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LECTURE 20

THE TIME FREQUENCY PLANE AND ITS TILINGS

Time bandwidth product

Time variance (σ_t^2) x Frequency variance (σ_Ω^2)

Time bandwidth product $\sigma_t^2 \sigma_\Omega^2$ for any $x \in \mathcal{L}_2(\mathbb{R}) \geq 0.25$

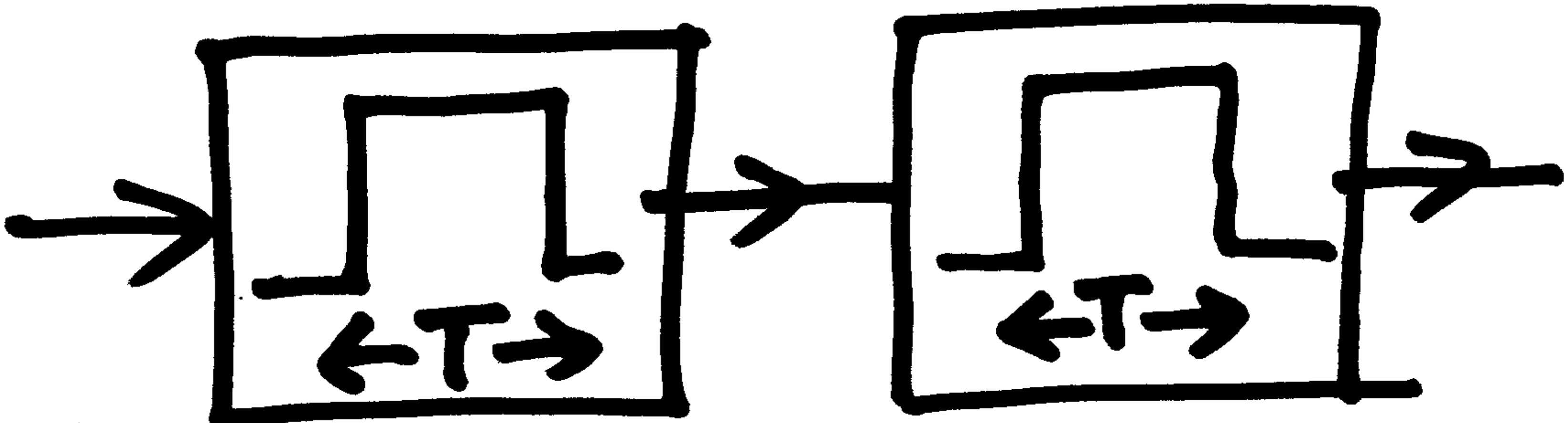
$$x(t) = e^{-t^2/2}$$

is an example of "optimal" function

More general optimal function

$$+\gamma_0 t^2/2$$

