Prof Gadre
Lec NO. 19
9/2/10

LECTURE 19

EVALUATING AND BOUNDING $\sigma_t^2 \; \sigma_\Omega^2$

How small can

$\sigma_t^2 \sigma_\Omega^2$ be?

(Time bandwidth product)

Time bandwidth product

$$= \frac{\|t\,x(t)\|_2^2 \, \left\|\frac{dx(t)}{dt}\right\|_2^2}{\|x\|_2^2 \, \|x\|_2^2}$$

Numerator =

$$\|t x(t)\|_2^2 \left\| \frac{dx(t)}{dt} \right\|_2^2$$

Treat $t x(t)$ and $dx(t)/dt$ as 'vectors'

Call them $\vec{v}_1$ and $\vec{v}_2$.

$$\langle \vec{v}_1 \cdot \vec{v}_2 \rangle \equiv |\vec{v}_1||\vec{v}_2| \cos\theta$$

θ = angle bet $\vec{v}_1, \vec{v}_2$.

$$|\langle \vec{v}_1, \vec{v}_2 \rangle|^2 = |\vec{v}_1|^2 |\vec{v}_2|^2 \cos^2\theta$$

$$0 < \cos^2\theta < 1$$

$$|\langle \vec{v}_1 \cdot \vec{v}_2 \rangle|^2 \leq |\vec{v}_1|^2 |\vec{v}_2|^2$$

$\langle f_1, f_2 \rangle$

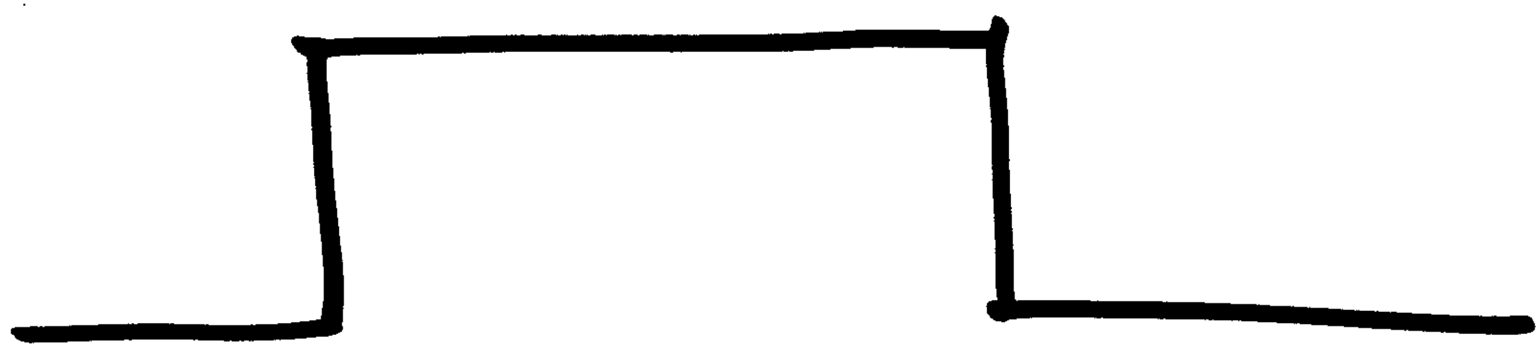
$f_1, f_2 \in \mathcal{L}_2(\mathbb{R})$

