LECTURE 15

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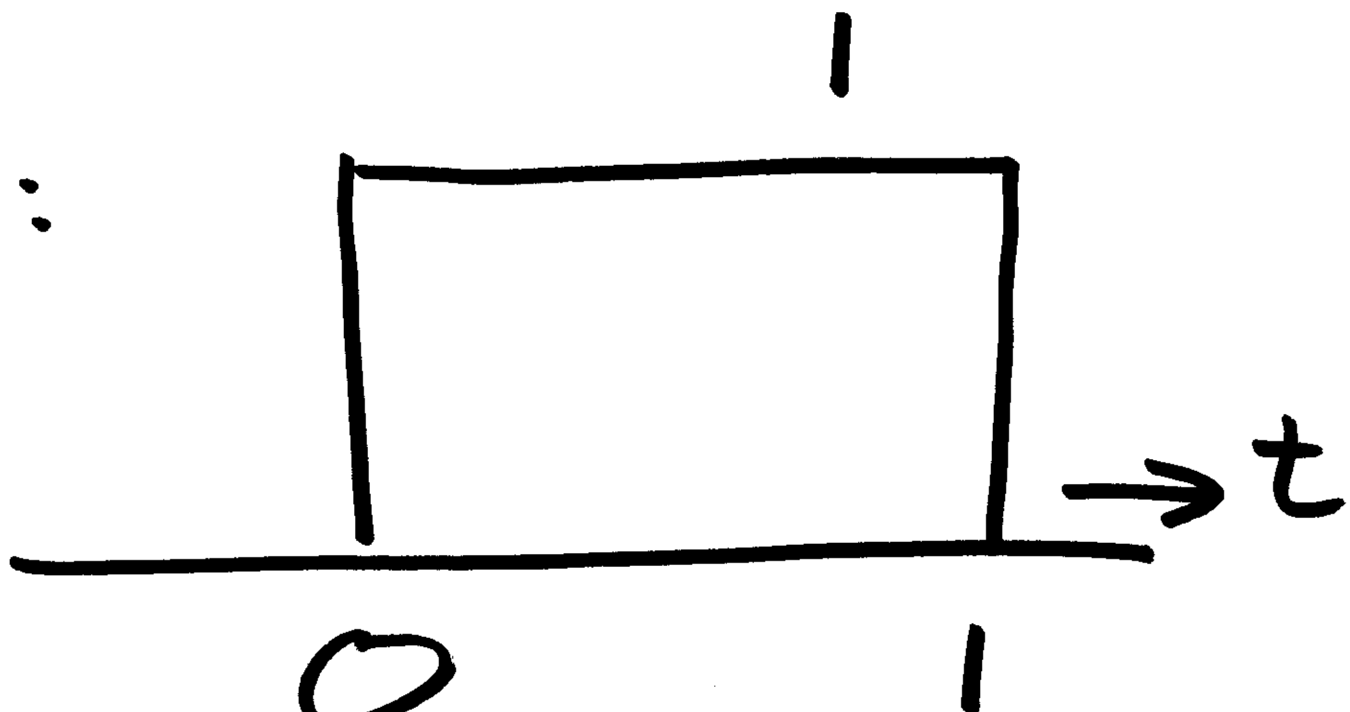
TIME AND FREQUENCY- JOINT PERSPECTIVE

Haar MRA:

$\phi(t)$:



$$\hat{\Phi}(\Omega) = \int_{0\,(-\infty)}^{1\,(+\infty)} \phi(t)\, e^{-j\Omega t}\, dt$$

