculate (if need be numerically/approx) TBP $\sigma_t \sigma_\Omega^2$ of $\Omega e^{-\Omega^2}$

$$\frac{d}{d\Omega} e^{-\Omega^2}$$ (to within constants)

$\frac{d}{d\Omega}$ is equivalent to multiply by t

The inverse Fourier transform of $\Omega e^{-\Omega^2}$ has the same form.

(exercise)

2. Exercise:

Consider

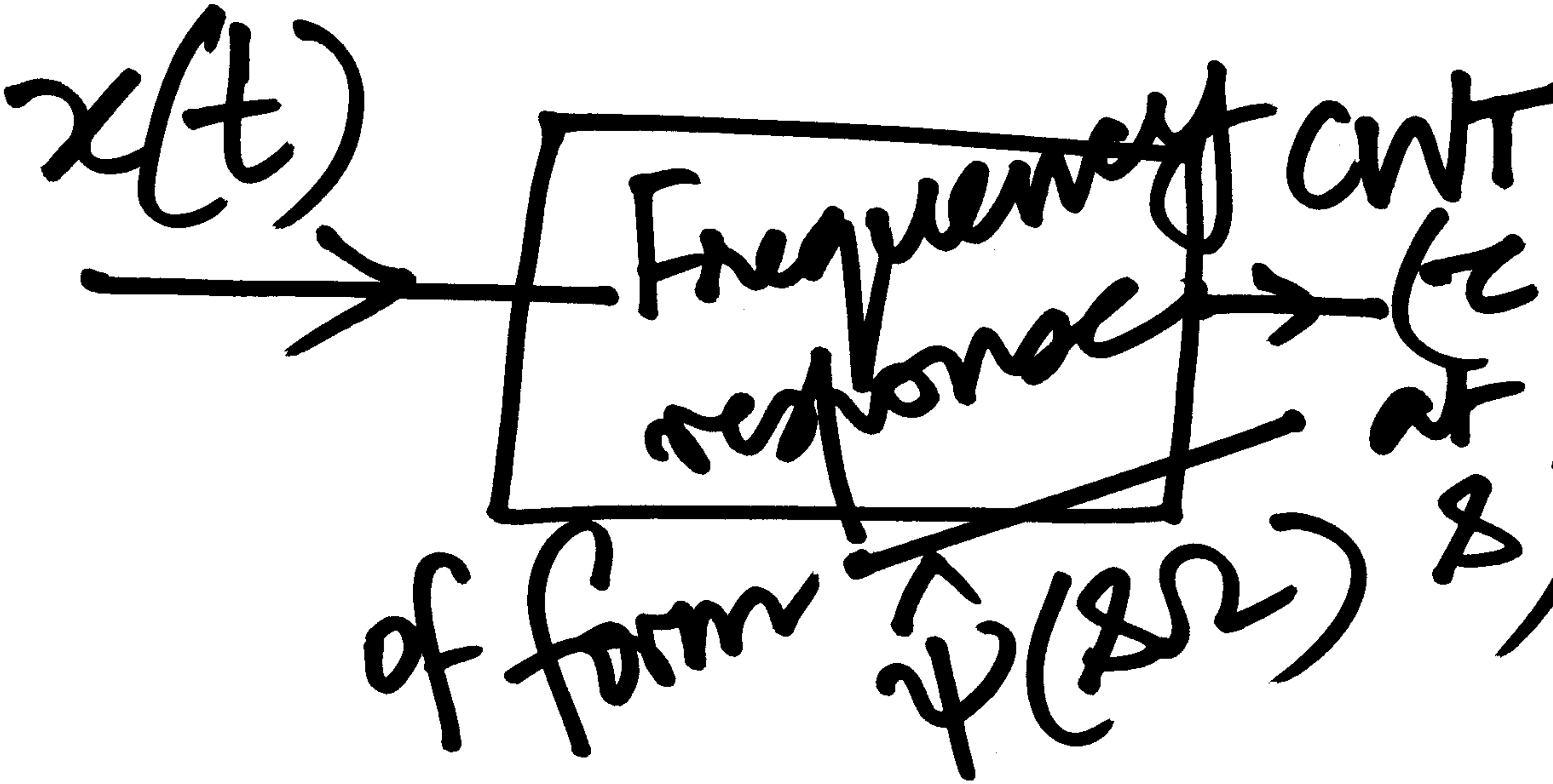
$$\frac{d^2}{d\Omega^2} e^{-\Omega^2}$$

$$\frac{d}{dr}\left(e^{-r^2}\right)$$

↓ derivative would look like a 'Mexican Hat'

