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LECTURE 17

THE UNCERTAINTY PRINCIPLE

Context:

Consider

$$x(t) \in \mathcal{L}_2(\mathbb{R}) \cap \mathcal{L}_1(\mathbb{R})$$

(both square integrable and absolutely integrab

$$x(t) \xrightarrow[\text{Fourier Transform}]{} \hat{x}(\Omega)$$

$\hat{x}(\Omega) \in \mathcal{L}_2(\mathbb{R})$ as well

$$\int_{-\infty}^{+\infty} |x(t)|^2 \text{ finite} = \|x\|_2^2$$

Define $P_x(t) = \dfrac{|x(t)|^2}{\|x\|_2^2}$

$$P_x(t) = \frac{|x(t)|^2}{\|x\|_2^2}$$

is a probability density! because: —

. . . .

(i) $P_x(t) \geq 0 \quad \forall t$

(density in t)

(ii) $\int_{-\infty}^{+\infty} P_x(t) = 1.$

(from definition)

Similarly define

$$P_{\hat{x}}(\Omega) = \frac{|\hat{x}(\Omega)|^2}{\|\hat{x}\|_2^2}$$

$P_{\hat{x}}(\Omega)$ is also a probability density

$P_{\hat{x}}(\Omega) \geq 0 \quad \forall \Omega$

(density in Ω)

$$\int_{-\infty}^{+\infty} P_{\hat{x}}(\Omega)\, d\Omega = 1$$

One could also take a "one-dimensional mass" perspective.

We could think of

$$p_x(t): \text{1-D mass in } t$$

Similarly $P_{\mathfrak{X}}(\Omega)$ is a "one-dimensional mass" in Ω.

If we choose "mass" perspective, consider "centre of mass" and "spread around centre"

If we choose the "probability density" perspective, consider "mean" and "variance"

Let us take the probability density perspective.

$P_x(t)$, $P_{\hat{x}}(\Omega)$: probability densities.

Let $p_x(t)$ have the mean t_0:

$$t_0 = \int_{-\infty}^{+\infty} t\, p_x(t)\, dt$$

Similarly let $P_{\hat{x}}(\Omega)$ have the mean Ω_0

$$\Omega_0 = \int_{-\infty}^{+\infty} \Omega P_{\hat{x}}(\Omega)\, d\Omega$$

Variances:

variance in t:

$$\sigma_t^2 = \int_{-\infty}^{+\infty} (t-t_0)^2 P_x(t)\,dt$$

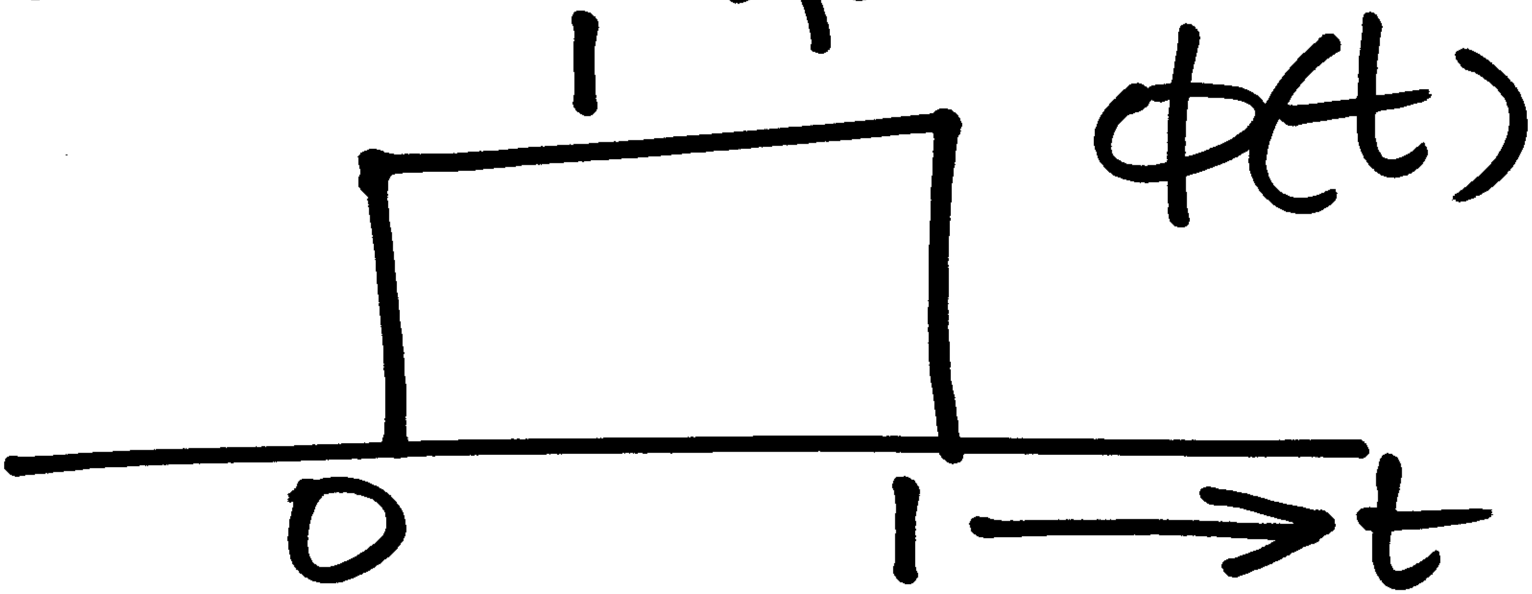
Variance in angular frequency:

$$\sigma_\Omega^2 = \int_{-\infty}^{+\infty} (\Omega - \Omega_0)^2 P_\Omega(\Omega)\, d\Omega$$

Containment in a given domain refers to the variance in that domain

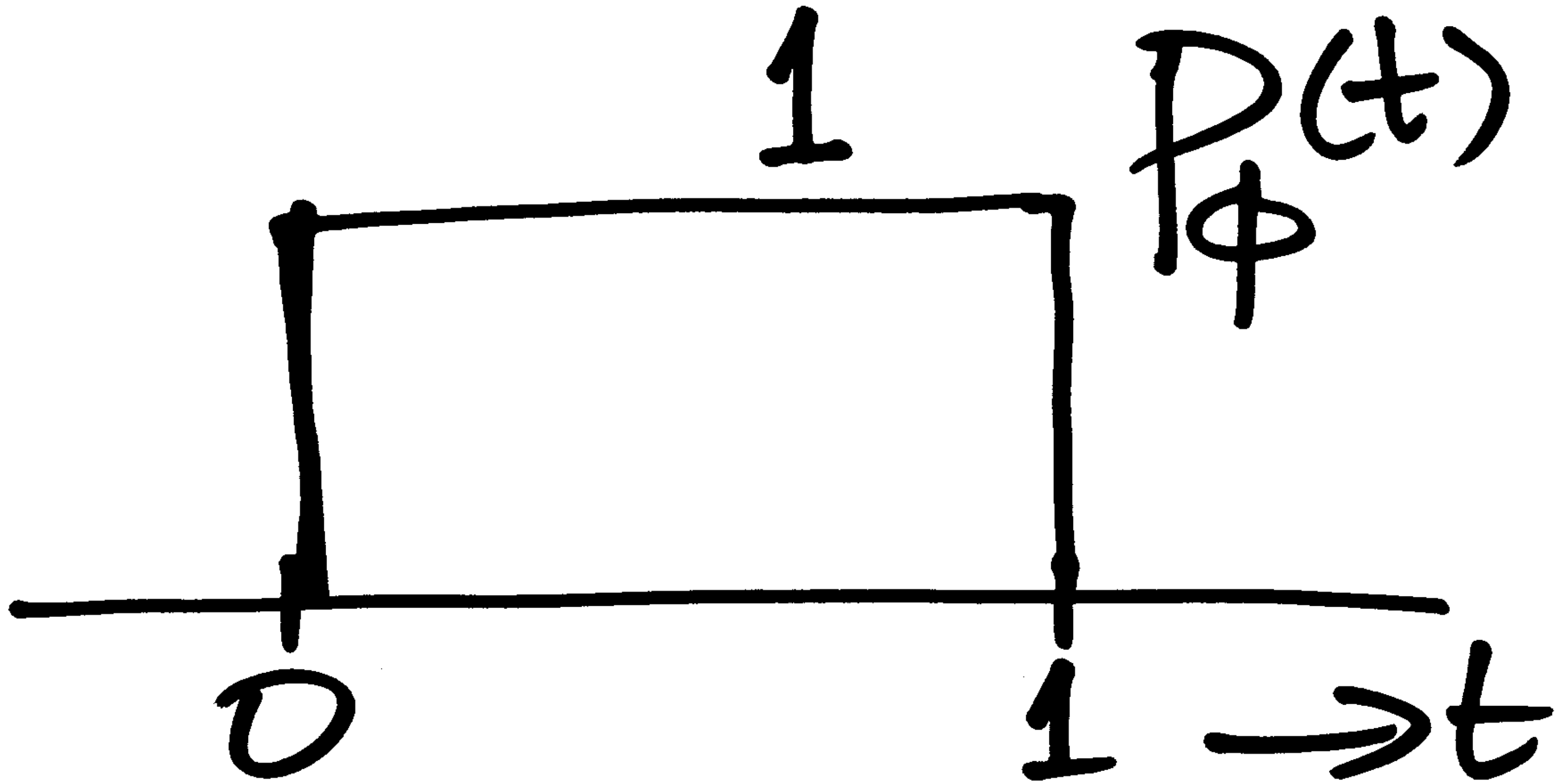
Example:

Haar scaling function:



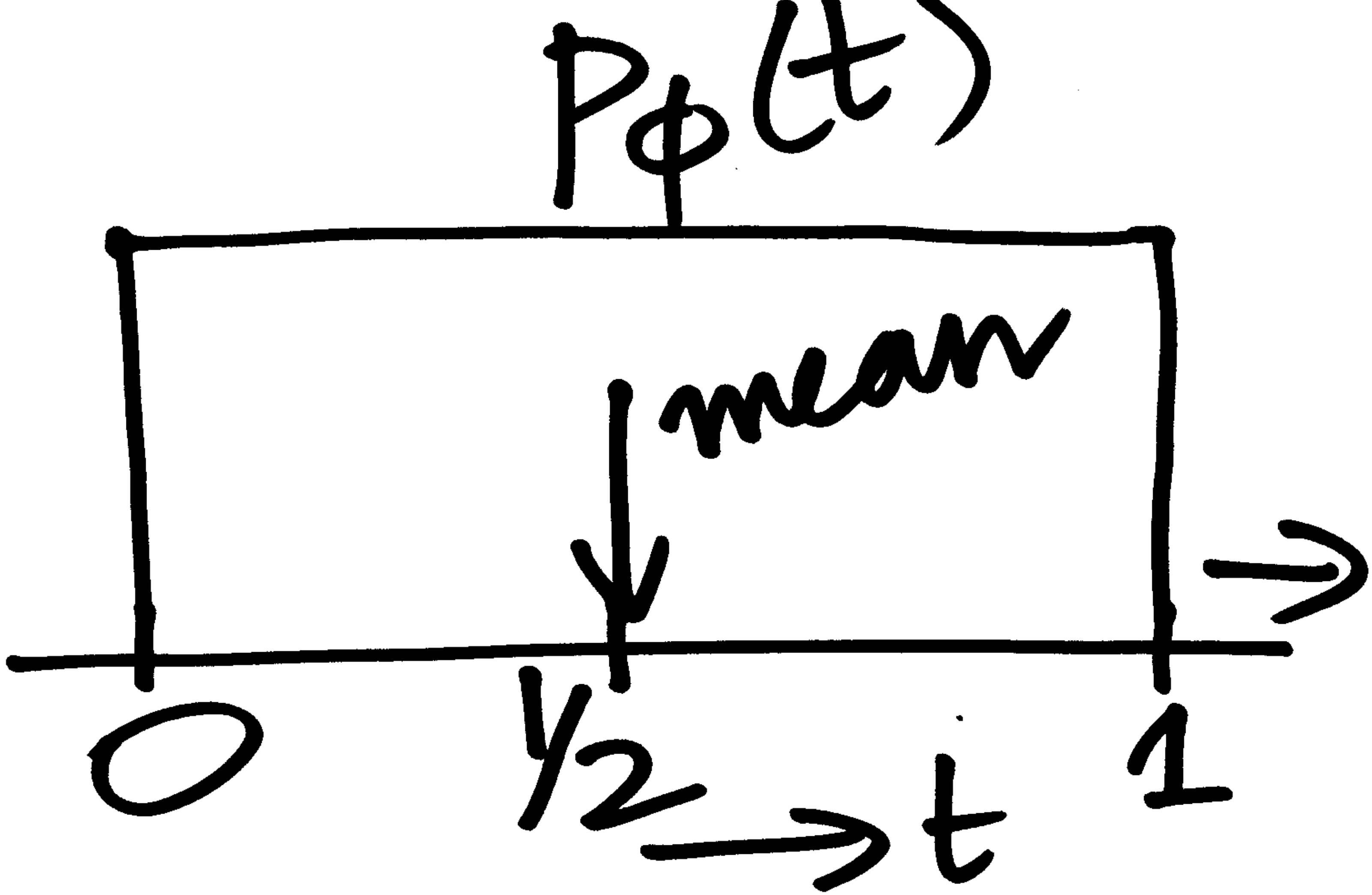
$$P_{\phi}(t) = \frac{|\phi(t)|^2}{\|\phi\|_2^2}$$