analog radian (angular) frequency

$$= \int_0^1 e^{-j\Omega t}\, dt$$

$$= \left. \frac{e^{-j\Omega t}}{-j\Omega} \right|_0^1$$

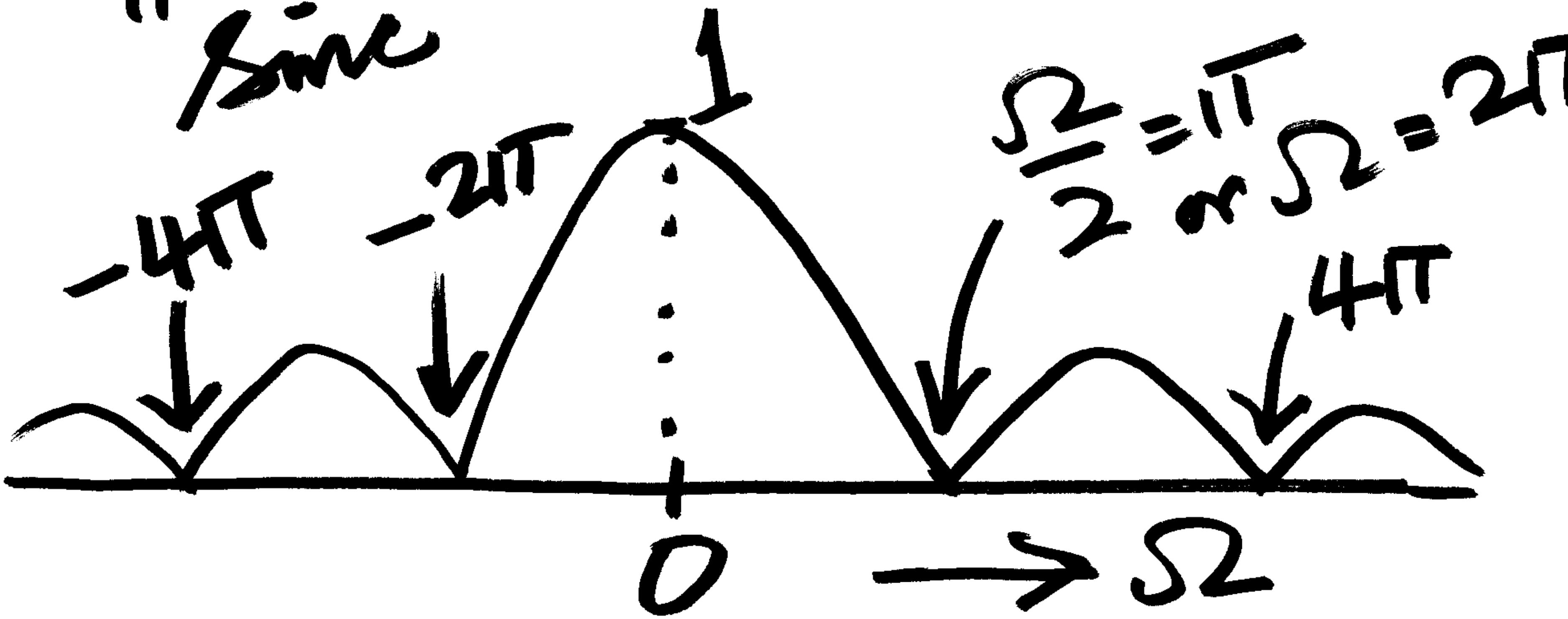
$$= \frac{1 - e^{-j\Omega}}{j\Omega}$$

$$= \frac{e^{-j\Omega/2}\left(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}\right)}{2 \cdot j\Omega/2}$$

$$= e^{-j\frac{\Omega}{2}} \cdot \left(\frac{\sin\frac{\Omega}{2}}{\Omega/2}\right)$$

$$\left|\hat{\Phi}(\Omega)\right| = \left|\frac{\sin\frac{\Omega}{2}}{\Omega/2}\right|$$

"sinc"