$$= e\ , \ \mathrm{Re}(\gamma_0) \text{ negative}$$



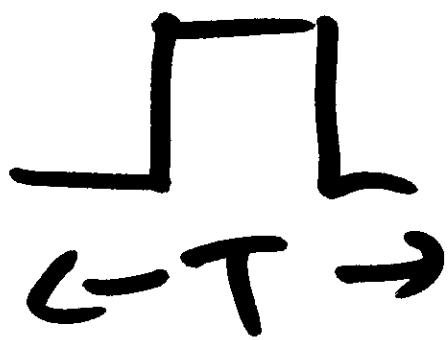
LSI system

LSI system

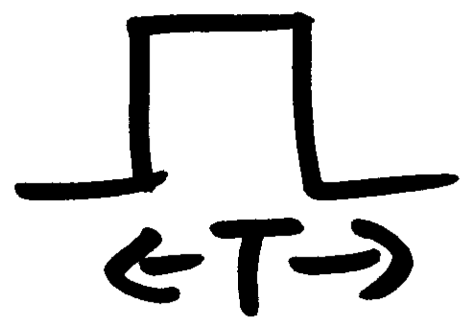
Composite LSI system

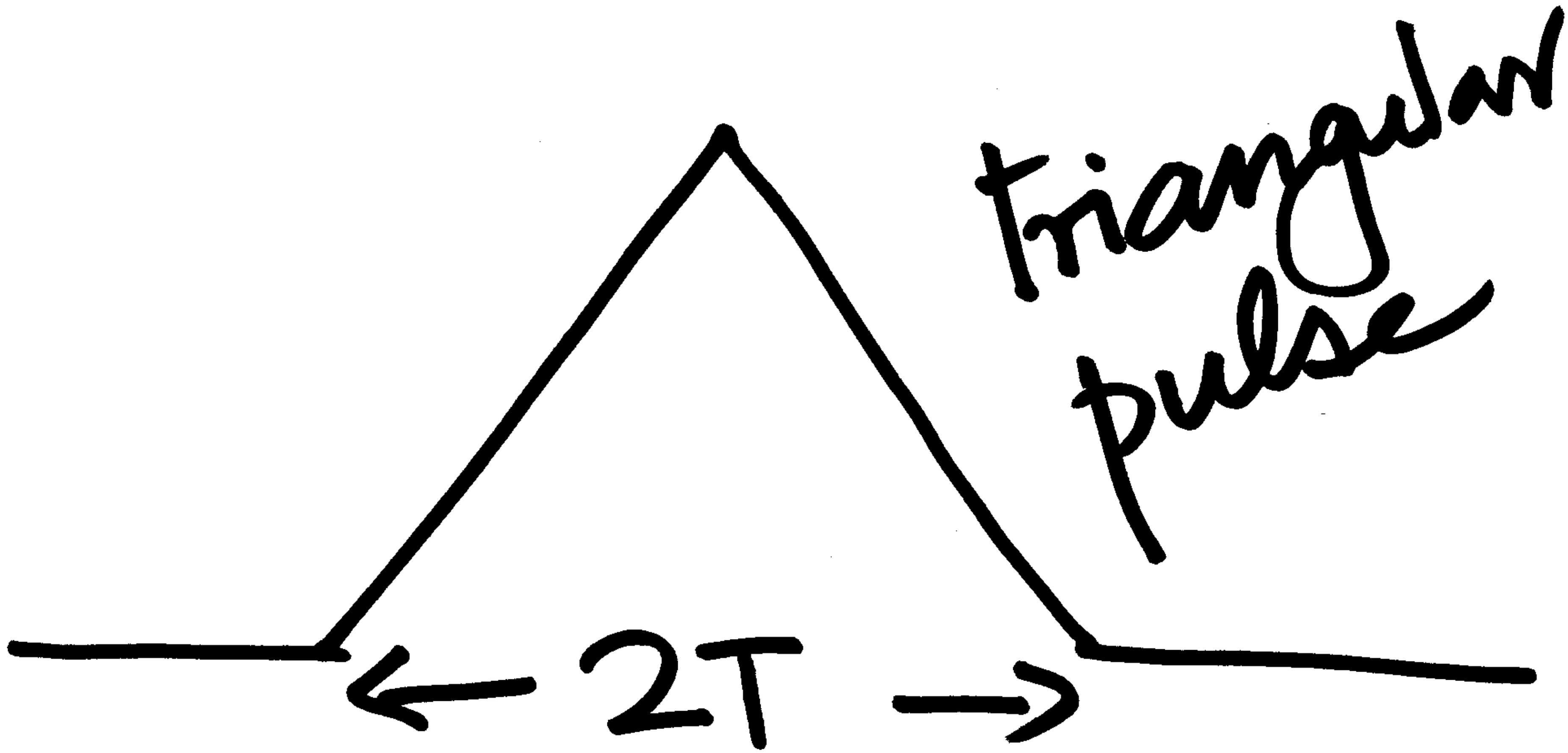
Impulse response of composite system

= Convolution of



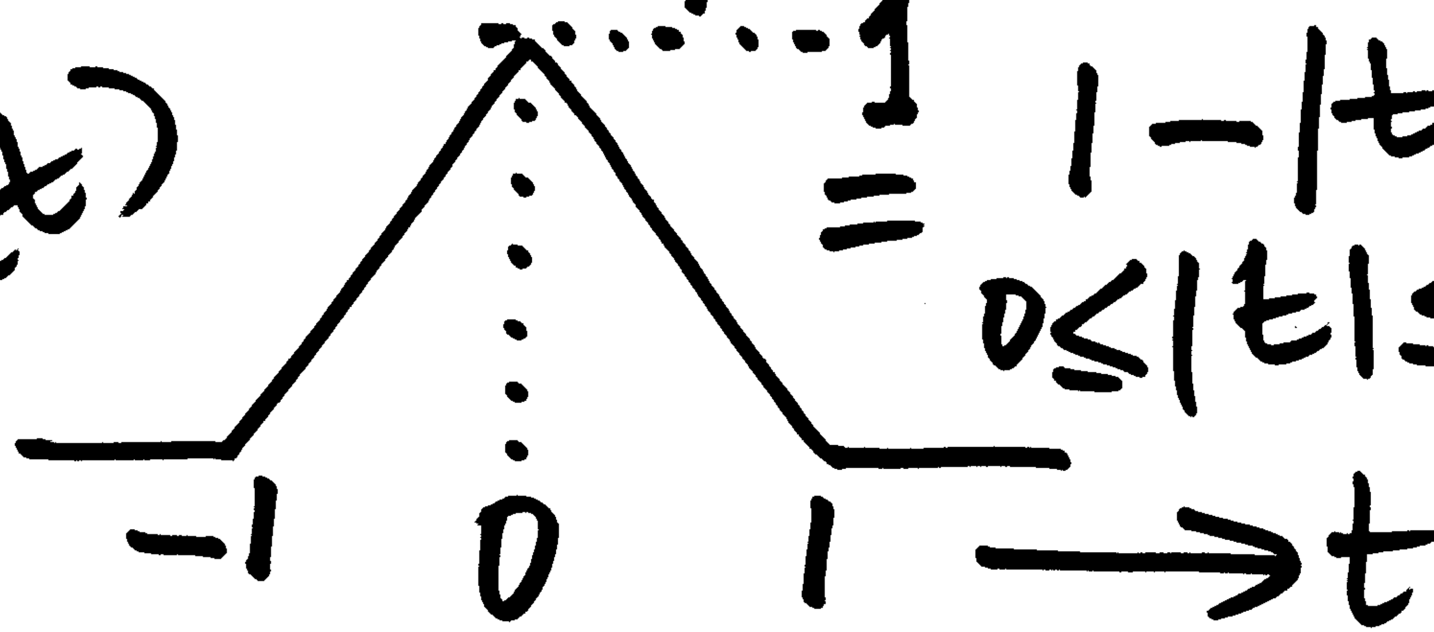
with




Triangular pulse
2T

Time bandwidth product of

$x(t) = 1 - |t|$, $0 \le |t| \le 1$



Time variance:

$x(t)$ centre $= 0$

$$\frac{\|t\,x(t)\|_2^2}{\|x(t)\|_2^2}$$

$$\|x(t)\|_2^2$$

$$= 2\int_0^1 (1-t)^2\,dt$$

$\lambda = 1-t$

$$\|x(t)\|_2^2 =$$

$$2 - \int_1^0 \lambda^2 \, d\lambda = 2 \int_0^1 \lambda^2 \, d\lambda = \frac{2}{3}$$

$$=$$

$$\|t\,x(t)\|_2^2$$

$$= 2\int_0^1 t^2(1-t)^2\,dt$$

using symmetry

$$= 2\int_0^1 t^2(1-2t+t^2)\,dt$$

$$= 2\int_0^1 (t^2-2t^3+t^4)\,dt$$

$$= 2\left\{\frac{t^3}{3} - 2\cdot\frac{t^4}{4} + \frac{t^5}{5}\right\}_0^1$$

$$= 2\left\{\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right\}$$

$$= 2 \cdot \left\{ \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right\}$$

$$= 2 \cdot \frac{10 - 15 + 6}{2 \times 15} = \frac{1}{15}$$

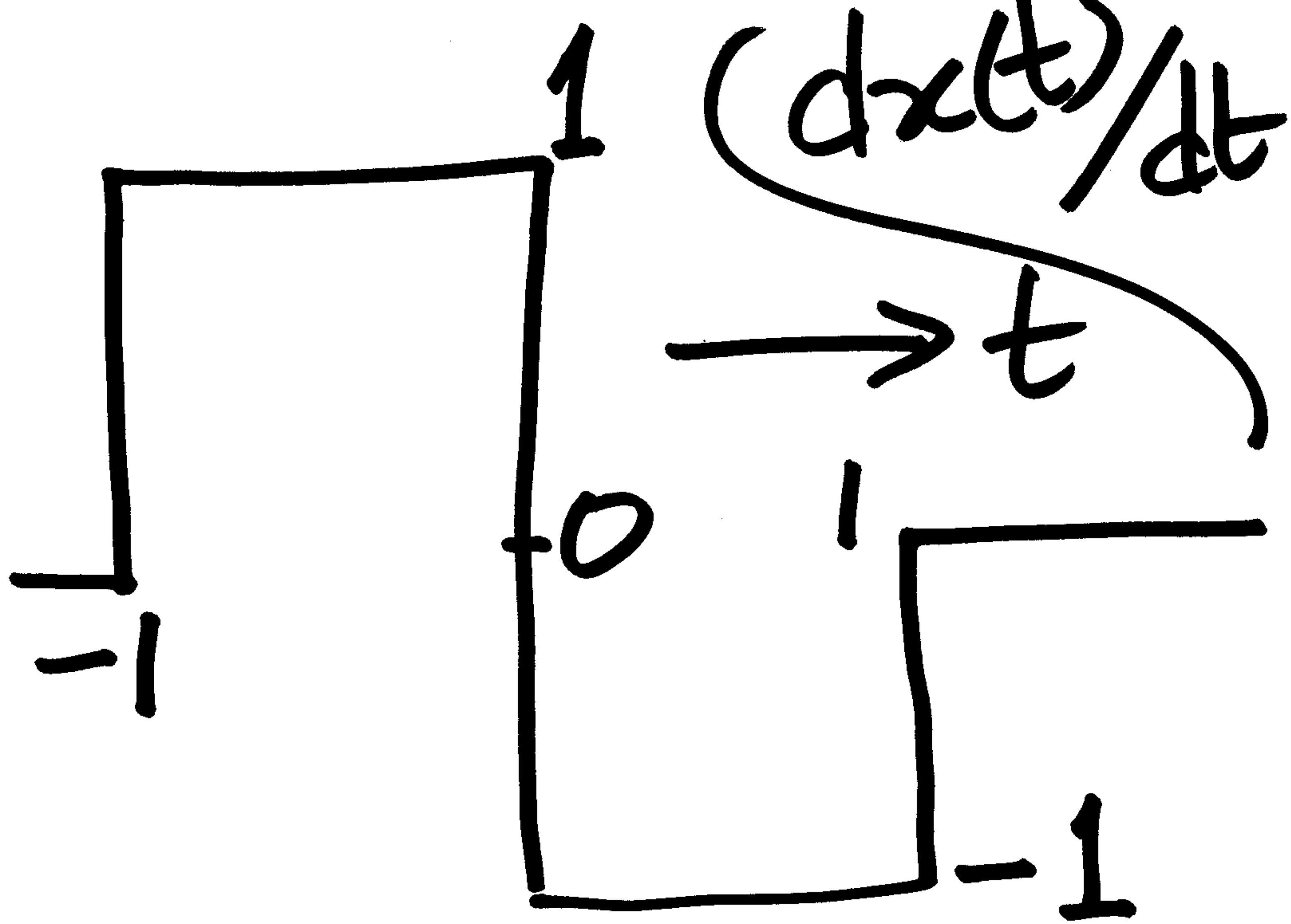
Time variance

$$= \frac{1/15}{2/3} = \frac{1}{15} \times \frac{3}{2}$$

$$= 0.1$$

Frequency variance

$$= \frac{\left\| \frac{dx(t)}{dt} \right\|_2^2}{\| x(t) \|_2^2}$$



1
(dx(t)/dt
t
0
1
-1
-1

$$\left\| \frac{dx(t)}{dt} \right\|_2^2 =$$

$$1^2 \times 1 + 1^2 \times 1$$

$$= 2$$

$$\|x\|_2^2 = 2/3$$

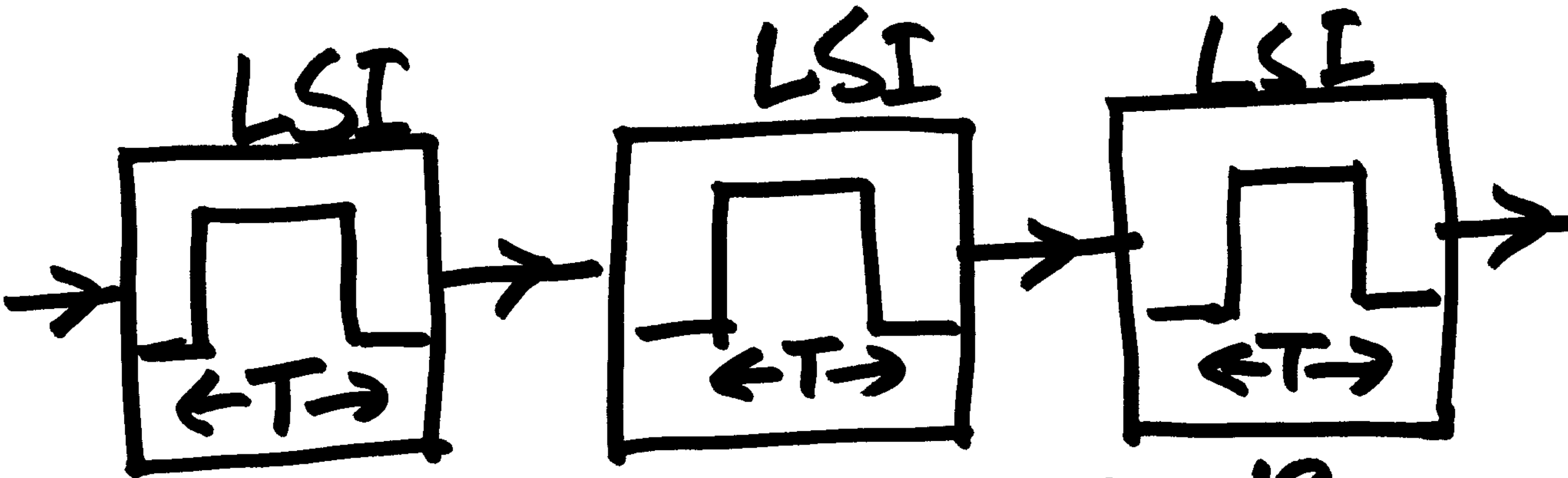
Frequency variance $= \dfrac{2}{2/3}$

$= 3$

Time bandwidth product

$$= \underset{\text{Time variance}}{0.1} \times \underset{\text{Frequency variance}}{3}$$

