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Lec. No. 22
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LECTURE 22

RECONSTRUCTION AND ADMISSIBILITY

Reconstruction of $x(t)$ from $STFT(x,v)(\tau_0,\Omega_0)$?

$$\mathrm{STFT}(x, v)(\tau_0, \Omega_0)$$

$$= \int_{-\infty}^{+\infty} x(t)\,\overline{v(t-\tau_0)\,e^{j\Omega_0 t}}\,dt$$

"component" of $x(t)$
"along" $v(t-\tau_0)\,e^{j\Omega_0 t}$

$$\int_{-\infty}^{+\infty} STFT(x,v)(\tau_0,\Omega_0) \cdot v(t-\tau_0)\, e^{j\Omega_0 t}\, d\tau_0\, d\Omega_0$$

This should give us $c_0 x(t)$

Expanding:

$$\iiint_{-\infty}^{+\infty} \left\{ x(t_1)\,\overline{v(t_1-\tau_0)}\, e^{-j\Omega_0 t_1}\, dt_1 \right\} \cdots$$

$$\cdots v(t-\tau_0)\, e^{j\Omega_0 t}\, d\tau_0\, d\Omega_0$$

Considering first the $d\Omega_0$ integral:

$$\int_{-\infty}^{+\infty} e^{-j\Omega_0 t_1} \cdot e^{j\Omega_0 t} \, d\Omega_0$$

$$= 2\pi \cdot \underbrace{\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\Omega_0 (t - t_1)} \, d\Omega_0}_{\text{Inverse Fourier Transform of } 1 \ \forall \Omega_0.}$$

$$= 2\pi\delta(t - t_1)$$

$\delta(\cdot)$: Continuous impulse

We now dispense of the $\int_{-\infty}^{+\infty} dt_1$

Essentially putting $t_1 = t$.

That leaves only:

$$2\pi \int_{-\infty}^{+\infty} x(t)\, \overline{v(t-\tau_0)}\, v(t-\tau_0)\, d\tau_0$$

independent of τ_0

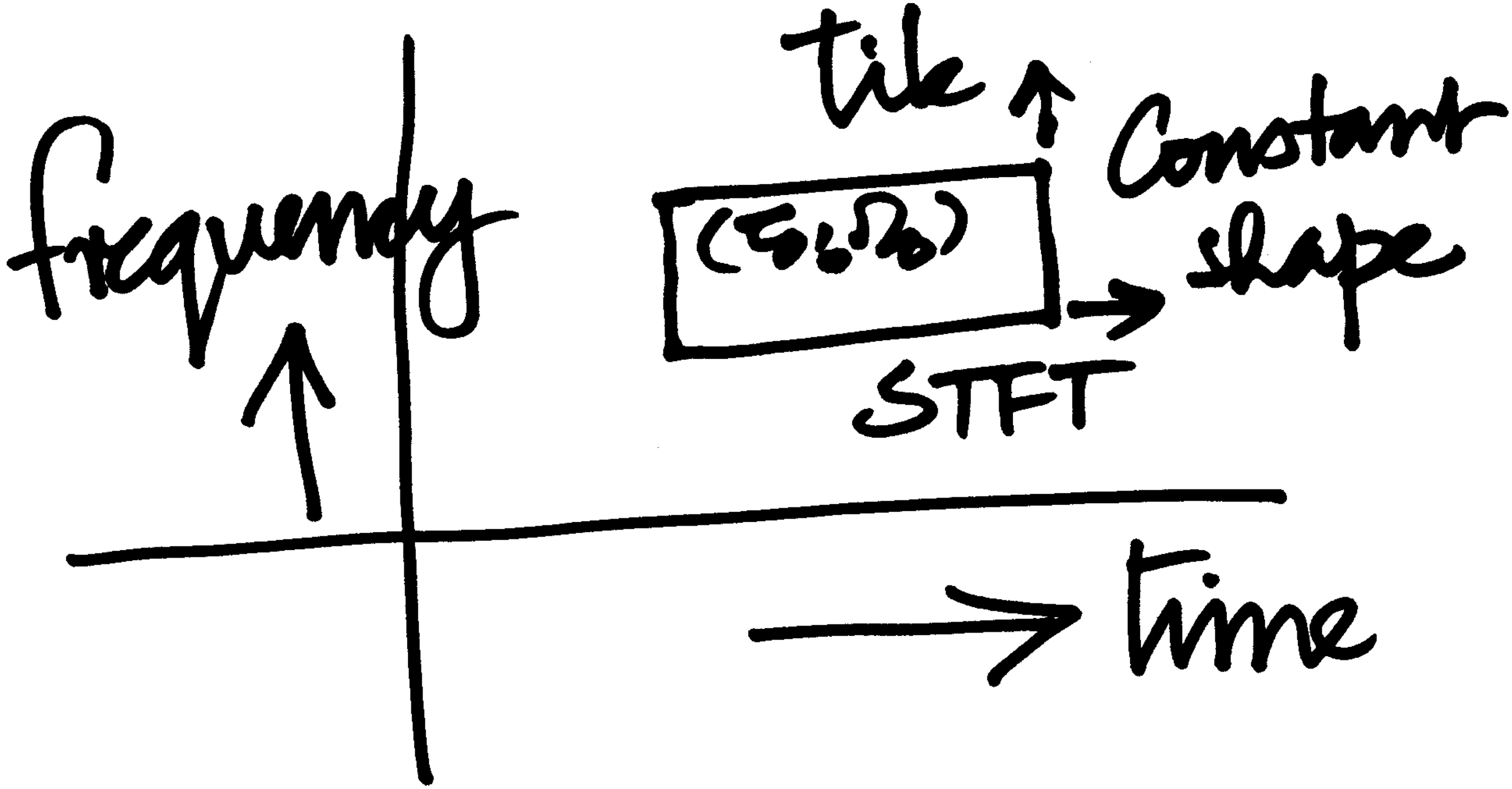
$$= 2\pi x(t) \int_{-\infty}^{+\infty} |v(t-\tau_0)|^2 \, d\tau_0$$