In general consider $\phi(2^m t - n)$

$m, n \in \mathbb{Z}$

$$\phi(t) \xrightarrow[\text{Fourier transform}]{} \hat{\phi}(\Omega)$$

$$\phi(\alpha t) \longrightarrow \frac{1}{|\alpha|}\hat{\phi}\left(\frac{\Omega}{\alpha}\right)$$

$$\underline{\forall \alpha \in \mathbb{R}-\{0\}}$$

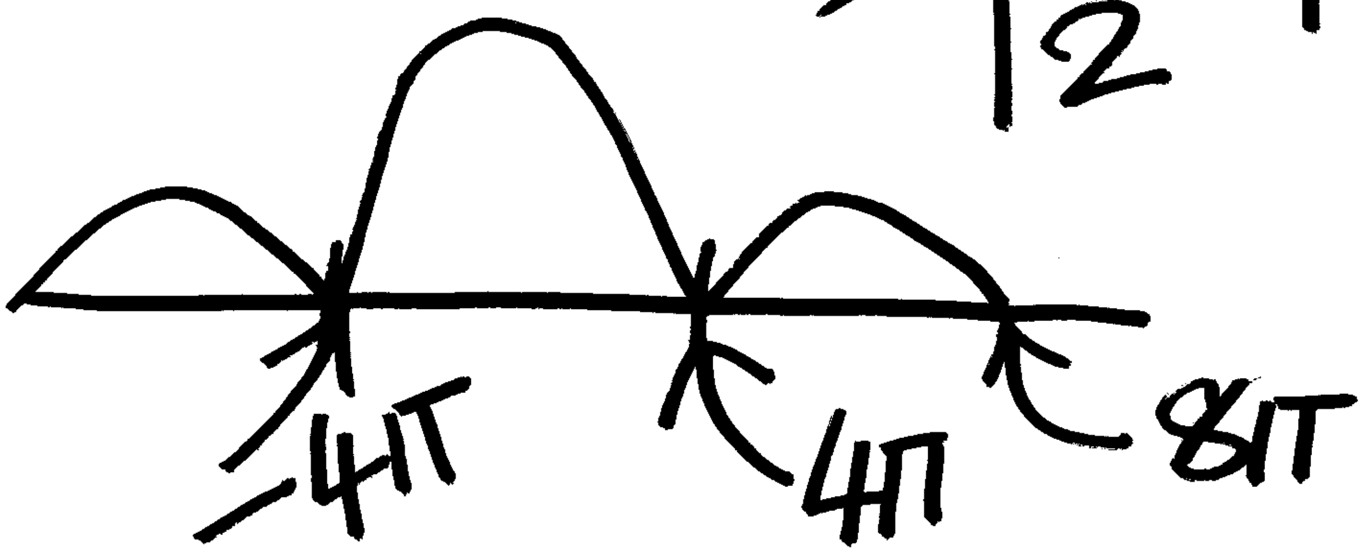
Magnitude only:

$$\phi(t) \xrightarrow[\text{Fourier Trans}]{} \left|\hat{\phi}(\Omega)\right|$$

$$\phi(2^m t - n) \longrightarrow \left|\frac{1}{2^m}\hat{\phi}\left(\frac{\Omega}{2^m}\right)\right|$$