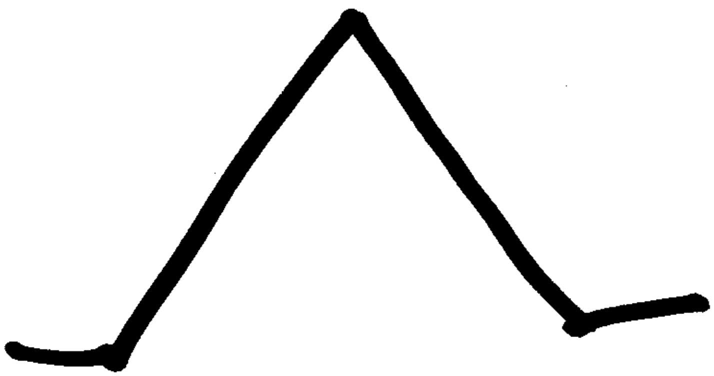
$$= 0.3$$



Exercise: (1) Evaluate the overall impulse response

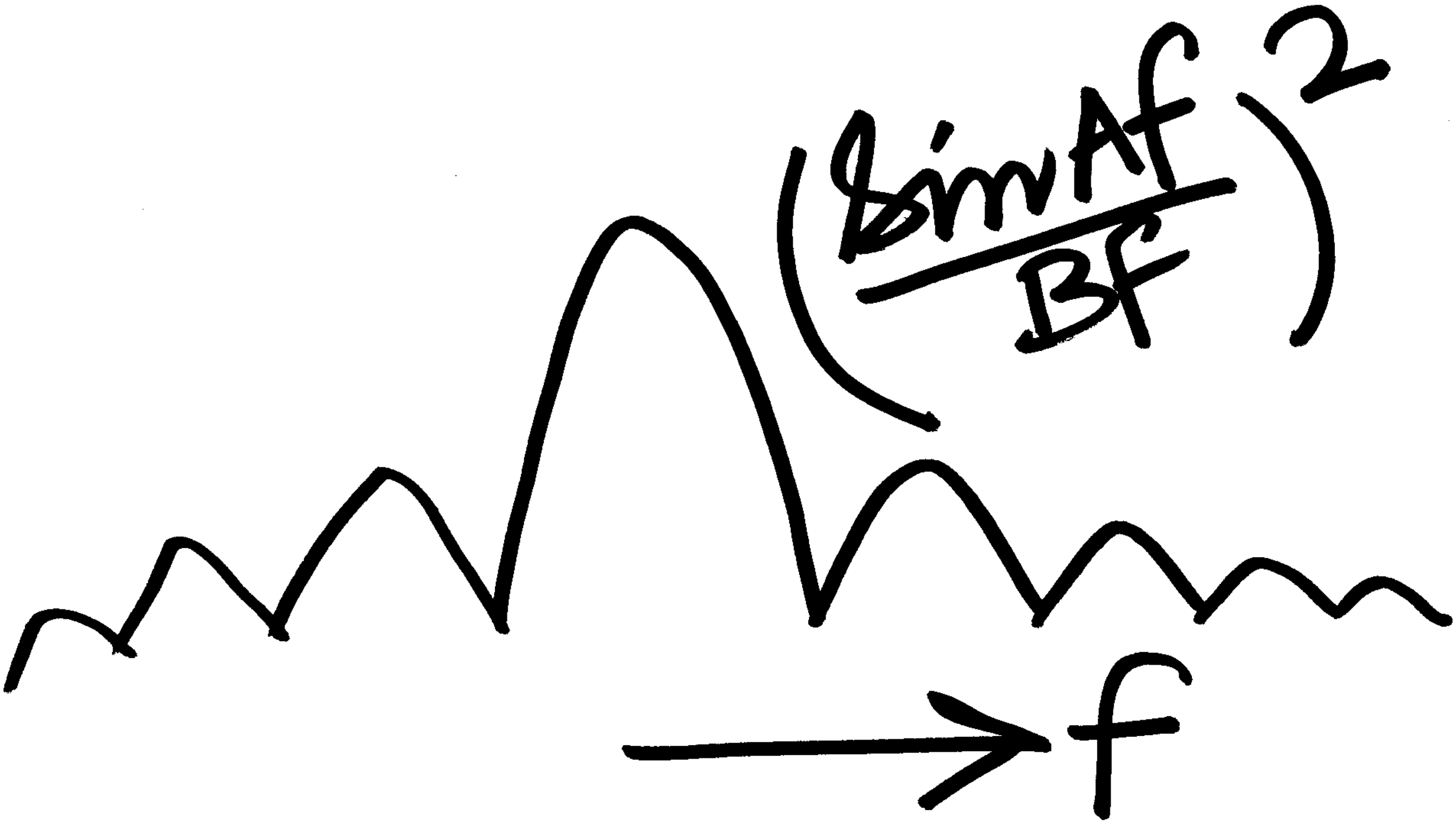
Exercise 2 :
Obtain the time bandwidth product of this impulse response

Fourier duality:

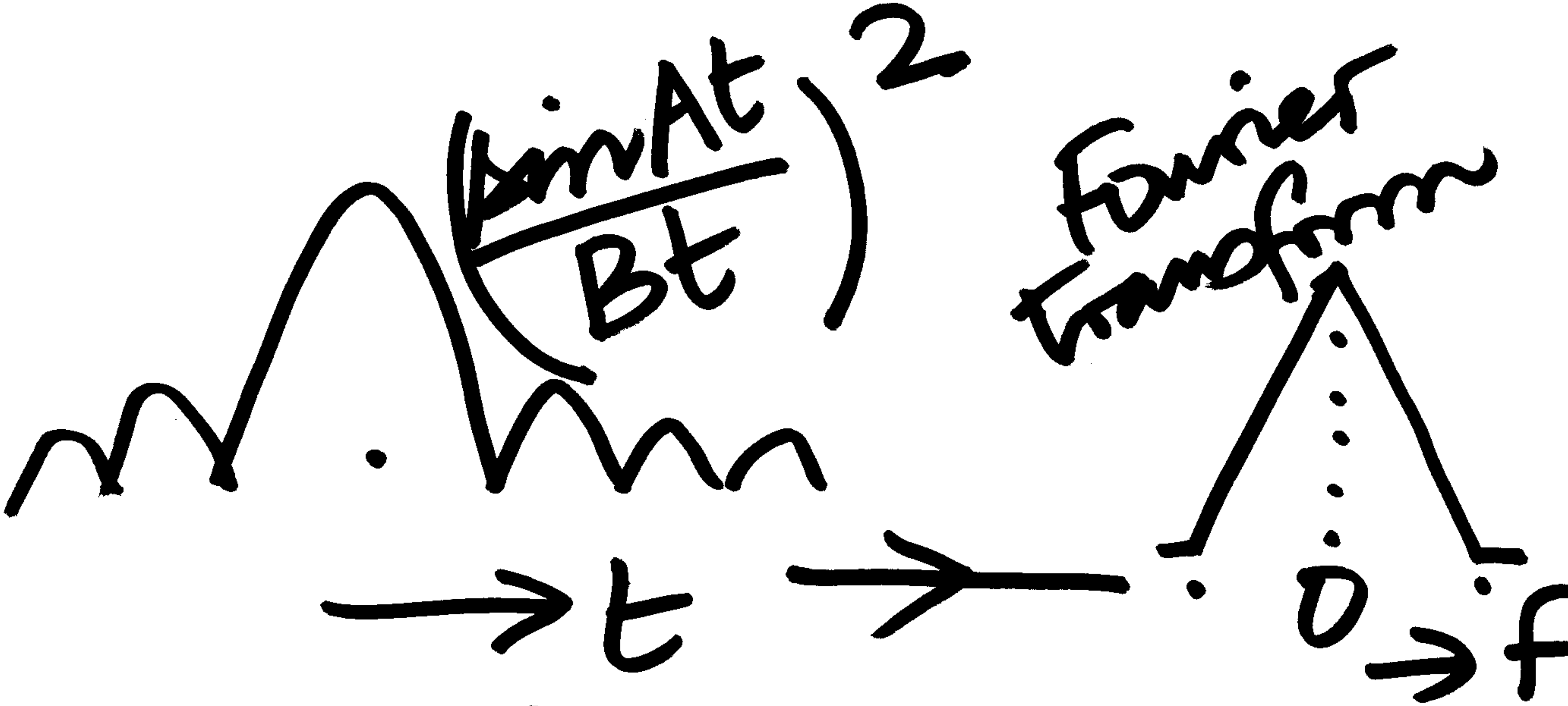


$\xrightarrow{\text{Fourier transform}}$ $\left(\dfrac{\sin Af}{Bf}\right)^2$

A, B constants



$\left(\frac{\sin Af}{Bf}\right)^2$
f

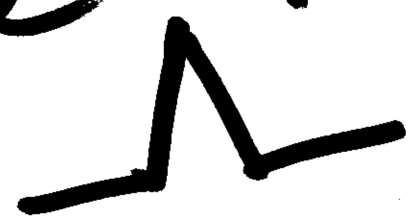


as a consequence of Fourier duality.

For the function

$$\frac{(\sin At)^2}{(Bt)^2},$$

time-variance = frequency variance of



Frequency variance = time variance of Λ

$\Rightarrow$ Time bandwidth product = <u>0.3</u>

The time bandwidth product is invariant to Fourier transformation

From this example, we see: it is possible to have two functions, one compactly supported and one NOT, with the SAME $\sigma_x^2 \sigma_\Omega^2$

"Time-Frequency Plane"