$$\|\phi\|_2^2 = 1 = \int_{-\infty}^{+\infty} |\phi(t)|^2 \, dt$$


1
$P_\phi(t)$
0
1
$\rightarrow t$

$$t_0 = \int_{-\infty}^{+\infty} t\, P_\phi(t)\, dt$$

$$= \int_{\cancel{-\infty}\; 0}^{\cancel{+\infty}\; 1} t \cdot dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$$



$p_\phi(t)$
mean
0
1/2
1
t

Variance:

$$\int_{-\infty}^{+\infty}\left(t-\frac{1}{2}\right)^2 P_{\phi}(t)\,dt$$

$$= \int_0^1 \left(t - \frac{1}{2}\right)^2 dt \qquad t - \tfrac{1}{2} = \lambda$$

$$= \int_{-1/2}^{1/2} \lambda^2 \, d\lambda$$

$$= \frac{x^3}{3}\Big|_{-1/2}^{1/2}$$

$$= \frac{(1/2)^3}{3} - \left(-\frac{(1/2)^3}{3}\right)$$

$$= \frac{2 \cdot \left(\frac{1}{2}\right)^3}{3}$$

$$= \frac{2}{3} \cdot \frac{1}{8} = \frac{1}{12}$$

$$\sigma_t^2 = 1/12$$

$$\sigma_t = +\sqrt{1/12} = \frac{1}{2\sqrt{3}}$$

Fraction of energy contained in $[t_0 - \sigma_t, t_0 + \sigma_t]$

$$= \int_{t_0-\sigma_t}^{t_0+\sigma_t} P_\phi(t)\, dt$$

$$= \int_{\frac{1}{2}-\frac{1}{2\sqrt{3}}}^{\frac{1}{2}+\frac{1}{2\sqrt{3}}} 1 \, dt$$

$$= \ldots$$

$$= t \Big|_{\frac{1}{2} - \frac{1}{2\sqrt{3}}}^{\frac{1}{2} + \frac{1}{2\sqrt{3}}}$$

$$= 2 \cdot \frac{1}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$|\hat{\Phi}(\Omega)|^2$$

$$= \left| \frac{\sin \frac{\Omega}{2}}{\Omega/2} \right|^2$$

$$\frac{1}{2\pi}\int |\hat{\phi}(\Omega)|^2 \, d\Omega$$

$$= \|\phi\|_2^2 = 1$$

$$\int_{-\infty}^{+\infty} |\hat{\phi}(\Omega)|^2 \, d\Omega = 2\pi$$

$$P_{\hat{\phi}}(\Omega) = \frac{|\hat{\phi}(\Omega)|^2}{2\pi}$$

For real functions $x(t)$, $\hat{x}(\Omega)$ is magnitude symmetric

Therefore $\Omega_0 = 0$

in general for real $x(t)$

Variance of $\hat{\phi}$:

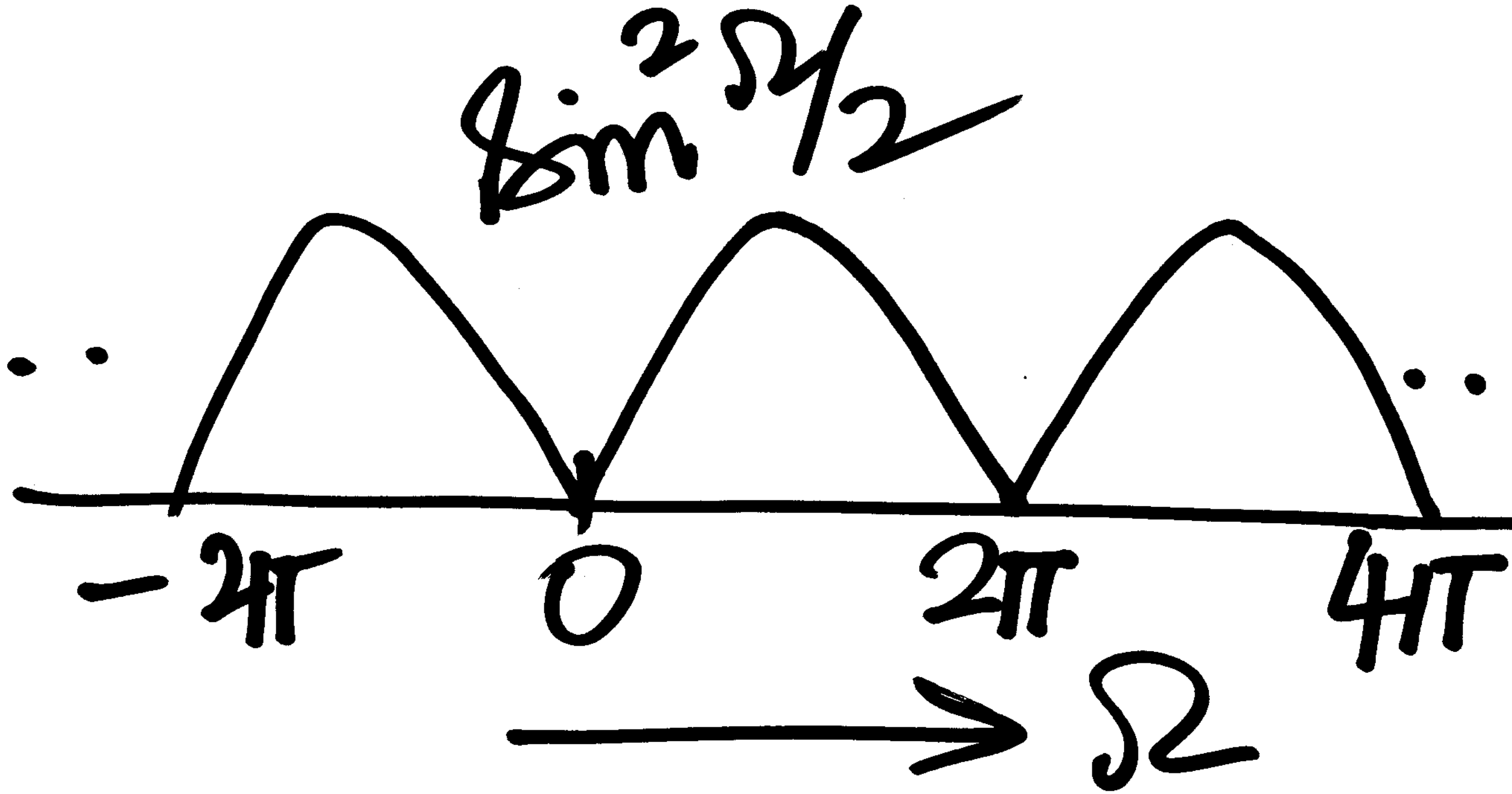
$$\int_{-\infty}^{+\infty} (\Omega - \Omega_0)^2 \, P_{\hat{\phi}}(\Omega) \, d\Omega$$

$$= \int_{-\infty}^{+\infty} \frac{\Omega^2}{2\pi} \cdot \left(\frac{\sin \Omega/2}{\Omega/2} \right)^2 d\Omega$$

$$= \int_{-\infty}^{+\infty} \frac{4}{2\pi} \cdot \sin^2 \frac{\Omega}{2} \, d\Omega$$

not important

trouble!



$\sin^2 \Omega/2$
-2π
0
2π
4π
Ω

The variance of

$\hat{\phi}$ is INFINITE!

$\phi(t)$ is not at all confined in the frequency (Ω) domain.

Variance of $\hat{\phi}$:

$$\int_{-\infty}^{+\infty} \Omega^2 P_{\hat{\phi}}(\Omega)\, d\Omega$$

$$= \int_{-\infty}^{+\infty} \frac{\Omega^2 \cdot |\hat{\phi}(\Omega)|^2 \, d\Omega}{\|\hat{\phi}\|_2^2}$$

$$= \frac{1}{\|\hat{\Phi}\|_2^2} \int_{-\infty}^{+\infty} \left| j\Omega \hat{\Phi}(\Omega) \right|^2 d\Omega$$

Fourier transform of $\frac{d\Phi(t)}{dt}$

$$= \frac{2\pi\,(\text{energy in derivative})}{2\pi\,(\text{energy in function})}$$

For real $x(t)$,

Ω-variance, σ_x^2

$$= \frac{\text{energy in } dx/dt}{\text{energy in } x}$$