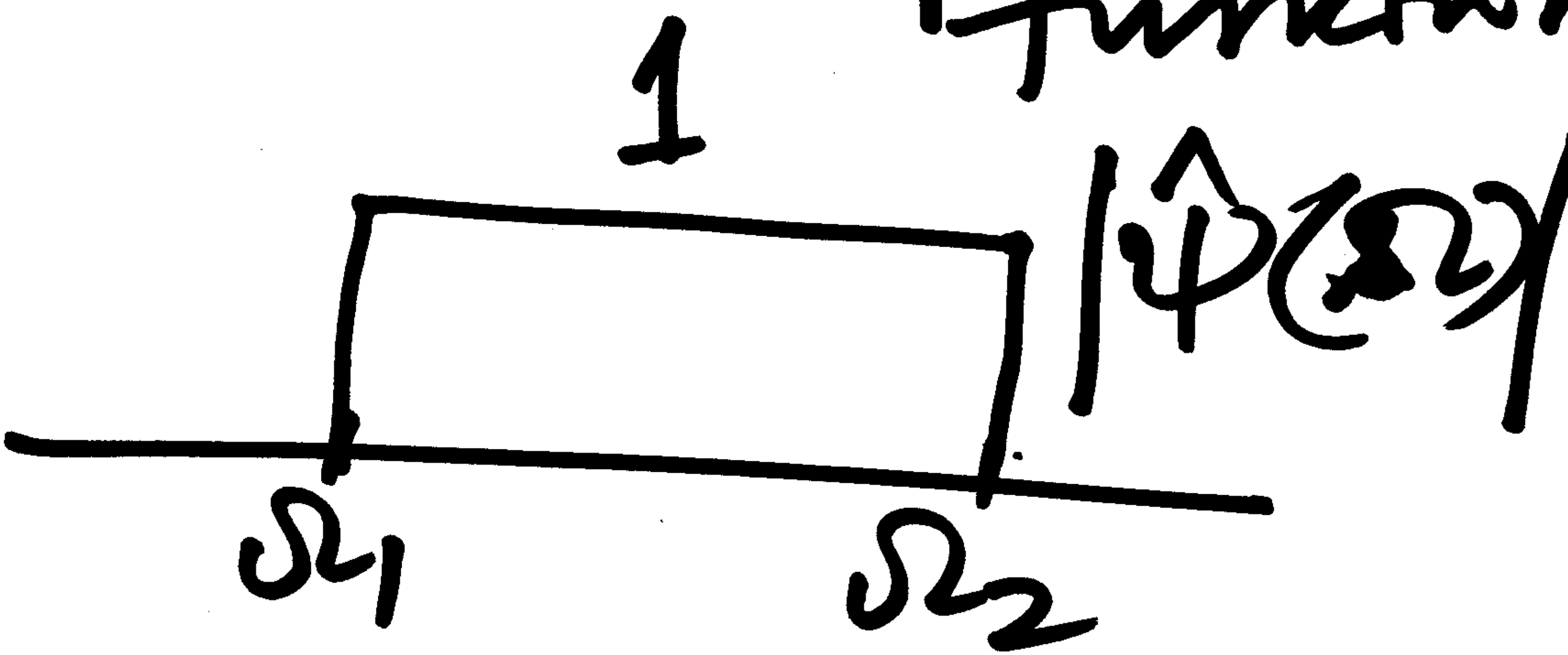
Admissibility is essentially a "Bandpass" <u>requirement</u>