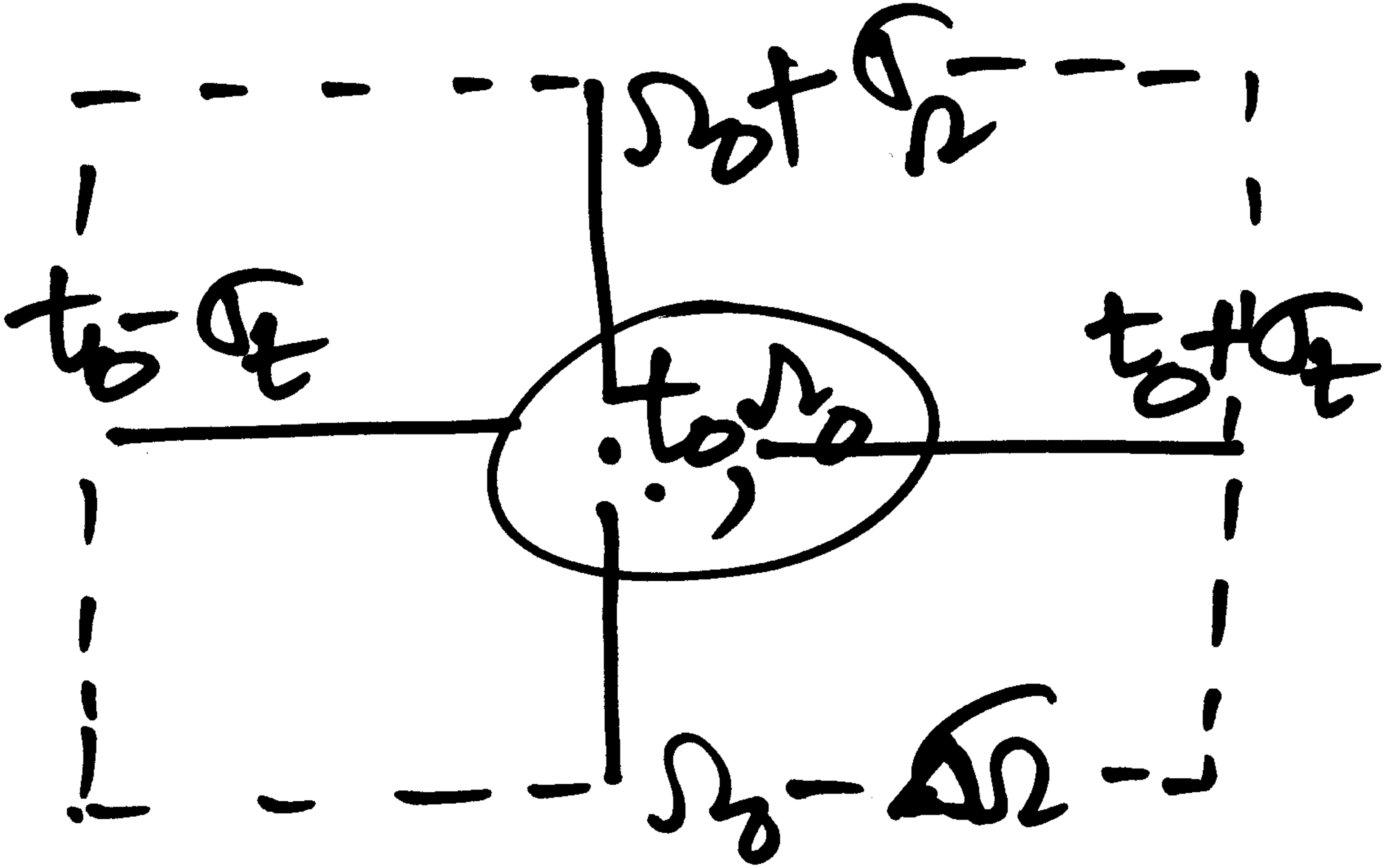
Frequency

time

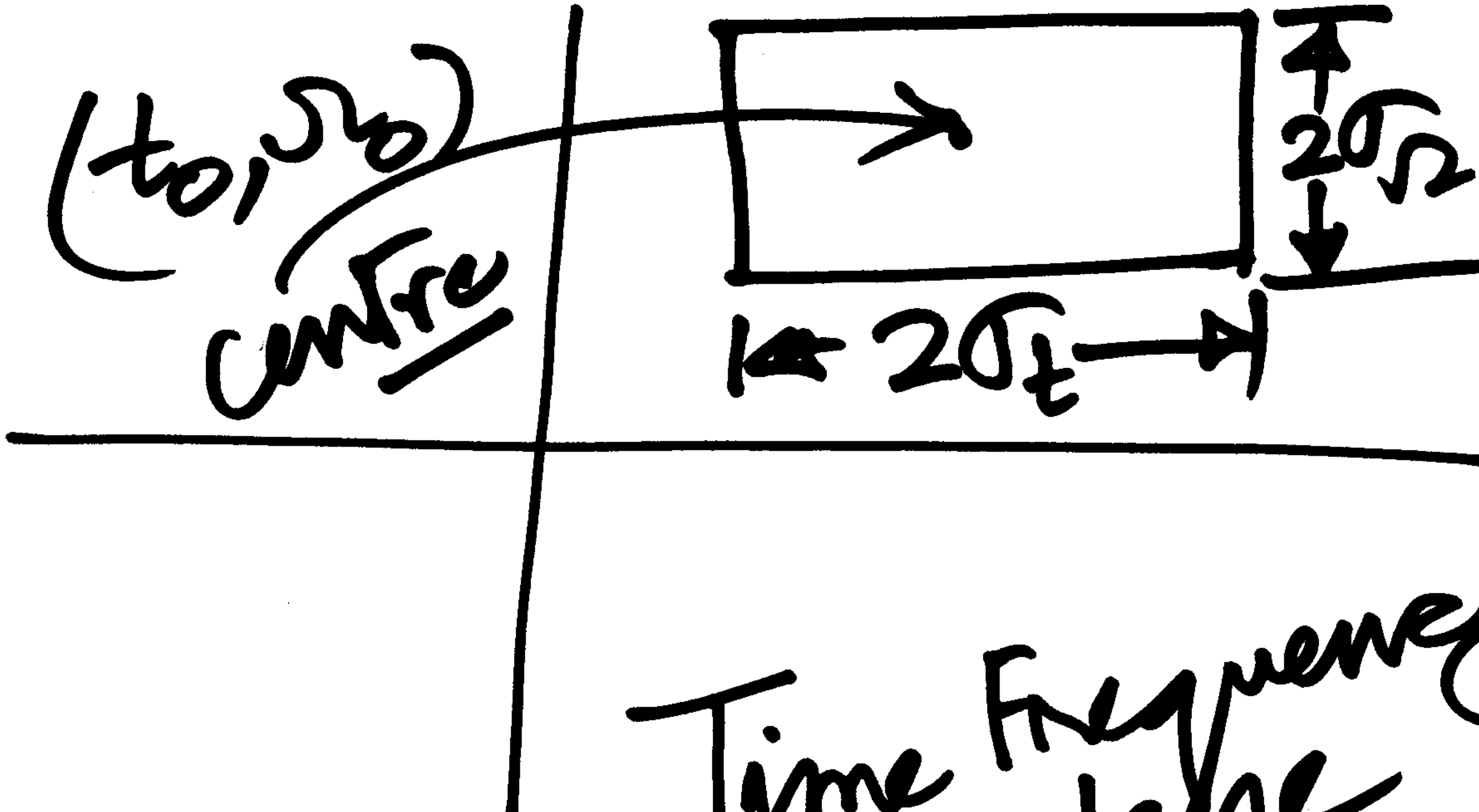
"Occupancy" of

$x(t) \in \mathcal{L}_2(\mathbb{R})$

in this time-frequency plane



$\Omega_0 + \sigma_\Omega$
$t_0 - \sigma_t$
t_0, Ω_0
$t_0 + \sigma_t$
$\Omega_0 - \sigma_\Omega$