$$t - \tau_0 = \lambda . \quad d\lambda = -d\tau_0 .$$

$$= 2\pi x(t) \underbrace{\int_{-\infty}^{+\infty} |v(\lambda)|^2 \, d\lambda}_{\text{squared } L_2 \text{ norm of } v}$$

$$= \underbrace{\left\{ 2\pi \|v\|_2^2 \right\}}_{c_0 \text{ desired.}} x(t)$$

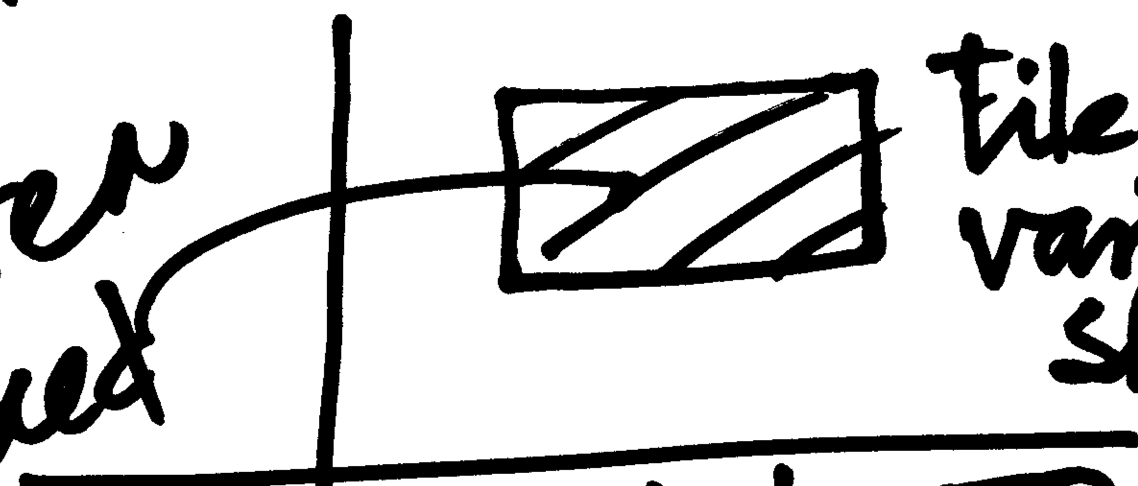
Done!



frequency
tile
(τ₀, Ω₀)
Constant shape
STFT
time

Continuous Wavelet Transform (CWT)

