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LECTURE 21

SHORT TIME FOURIER TRANSFORM AND WAVELET TRANSFORM IN GENERAL

Short Time Fourier Transform:

Choose a <u>window</u> function: $v(t)$

Finite time variance:

$$t\,v(t) \in \mathcal{L}_2(\mathbb{R})$$

$$v(t) \in \mathcal{L}_2(\mathbb{R})$$

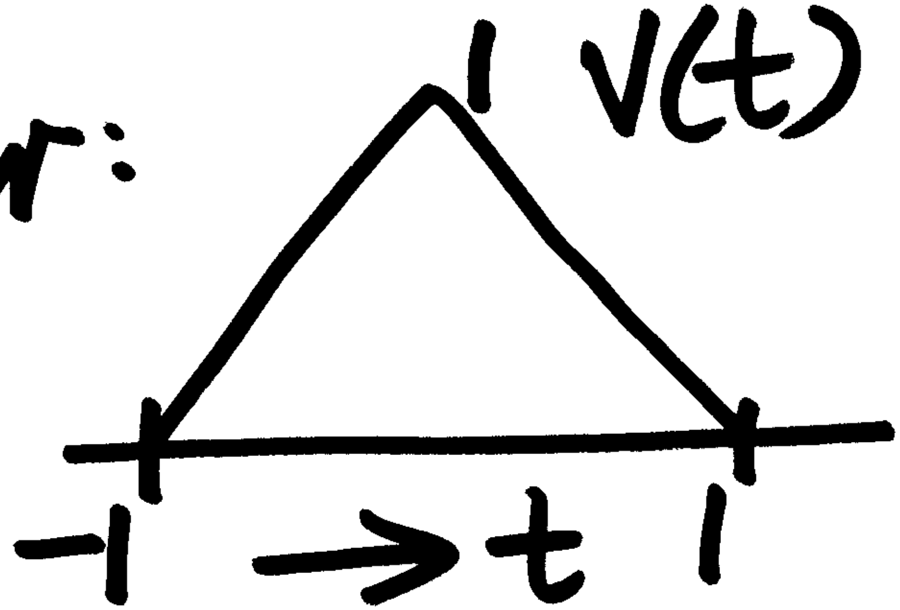
Finite frequency variance:

$$\frac{dv(t)}{dt} \in \mathcal{L}_2(\mathbb{R})$$

$$v(t) \in \mathcal{L}_2(\mathbb{R}).$$

Examples of windows:

Triangular:



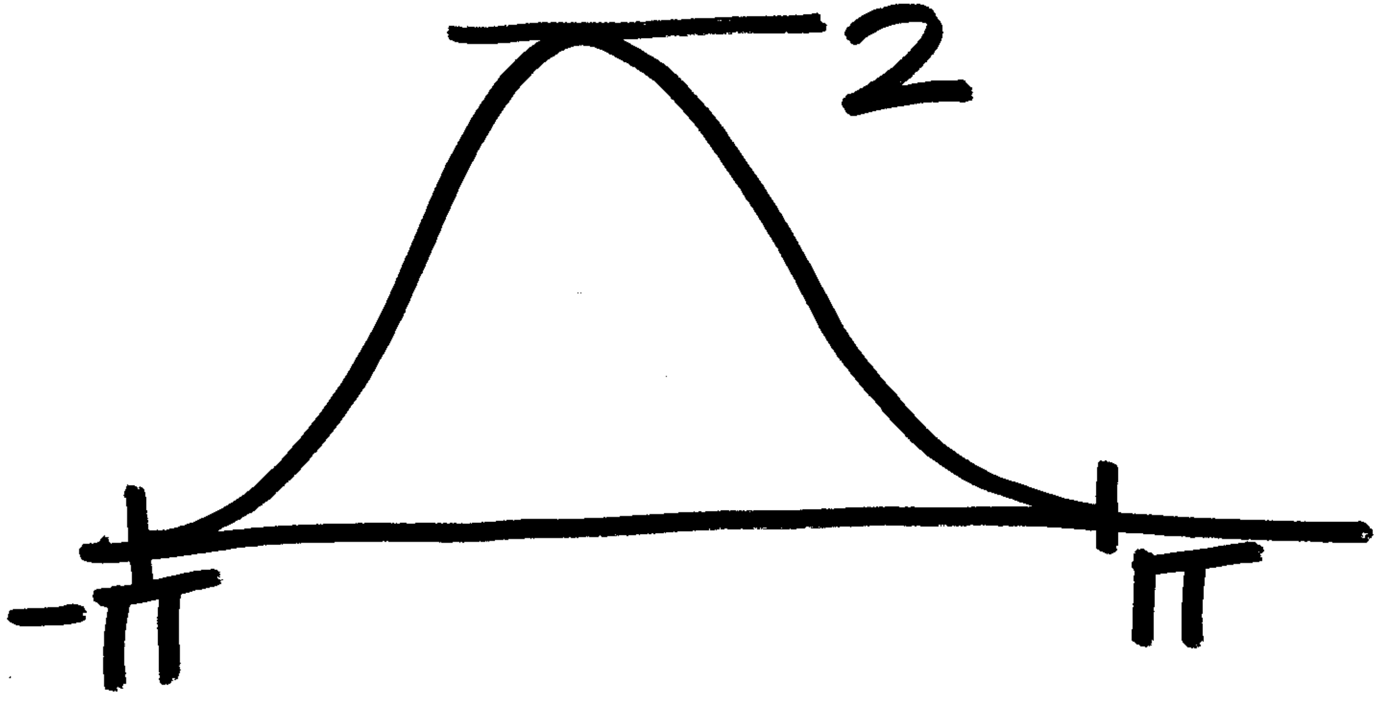
Gaussian:

$$v(t) = e^{-t^2/2}$$

Raised cosine

$(1 + \cos t)$

over $[-\pi, \pi]$



$x(t) \in \mathcal{L}_2(\mathbb{R})$

$v(t) \in \mathcal{L}_2(\mathbb{R})$

window.

STFT $\equiv$ Short Time Fourier Transform

$$STFT(x, V)(\tau_0, \Omega_0)$$

secondary arguments

primary arguments

= dot product of (inner product) $x(t)$ with $v(t-\tau_0)\, e^{j\Omega_0 t}$

$$= \int_{-\infty}^{+\infty} x(t)\,\overline{v(t-\tau_0)\,e^{-j\Omega_0 t}}\,dt$$

$$= \int_{-\infty}^{+\infty} x(t)\,\overline{v(t-\tau_0)}\,e^{-j\Omega_0 t}\,dt$$

Parseval's theorem

= inner product of Fourier Transforms of x, and $\underline{v(t-\tau_0)e^{j\Omega_0 t}}$

Fourier Transform

of $v(t-\tau_0)\, e^{j\Omega_0 t}$

$$= \int_{-\infty}^{+\infty} v(t-\tau_0)\, e^{j\Omega_0 t}\, e^{-j\Omega t}\, dt$$

Let $t - \tau_0 = \lambda$

$$= \int_{-\infty}^{+\infty} v(\lambda)\, e^{j(\Omega_0 - \Omega)(\lambda + \tau_0)}\, d\lambda$$

$$= e^{j(\Omega_0 - \Omega)\tau_0} \int_{-\infty}^{+\infty} \cdots\cdots$$

$$\cdots \underbrace{\int_{-\infty}^{+\infty} v(\lambda)\, e^{-j(\Omega-\Omega_0)\lambda}\, d\lambda}_{\hat{V}(\Omega-\Omega_0)}$$

