LECTURE 18

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TIME BANDWIDTH PRODUCT – UNCERTAINTY BOUND

$x(t) \in \mathcal{L}_2(\mathbb{R})$

Time and frequency variances

Time centre

$$t_0 = \frac{\int_{-\infty}^{+\infty} t\,|x(t)|^2\,dt}{\|x\|_2^2}$$

Time variance

$$= \frac{\int_{-\infty}^{+\infty} (t-t_0)^2 |x(t)|^2 \, dt}{\int_{-\infty}^{+\infty} |x(t)|^2 \, dt}$$

Frequency centre

$$\Omega_0 = \frac{\displaystyle\int_{-\infty}^{+\infty} \Omega \, |\hat{x}(\Omega)|^2 \, d\Omega}{\|\hat{x}\|_2^2}$$

If $x(t)$ is real

$|\hat{x}(\Omega)|^2$ is an even function of Ω

$\Rightarrow \Omega_0 = 0.$

Frequency variance

$$\sigma_\Omega^2 = \frac{\int_{-\infty}^{+\infty} (\Omega - \Omega_0)^2 |\hat{x}(\Omega)|^2 d\Omega}{\int_{-\infty}^{+\infty} |\hat{x}(\Omega)|^2 d\Omega}$$

Effect of shifting in time:

$$y(t) = x(t - t_1).$$

Exercise: Show:

Time centre of

$$y = t_0 + t_1$$

Time variance of

y = Time variance of x

$$y(t) = x(t - t_1)$$

$$\hat{y}(\Omega) = e^{-j\Omega t_1} \cdot \hat{x}(\Omega)$$

$$|\hat{y}(\Omega)|^2 = |\hat{x}(\Omega)|^2$$

Shifting in the frequency domain

or

modulation!

$$y(t) = e^{j\Omega_1 t} x(t)$$

$$\hat{y}(\Omega) = \hat{x}(\Omega - \Omega_1)$$

Exercise:

Time centre of y

= Time centre of x

$$|y(t)|^2 = |x(t)|^2$$

Time variance of

y = Time variance of x !

Frequency centre of y = (Frequency centre of x) + Ω_1

Frequency variance of y

= Frequency variance of x

Time bandwidth product

$$= \sigma_t^2 \sigma_\Omega^2$$

Time bandwidth product =
Time variance
× Frequency variance

The time bandwidth product is unchanged when we shift in time or frequency (modulate)

$$y(t) = c_0\, x(t)$$

c_0 : constant

$\neq 0$

Centres or variances always have a ratio of $|x|^2$ or $|\hat{x}|^2$ involved.

$$|y|^2 = |G_0|^2 |x|^2$$

$$|\hat{y}|^2 \quad |G_0|^2 |\hat{x}|^2$$

$|c_0|^2$ term (>0)
cancels from
the ratio

Scaling dependent variable (multiplying signal by constant) leaves centres and variances <u>UNCHANGE</u>

Scaling the independent variable:

$$y(t) = x(\alpha t)$$
$$\alpha \in \mathbb{R}, \alpha \neq 0$$

$$x(t) \longrightarrow \hat{x}(\Omega)$$

$$x(\alpha t) \longrightarrow \frac{1}{|\alpha|}\hat{x}\left(\frac{\Omega}{\alpha}\right)$$

$\alpha \in \mathbb{R}$

$\alpha \neq 0$

Time domain:

Time centre:

$$y(t) = x(\alpha t)$$

$$\alpha \in \mathbb{R},\ \alpha \neq 0$$

$$\|y\|_2^2 = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt$$

$$= \int_{-\infty}^{+\infty} |x(\alpha t)|^2 \, dt$$

$$\lambda = \alpha t$$

$$d\lambda = \alpha \, dt$$

$$\Rightarrow dt = \frac{1}{\alpha} d\lambda$$

$$\Rightarrow \|y\|_2^2 = \frac{1}{|\alpha|} \int_{-\infty}^{+\infty} |x(\lambda)|^2 d\lambda$$

$$\|y\|_2^2 = \frac{1}{|\alpha|} \cdot \|x\|_2^2$$

Time centre of y

$$= \frac{\int_{-\infty}^{+\infty} t \cdot |y(t)|^2 \, dt}{\int_{-\infty}^{+\infty} |y(t)|^2 \, dt}$$

Exercise:

Time centre of y

$$= \frac{1}{[\alpha]} \cdot \text{Time centre of } x$$

not modulus.