'TILING'



tile of variable shape

area fixed

related to TBP of ψ

Increasing δ_0

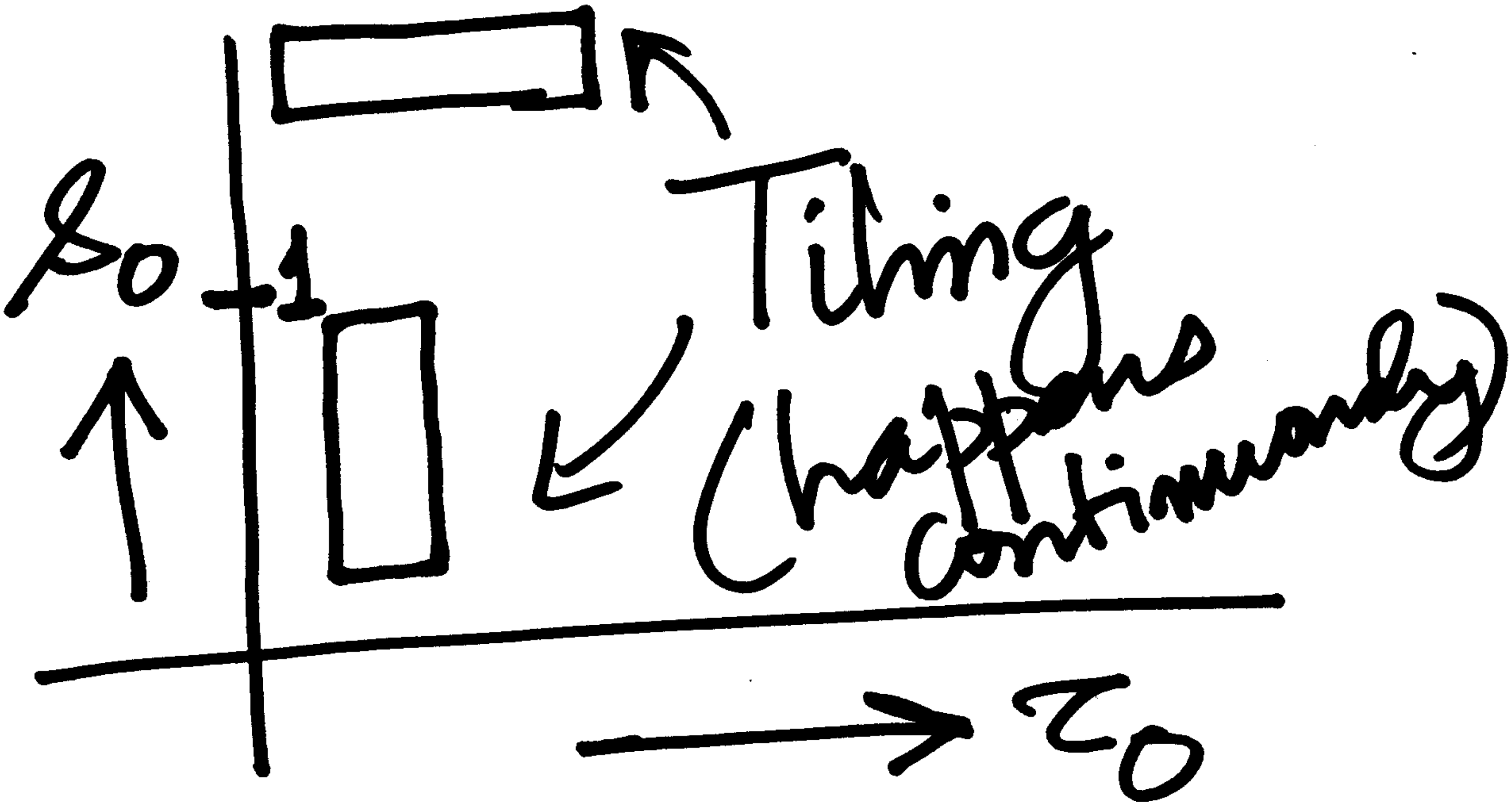