Fourier Transform

of $v(t-\tau_0)\, e^{j\Omega_0 t}$

$$= \left\{ \hat{V}(\Omega - \Omega_0) \right\} \left\{ e^{j(\Omega_0 - \Omega)\tau_0} \right\}$$

$$\underline{STFT(x,v)(\tau_0,\Omega_0)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{x}(\Omega)\cdot\hat{v}(\Omega-\Omega_0)\, e^{j(\Omega_0-\Omega)\tau_0}\, d\Omega$$

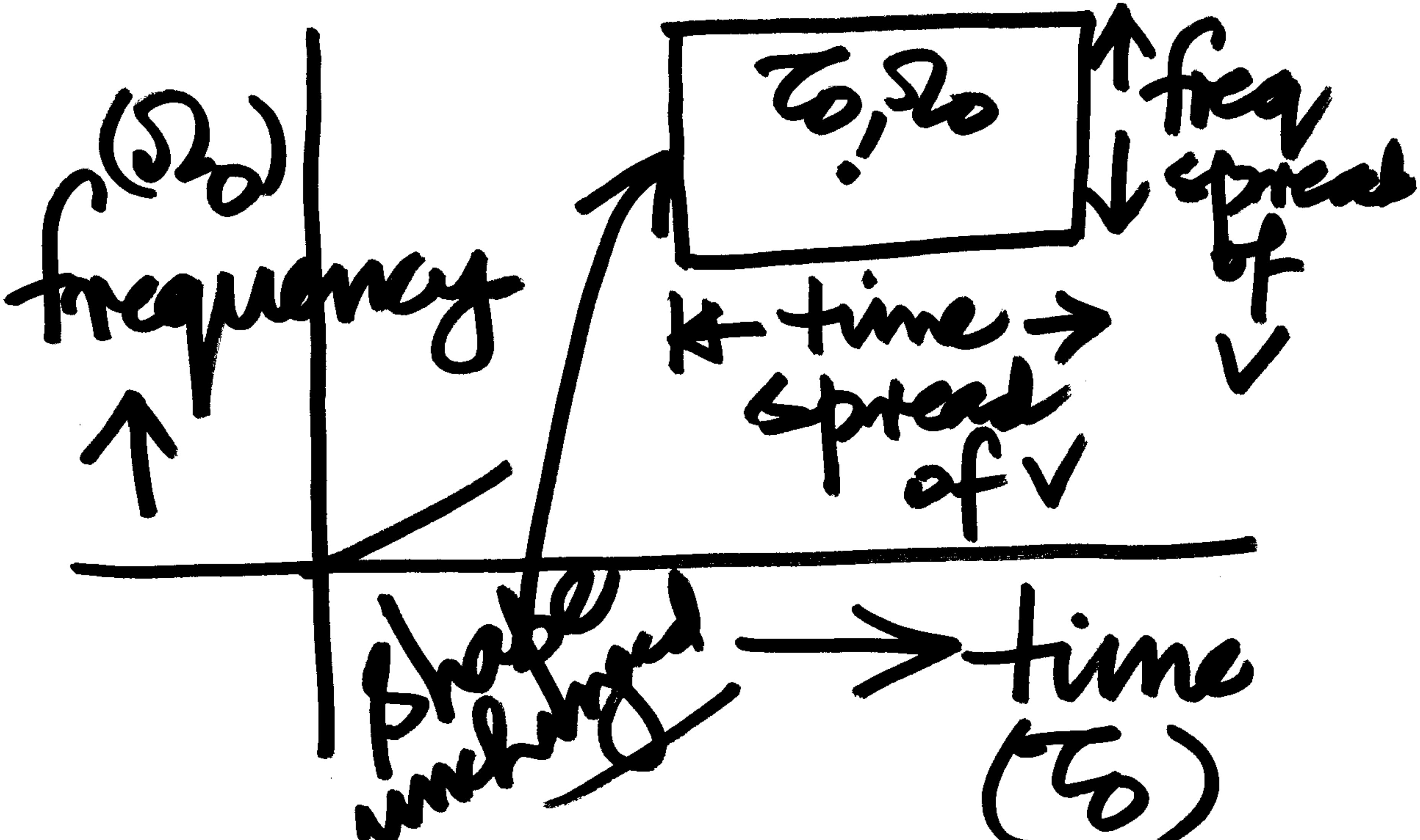
$$= e^{-j\Omega_0\tau_0} \cdots$$

$$\cdots \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{x}(\Omega)\overline{\hat{v}(\Omega-\Omega_0)}\, e^{j\Omega\tau_0}\, d\Omega$$