$$|\langle f_1, f_2 \rangle|^2 \leq \|f_1\|_2^2 \, \|f_2\|_2^2$$

Cauchy-Schwarz inequality

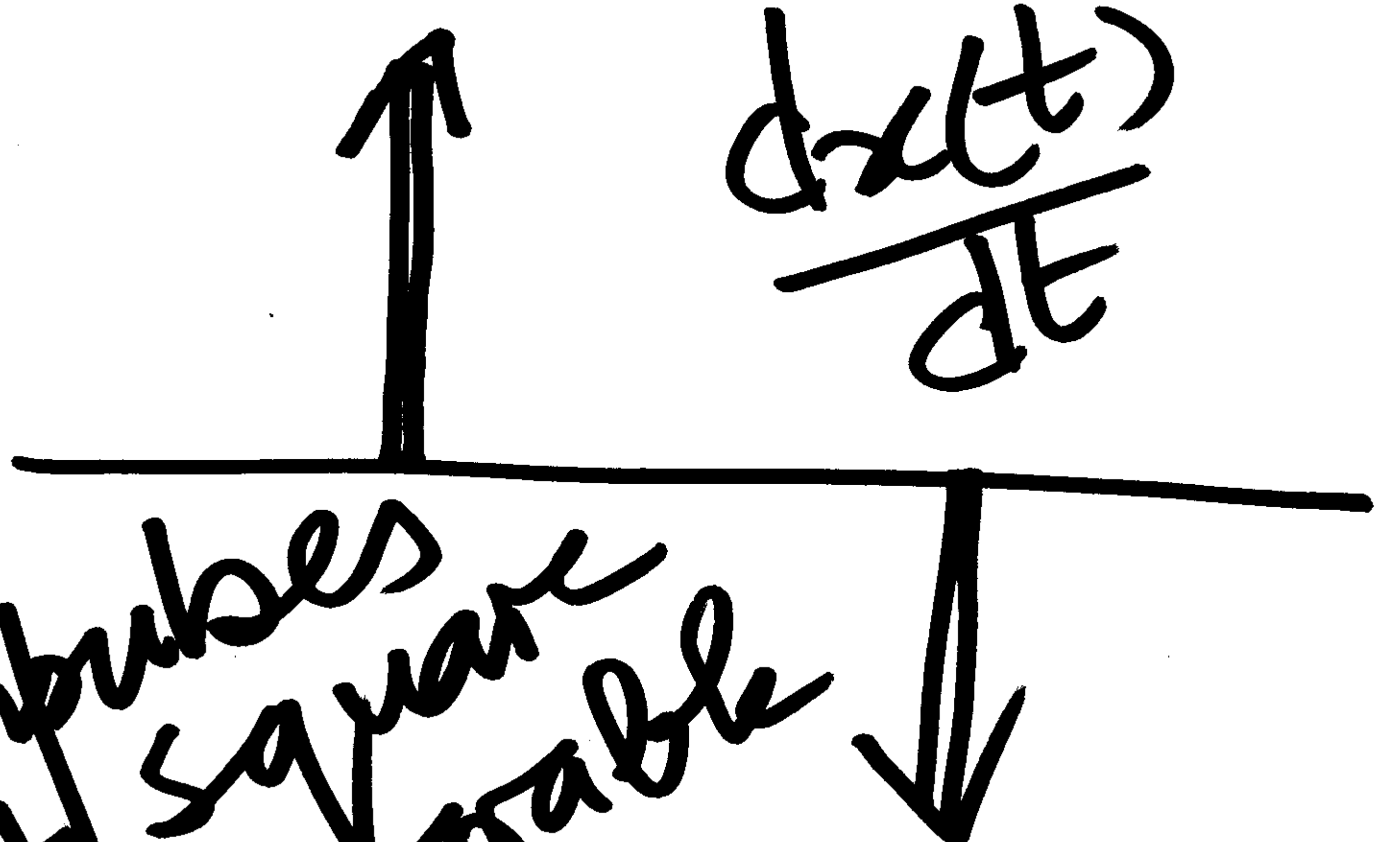
Consider $x(t)$ so that $t\,x(t) \in \mathcal{L}_2(\mathbb{R})$