Time Frequency plane

Uncertainty principle:
Rectangle area
cannot be smaller
than $2\sigma_t 2\sigma_\Omega$ $4\sigma_t\sigma_\Omega$
$= \geq 4\sqrt{0.25}$
$= 4 \times 0.5$

"Tiling" the time frequency plane: Covering this plane with rectangular "tiles" corresponding to such functions

Take any other function to be analyzed: $y(t)$

"Tool" function

$$= x(t)$$

$$\int_{-\infty}^{+\infty} y(t)\overline{x(t)}\,dt$$

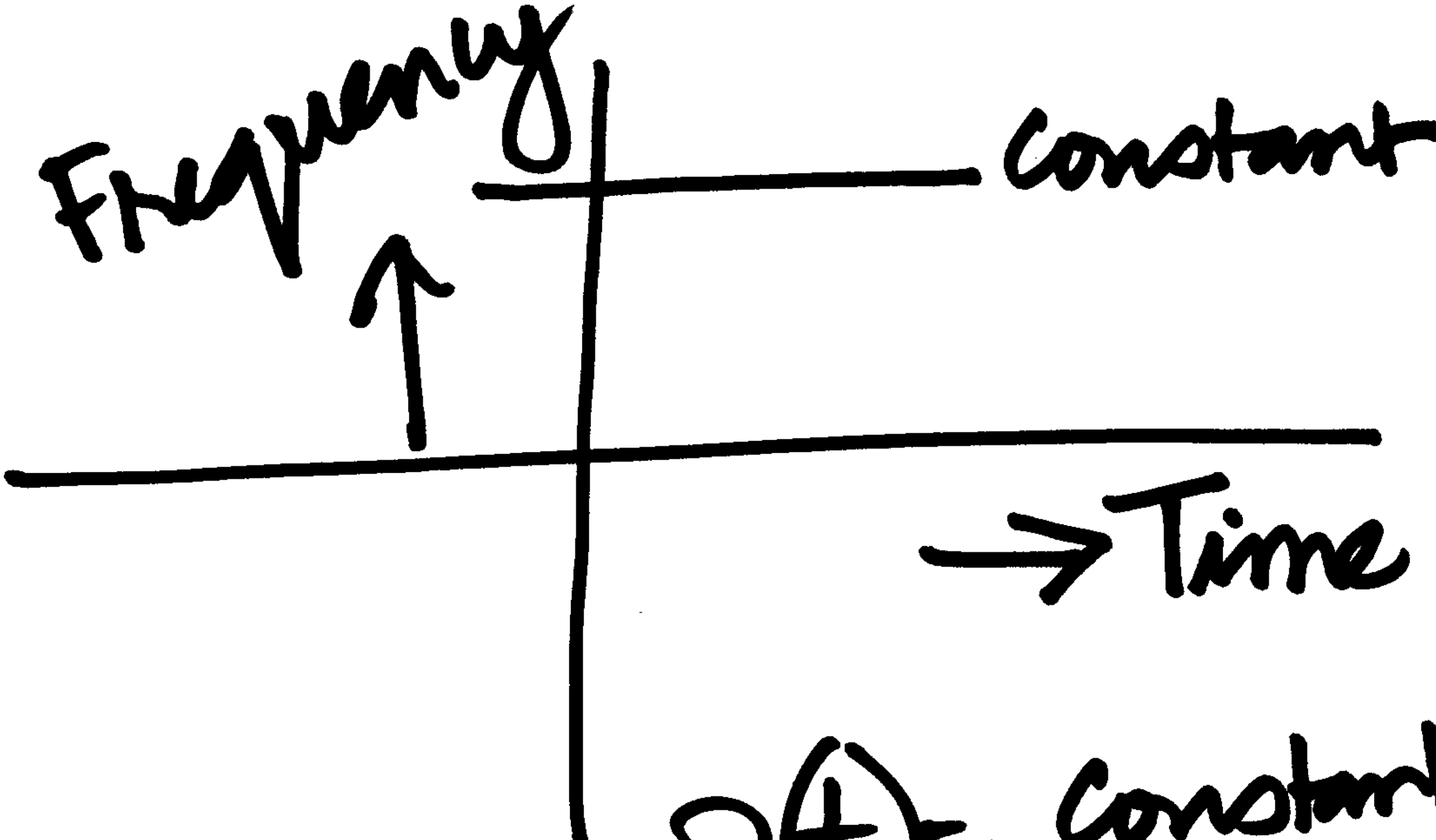
From Parseval's theorem

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(\Omega)\overline{X(\Omega)}\,d\Omega$$

"Chirp" function

$= \sin \Omega(t) \cdot t$

"instantaneous frequency"



$\Omega(t) = \text{constant}$