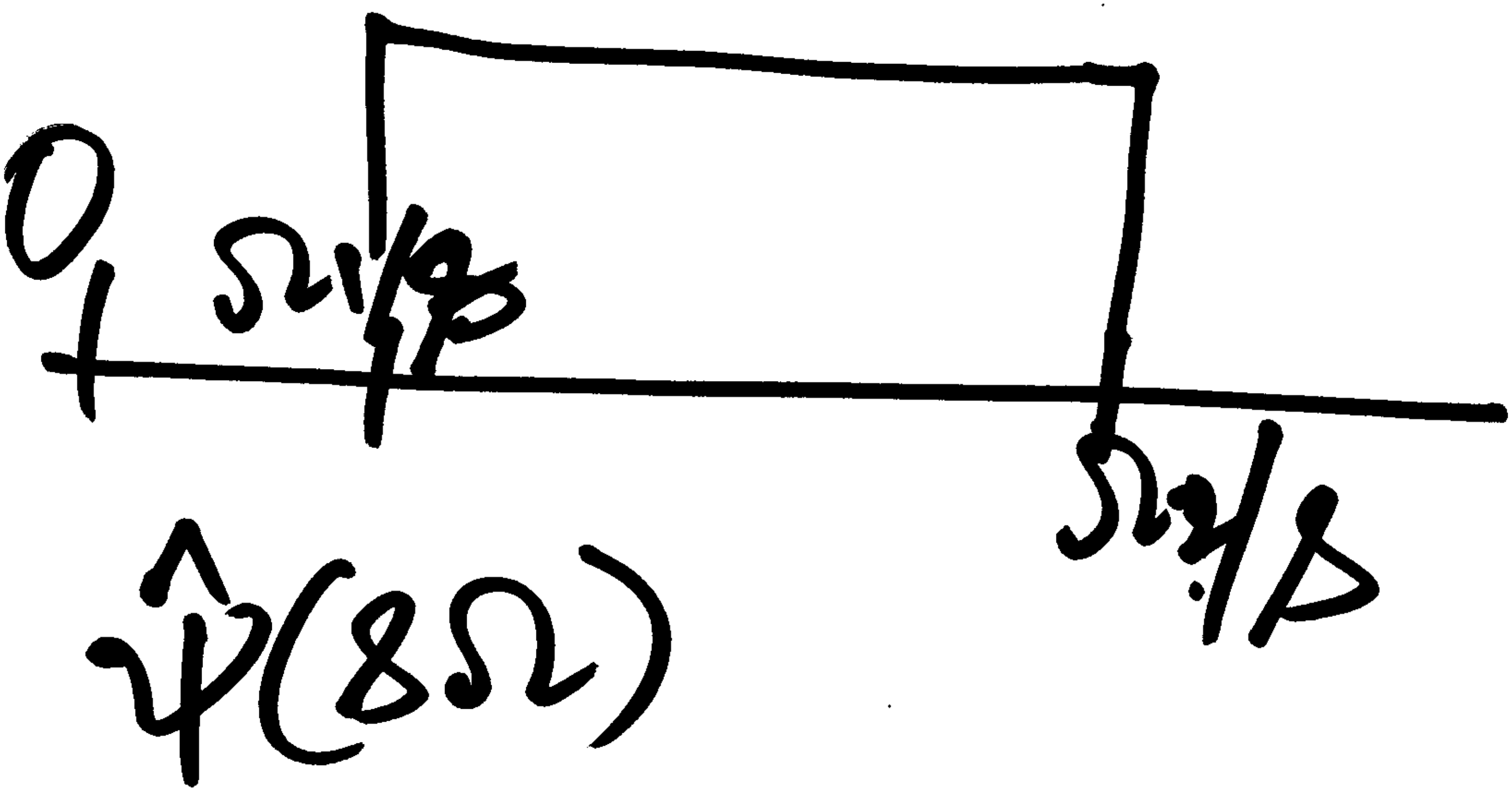
When can we discretize $\mathcal{S}$?



x(t)
Frequency response
of form $\hat{\psi}(s\Omega)$
CWT
(τ at s)

Ideal bandpass function




0
p̂(8Ω)

We could, in general allow

$$\delta = a_0^k$$

k ; all integers; $a_0 > 1$