$$\frac{dx(t)}{dt} \in \mathcal{L}_2(\mathbb{R})$$

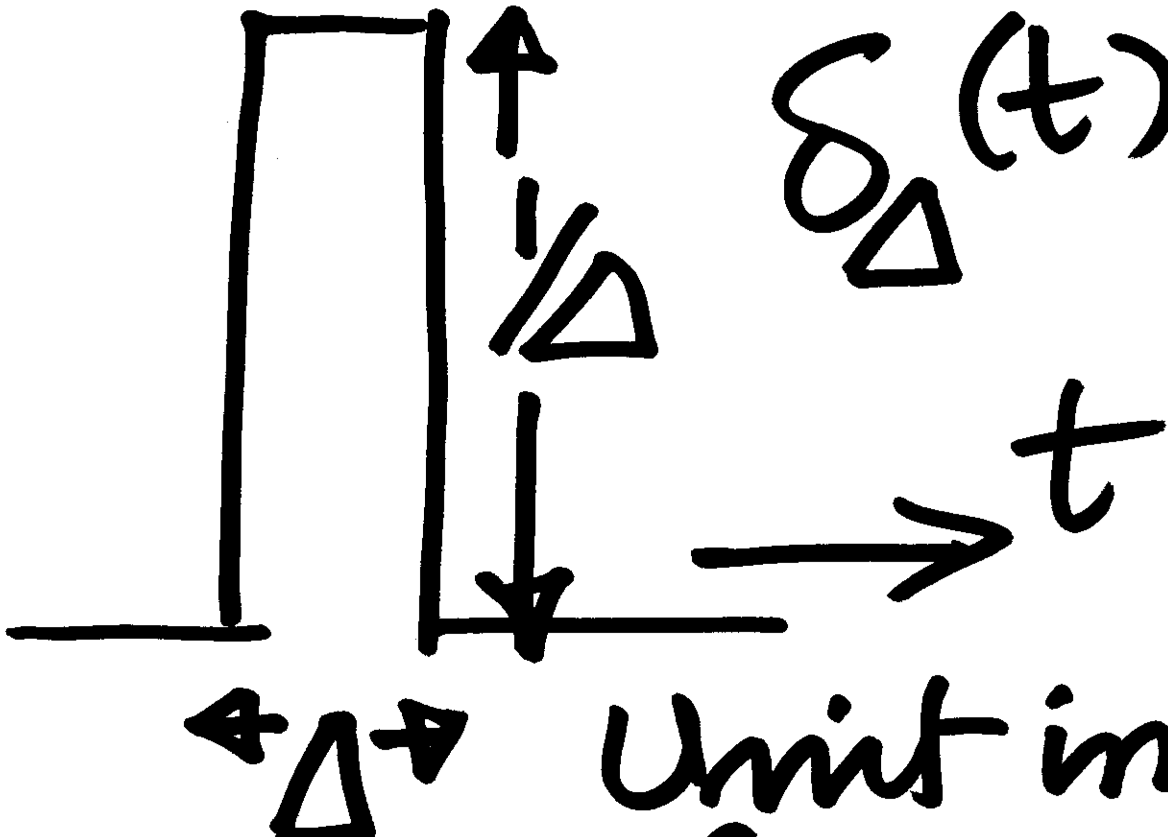


Not true for this!

$$\frac{dx(t)}{dt} \notin \mathcal{L}_2(\mathbb{R})$$

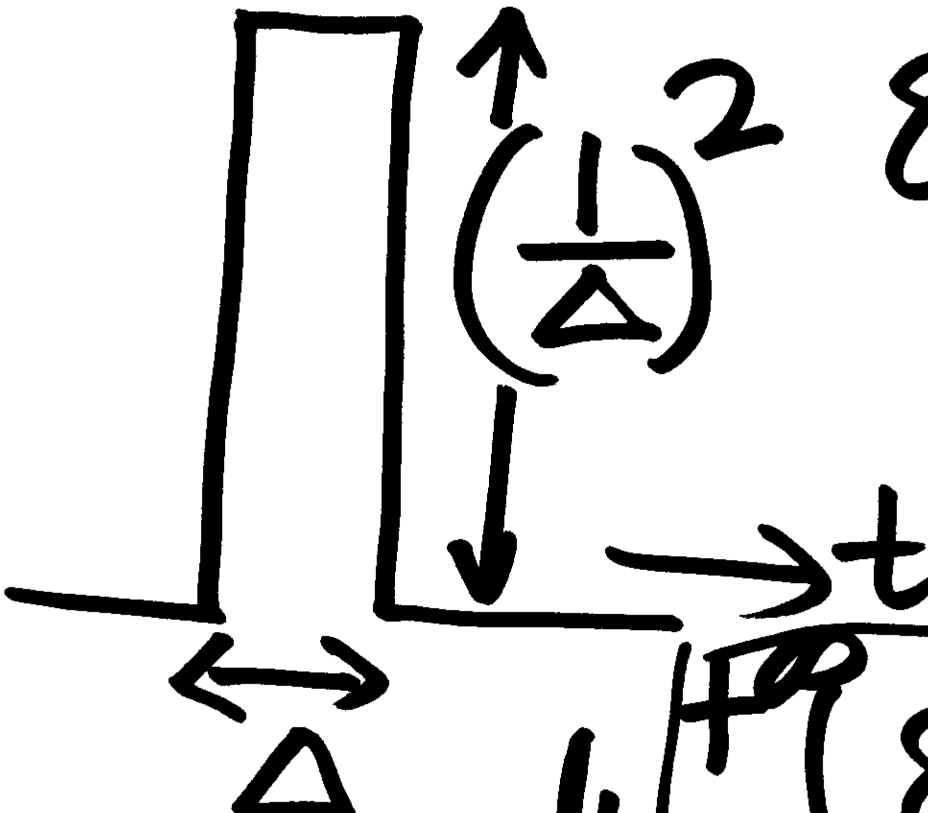


Impulses not square integrable



Unit impulse

$= \mathrm{Lt}_{\Delta \to 0}\, \delta_\Delta(t)$



$\delta_\Delta^2(t)$

$$\int_{-\infty}^{+\infty} \delta_\Delta^2(t)\,dt = \frac{1}{\Delta}$$

dt

$\Delta \to 0$ diverges

The impulse is not square integrable (infinite energy)

Using Cauchy Schwarz inequality

$$\|t\,x(t)\|_2^2 \, \left\| \frac{dx(t)}{dt} \right\|_2^2 \cdots$$

$\cdots$

$$\geq \left| \left\langle t\,x(t), \frac{dx(t)}{dt} \right\rangle \right|^2$$

$$\int_{-\infty}^{+\infty} t\,x(t) \cdot \overline{\frac{dx(t)}{dt}}\, dt$$

(Complex no)