Time variance of y

$$= \frac{1}{|\alpha|^2}\left(\text{Time variance of } x\right)$$

Frequency domain:

$$x(\alpha t) \longrightarrow \frac{1}{|\alpha|}\hat{x}\left(\frac{\Omega}{\alpha}\right)$$

$(y(t))$

no effect

Frequency centre of y

$$= \alpha \left(\begin{array}{l} \text{Frequency} \\ \text{centre} \\ \quad\text{of } x \end{array} \right)$$

Frequency variance of y

$= |\alpha|^2$ (Frequency variance of x)

Time bandwidth product of y

$$= \left(\text{Time variance of } y\right)\left(\text{Frequency variance of } y\right)$$

$$= \frac{1}{|\alpha|^2}\left(\text{Time variance of } x\right) \cdot$$

cancel

$$\cdot\, |\alpha|^2 \cdot \left(\text{Frequency variance of } x\right)$$

= Time bandwidth product of x !

The time bandwidth product is invariant to

(i) shift in time

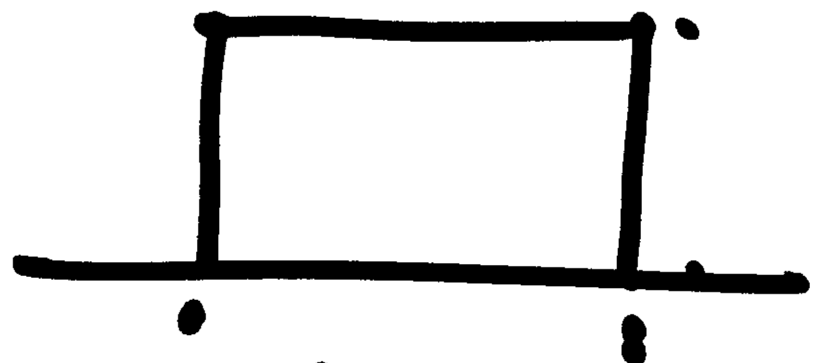
(ii) Shift in frequency or modulation in time

(iii) multiplying function by constant

(iv) Scaling the independent variable (<u>dilation</u>)

Challenge:

Can two differently shaped functions have the same time-bandwidth prod?

We had noted:
Time bandwidth product of 

$\longrightarrow \infty$!

Basic fundamental question: How small can the time bandwidth product be?

Assume without loss of generality that

time centre $t_0 = 0$

frequency centre $\Omega_0 = 0$

Time bandwidth product =

$$\frac{\left\{\int_{-\infty}^{+\infty} t^2 |x(t)|^2 \, dt\right\}}{\|x\|_2^2} \cdot \frac{\left\{\int_{-\infty}^{+\infty} \Omega^2 |\hat{x}(\Omega)|^2 \, d\Omega\right\}}{\|\hat{x}\|_2^2}$$