$\Rightarrow$ expanding in time,

compressing in frequency

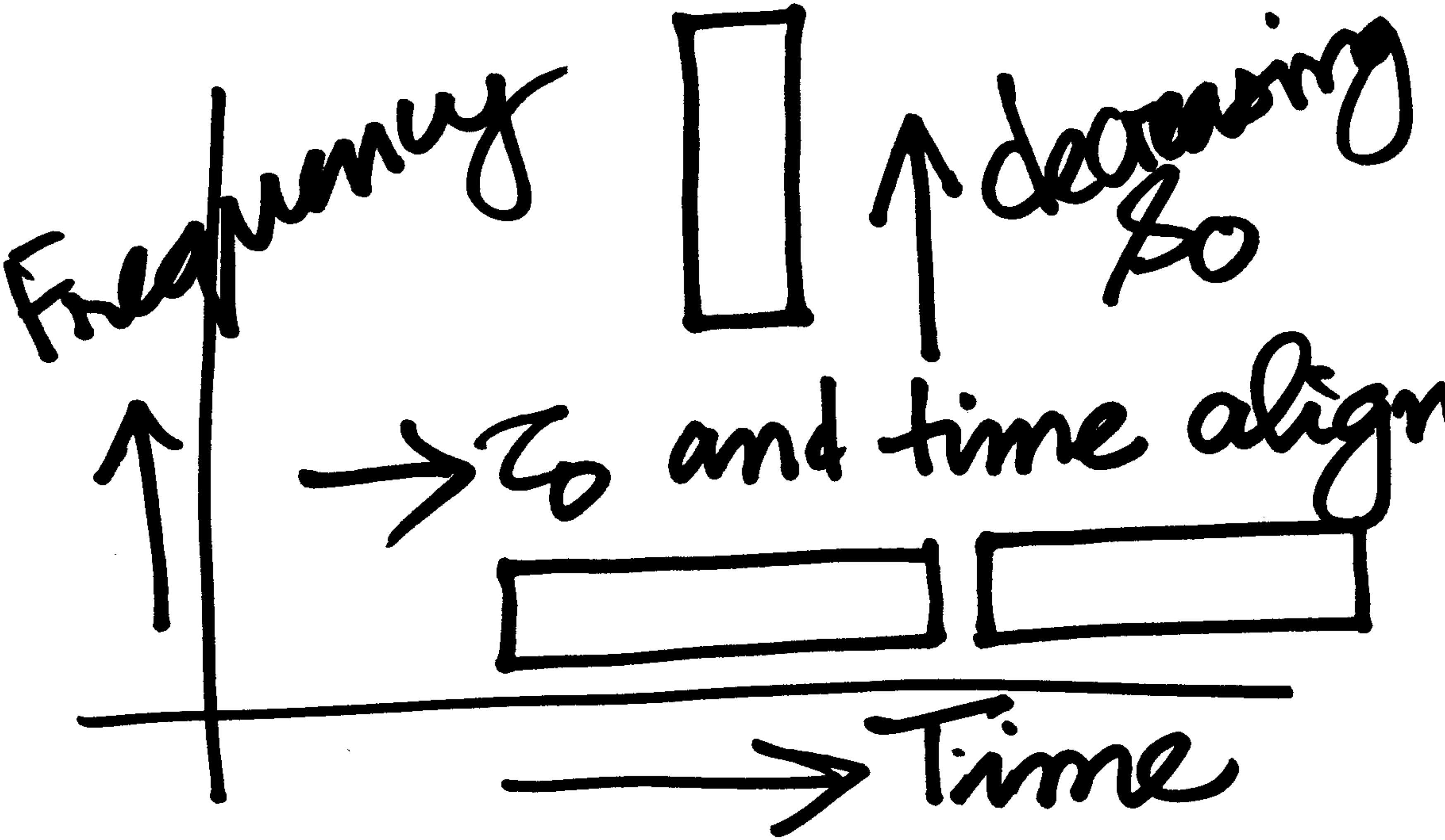
(vice versa).