Let that complex no be z

$$|z|^2 \geq |\operatorname{Re} z|^2$$

$$\text{or } |\operatorname{Im} z|^2$$

Numerator of time bandwidth product $\geq$

$$\left| \mathrm{Re} \left\{ \int_{-\infty}^{+\infty} t\, x(t) \frac{d\overline{x(t)}}{dt}\, dt \right\} \right|$$

Because t is real

$$\overline{\frac{dx(t)}{dt}} = \frac{d\,\overline{x(t)}}{dt}$$

numerator, thus $\geq$

$$\left| \mathrm{Re} \int_{-\infty}^{+\infty} t \, x(t) \, \frac{d\overline{x(t)}}{dt} \cdot dt \right|^2$$

This =

$$\left|\int_{-\infty}^{+\infty} t \,\mathrm{Re}\left\{x(t)\frac{d\overline{x(t)}}{dt}\right\} dt\right|^2$$

$$\mathrm{Re}\left\{x(t)\frac{d\overline{x(t)}}{dt}\right\}$$

$$=\frac{1}{2}\left\{x(t)\frac{d\overline{x(t)}}{dt}+\overline{x}(t)\frac{dx(t)}{dt}\right\}$$

$$= \frac{1}{2} \frac{d}{dt} \left\{ x(t) \overline{x(t)} \right\}$$

$$= \frac{1}{2} \frac{d}{dt} |x(t)|^2$$

Numerator $\geq$

$$\left| \frac{1}{2} \int_{-\infty}^{+\infty} t \cdot \frac{d}{dt} |x(t)|^2 \, dt \right|^2$$

evaluate by parts

$$\int_{-\infty}^{+\infty} t \frac{d}{dt}|x(t)|^2 \, dt$$

$$= t|x(t)|^2 \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

$$\parallel \qquad \qquad \parallel$$

$$t|x(t)|^2 \Big|_{-\infty}^{+\infty}$$

We have agreed

$$\int_{-\infty}^{+\infty} t^2 |x(t)|^2 \, dt$$ finite

In order that this integral converge

$$t^2 |x(t)|^2 \longrightarrow 0$$

as $t \to +\infty$

and $t \to -\infty$

$t > 1$ (certainly so as $|t| \to \infty$)

$t^2 > t$

$$t^2|x(t)|^2 \to 0$$

as $t \to +$ or $-\infty$

guarantees

$$t|x(t)|^2 \to 0$$

as $t \to +$ or $-\infty$.

We are thus left only with

$$-\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = -\|x\|_2^2$$

Numerator $\geqslant$

$$\left|\frac{1}{2}\left(-\|x\|_2^2\right)\right|^2$$

$$= \frac{1}{4}\|x\|_2^2\|x\|_2^2$$

Thus: Time bandwidth product

$$\geq \frac{\frac{1}{4}\|x\|_2^4}{\|x\|_2^4} = \frac{1}{4}$$

$$= \underline{0.25}$$

Time variance
involves $\|t\,x(t)\|_2^2$

Frequency variance
involves $\left\|\frac{dx(t)}{dt}\right\|_2^2$

"Optimal" function in the sense of time bandwidth product: $\bar{\imath}\bar{\imath}$
(means)

The Cauchy Schwarz inequality becomes an <u>equality</u>.

That is <u>"$\cos^2\theta$" = 1</u>

We need "vectors"
$t\,x(t)$ and $\frac{dx(t)}{dt}$
to be "COLLINEAR"
(linearly <u>dependent</u>)

Thus:

$$\frac{dx(t)}{dt} = \gamma_0 t x(t)$$

<u>Solve</u>

$$\frac{dx}{x} = \gamma_0 t \, dt$$

Integrate:

$$\int \frac{dx}{x} = e^{\left\{ b_0 \frac{t^2}{2} + c_0 \right\}}$$

$$e^{\ln x}$$

Const of integ

$$x(t) = \underbrace{e^{c_0}}_{\text{Constant } \tilde{c}_0} \cdot e^{\gamma_0 \frac{t^2}{2}}$$

$$x(t) = \tilde{c}_0 \cdot e^{\gamma_0 \frac{t^2}{2}}$$

$$x(t) \in \mathcal{L}_2(\mathbb{R})$$

$$|x(t)|^2$$

$$= |\tilde{c}_0|^2 \left| e^{\gamma t^2/2} \right|$$

to be integrable

Possible only if ζ_0 has a <u>negative</u> real part

One function which is 'optimal' is with $\gamma_0 = -1$

$\tau_0 = 1$

One 'optimal' function, i.e. with time bandwidth product = 0.25

is $e^{-\frac{t^2}{2}}$

Gaussian.