In the STFT, we have fixed-shape tiles:

τ_0: movement 'along time'

So: movement along 'angular frequency'



τ_0, Ω_0
freq spread of v
time spread of v
frequency (Ω_0)
shape unchanged
time (τ_0)

Continuous Wavelet Transform:

Inner product of $x(t)$ with $\psi\left(\frac{t-\tau_0}{s_0}\right)$

ψ: wavelet? $s_0 \in \mathbb{R}^+$

$$\left\langle x(t), \psi\left(\frac{t-\tau_0}{s_0}\right)\right\rangle$$

We need to normalize $\psi\left(\frac{t-\tau_0}{s_0}\right)$

i.e.

$$\overbrace{\frac{1}{+\sqrt{s_0}}\,\psi\left(\frac{t-\tau_0}{s_0}\right)}^{\text{Same norm as } \psi(\cdot)}$$

should be used

$$= \frac{1}{\sqrt{s_0}} \int x(t)\, \overline{\psi\left(\frac{t-\tau_0}{s_0}\right)}\, dt$$

$\psi(\cdot)$ is real, $\Rightarrow$

$\overline{}$ is redundant

Parseval's theorem:

= Inner product of $\hat{x}(\Omega)$ with FT of $\frac{1}{\sqrt{s_0}}\psi\left(\frac{t-\tau_0}{s_0}\right)$

Fourier Transform of

$$\frac{1}{\sqrt{s_0}}\psi\left(\frac{t-\tau_0}{s_0}\right)$$

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2 steps:

$$\psi(t) \rightarrow \frac{1}{\sqrt{s_0}}\psi\left(\frac{t}{s_0}\right)$$

Scaling independent variable

$$\frac{1}{\sqrt{s_0}}\psi\left(\frac{t}{s_0}\right) \longrightarrow$$

$$\sqrt{s_0}\,\hat{\psi}(s_0\Omega)$$

remember: $s_0 \in \mathbb{R}^+$

$$\frac{1}{\sqrt{s_0}}\,\psi\left(\frac{t-\tau_0}{s_0}\right)$$

Replace $t \leftarrow t-\tau_0$

$\Rightarrow$ multiply in F.T. by $e^{-j\Omega\tau_0}$

Fourier Transform

of $\frac{1}{\sqrt{s_0}}\psi\left(\frac{t-\tau_0}{s_0}\right)$

$$= \sqrt{s_0}\,\hat{\psi}(s_0\Omega)\,e^{-j\Omega\tau_0}$$

Continuous wavelet transform : CWT

$$\underbrace{CWT(x, \psi)}_{\text{Secondary arguments}}\underbrace{(\tau_0, s_0)}_{\text{Primary arguments}}$$