"movement" along the Ω-axis happens because centre frequency $\neq 0$

Centre frequency
<u>and</u> bandwidth
are both divided
by 8o.



s_0
1
Tiling
(happens continuously)
τ_0


Frequency
decreasing
δ_0
τ_0 and time align
Time

Reconstruction of $x(t)$ from $CWT(x,\psi)(\tau_0, s_0)$ should be ...

$$\int_0^{\infty}\int_{-\infty}^{+\infty} CWT(x,\psi)(\tau_0,s_0) \cdots \frac{1}{\sqrt{s_0}}\,\psi\left(\frac{t-\tau_0}{s_0}\right)\underbrace{f(s_0)}_{\text{"Weight"}}\, d\tau_0\, ds_0$$

$CWT(x, \psi)(\tau_0, s_0)$

using Parseval's theorem ...

$$\int_{0}^{\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{2\pi}\hat{x}(\Omega)\,\overline{\hat{\psi}(s_0\Omega)\cdots}$$

$$\cdots\overline{\sqrt{s_0}\,e^{j\Omega\tau_0}}\cdot\frac{1}{\sqrt{s_0}}\psi\left(\frac{t-\tau_0}{s_0}\right)\cdots$$

$$\ldots f(\theta_0)\,\cancel{e^{i\Omega\cdot\xi_0}}$$

already written

$$d\Omega\, d\xi_0\, d\theta_0$$

Let us first dispense

with $\int_{-\infty}^{+\infty} d\tau_0$

$$\int_{-\infty}^{+\infty} \psi\left(\frac{t-\tau_0}{s_0}\right) e^{j\Omega\tau_0}\, d\tau_0$$

$$\frac{t-\tau_0}{s_0} = \lambda$$

$$t - \tau_0 = s_0 \lambda$$

$$\tau_0 = t - s_0 \lambda .$$

$$d\tau_0 = - s_0 \, d\lambda .$$

$$= \int_{-\infty}^{+\infty} \underline{\Psi}(\lambda) \cdot e^{j\Omega(t - \underline{s_0}\lambda)} \; s_0 \, d\underline{\lambda}$$

$$= e^{j\Omega t} \cdot \delta_0 \underbrace{\int_{-\infty}^{+\infty} \psi(\lambda) \cdot e^{-j\delta_0 \Omega \lambda} \, d\lambda}$$

$$= e^{j\Omega t} \delta_0 \hat{\Psi}(\delta_0 \Omega)$$

Collapse $d\tau_0$.

$$\int \hat{x}(\Omega) \cdot \overline{\hat{\psi}(s_0\Omega)} \cdots$$

$$\cdots \sqrt{s_0}\, e^{j\Omega\tau_0}\, d\Omega$$

Substitute in reconst

What is left:

$$\int_0^{\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{x}(\Omega) \cdot \overline{\hat{\psi}(s_0 \Omega)} \; \hat{f}(s_0) \cdots$$

$$\cdots e^{j\Omega t}\, s_0 \cdot \hat{\psi}(s_0\Omega)\, d\Omega\, ds_0$$

Let us now dispense with $\displaystyle\int_0^\infty d\delta_0$.

$$\int_0^\infty \overline{\hat{\psi}(s_0\Omega)} \cdot f(s_0) \cdot s_0 \cdot \hat{\psi}(s_0\Omega)\, ds_0$$

$$= \int_0^\infty \left|\hat{\psi}(\gamma_0 \Omega)\right|^2 \left(\gamma_0 f(\gamma_0)\right) d\gamma_0 .$$

Make this <u>Ω-indep</u>.

If we could make $\delta_0 f(\delta_0)\, d\delta_0$

$$\equiv \left(d\delta_0 / \delta_0 \right)$$

$$\Rightarrow f(\delta_0) = \frac{1}{\delta_0^2}$$

(valid weight — positive).

$$\delta_0 f(\delta_0)\, d\delta_0$$

$$\equiv \frac{d\delta_0}{\delta_0}$$

$$\delta_1 = \Omega \delta_0 .$$
$$d\delta_1 = \Omega \, d\delta_0$$

$$\frac{d\delta_1}{\delta_1} = \frac{d\delta_0}{\delta_0}$$

$$(\Omega \neq 0)$$

The triple integral finally collapses to:

$$\left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{x}(\Omega)\, e^{j\Omega t}\, d\Omega \right\}$$

$\times$ some constant

The constant comes from

$$\int_0^\infty \left|\hat{\psi}(s_0\Omega)\right|^2 \frac{ds_0}{s_0}$$

$$= \int_0^\infty \left|\hat{\psi}(s_1)\right|^2 \frac{ds_1}{s_1}$$

when $\Omega > 0$

and $\displaystyle\int_0^{-\infty} |\hat{\psi}(\xi_1)|^2 \frac{d\xi_1}{\xi_1}$

when $\Omega < 0$.