Consider

$$\int_{-\infty}^{+\infty} \Omega^2 |\hat{x}(\Omega)|^2 d\Omega$$

$$= \int_{-\infty}^{+\infty} |j\Omega \hat{x}(\Omega)|^2 d\Omega$$

$$x(t) \longrightarrow \hat{x}(\Omega)$$

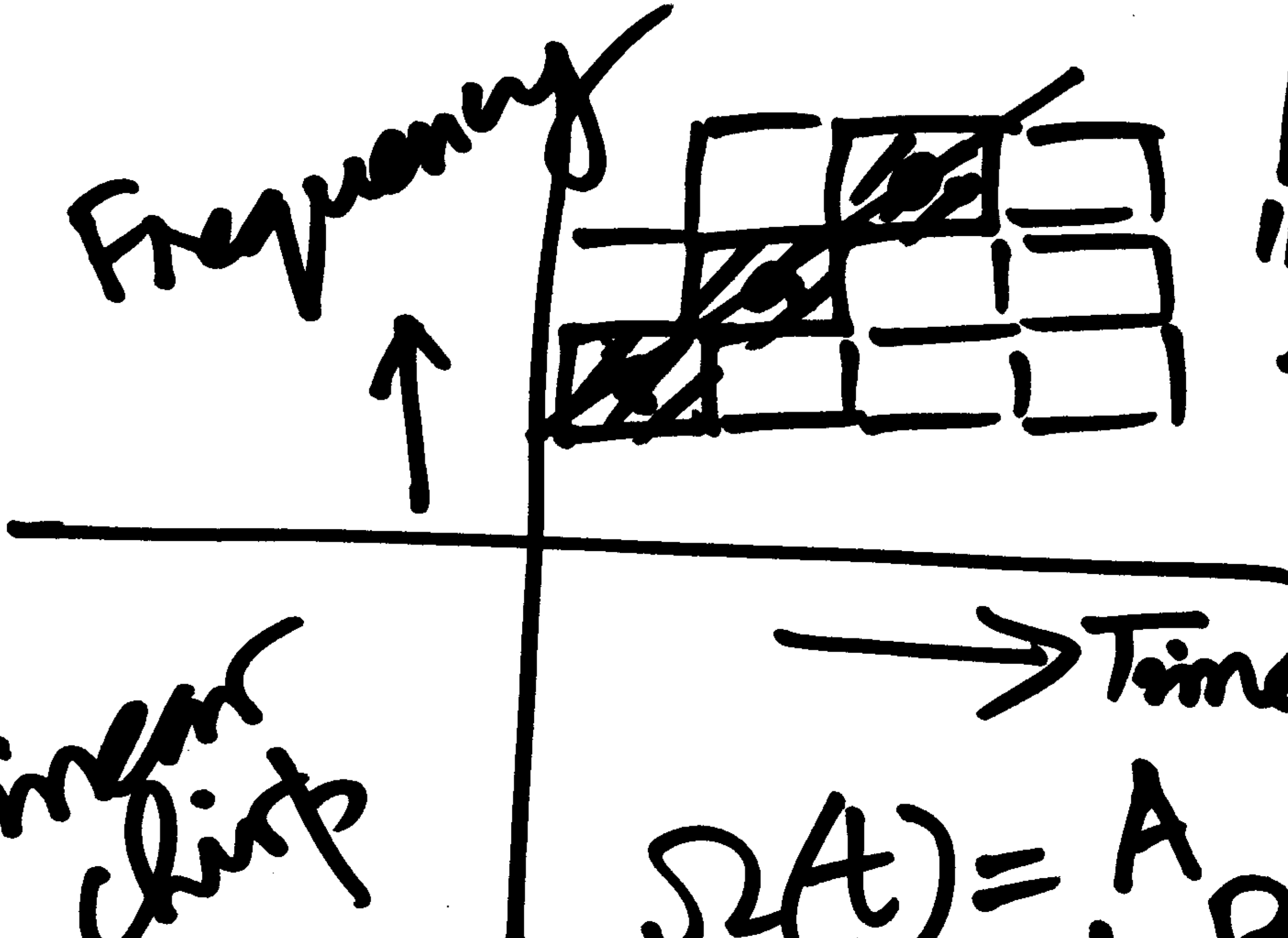
$$\frac{dx(t)}{dt} \longrightarrow j\Omega \cdot \hat{x}(\Omega)$$

$$\int_{-\infty}^{+\infty} |j\Omega \hat{x}(\Omega)|^2 \, d\Omega$$

$$= 2\pi \int_{-\infty}^{+\infty} \left|\frac{dx(t)}{dt}\right|^2 dt$$

$$\|\hat{x}\|_2^2 = 2\pi \|x\|_2^2$$

Thus:



"lighted up".

linear chirp

$$\Omega(t) = A + Bt$$