$$\mathrm{CWT}(x,\psi)(\tau_0, s_0)$$

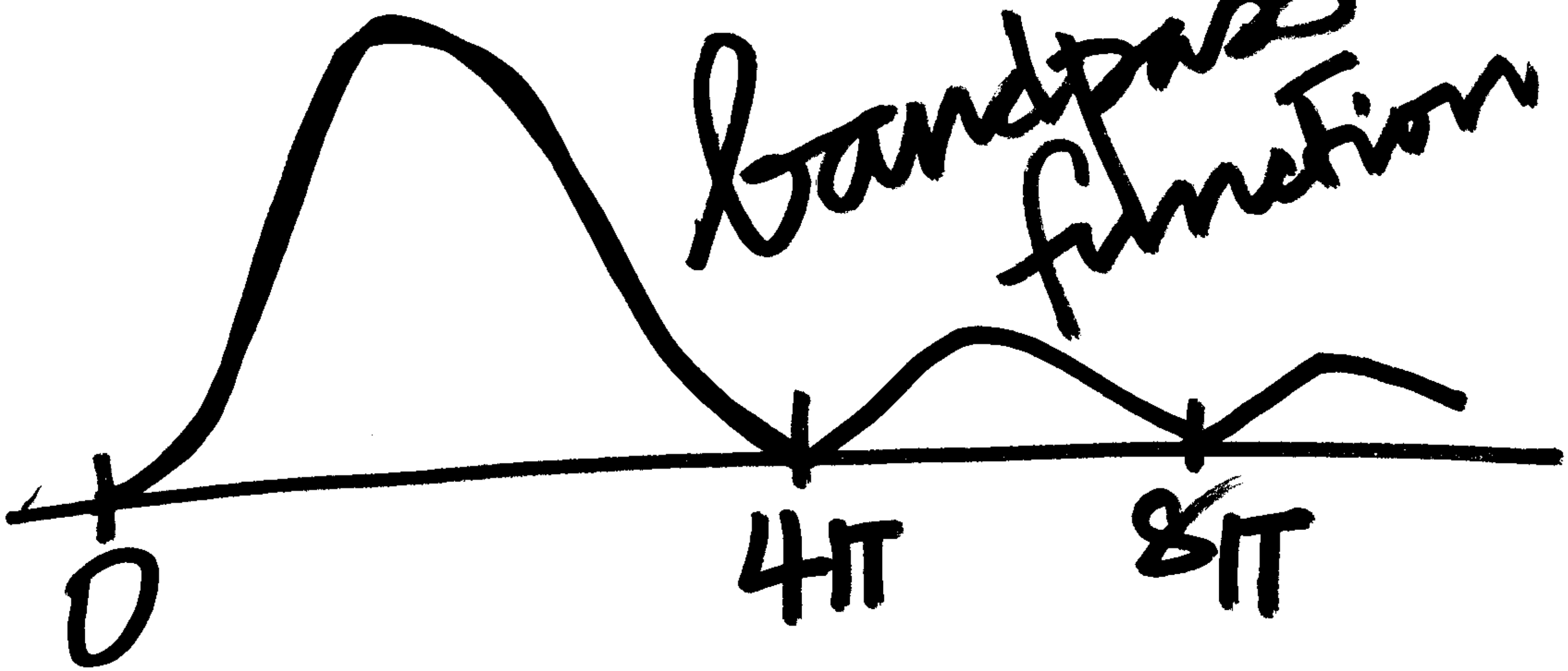
$$= \int_{-\infty}^{+\infty} x(t)\, \overline{\psi\left(\frac{t-\tau_0}{s_0}\right)} \frac{1}{\sqrt{s_0}}\, dt$$

or

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{x}(\Omega)\cdot\overline{e^{-j\Omega\tau_0}\sqrt{s_0}\,\hat{\psi}(s_0\Omega)}\,d\Omega$$

$$=\frac{1}{2\pi}\cdot\sqrt{s_0}\int_{-\infty}^{+\infty}\hat{x}(\Omega)\cdot\overline{\hat{\psi}(s_0\Omega)}\,e^{j\Omega\tau_0}\,d\Omega$$

Recall Haar $\Psi(\cdot)$:



If we accept bandpass $\psi(\cdot)$

Interpretation:

$x \to \boxed{\text{bandpass filter}} \to$ Output CWT as a fn of τ_0 at s_0

impulse response

$$\frac{1}{\sqrt{s_0}}\,\psi\!\left(\frac{-t}{s_0}\right)$$