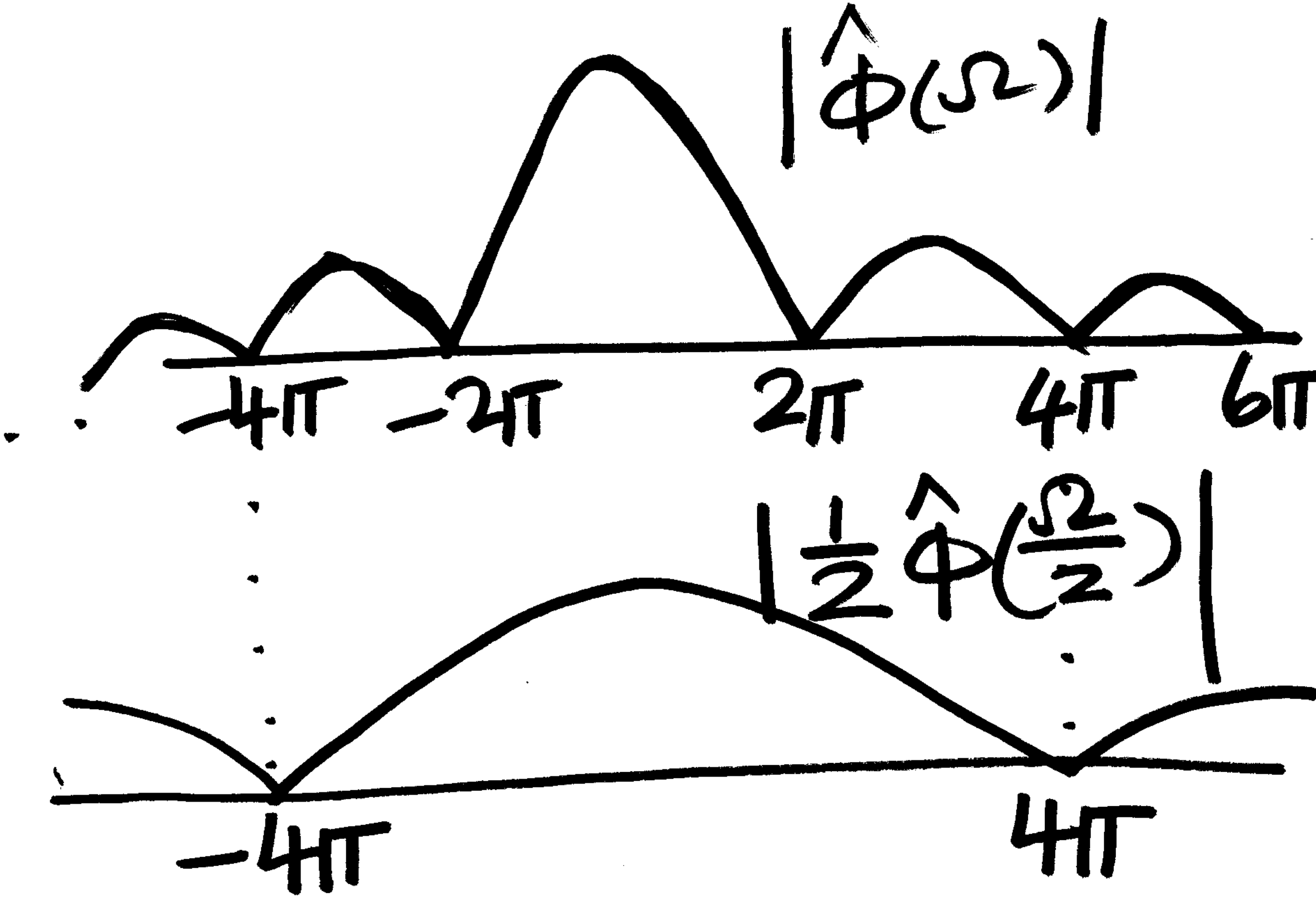
$m=1$:

$$\phi(2t-n) \longrightarrow \left|\frac{1}{2}\hat{\phi}\left(\frac{\Omega}{2}\right)\right| =$$



$m = -1$

$$\phi\left(\frac{t}{2} - n\right) \longrightarrow \left| 2\,\hat{\phi}(2\Omega) \right|$$



$|\hat{\phi}(\Omega)|$
-4π
-2π
2π
4π
6π
$|\frac{1}{2}\hat{\phi}(\frac{\Omega}{2})|$
-4π
4π

To calculate coefficients in a subspace V_m,

$$\langle x, \phi(2^m t - n) \rangle$$

$$\phi(2^m t - n)$$

translation does not affect $\|\cdot\|_2$

without loss of generality, $\underline{n = 0}$

$$\|\phi(2^m\cdot)\|_2^2 = \int_0^{2^{-m}} 1^2\,dt = 2^{-m}$$

To make $\phi(2^{m}t-n)$ unit norm

$$\frac{1}{\sqrt{2^{-m}}}\cdot\phi(2^{m}t-n)$$

$$=2^{-m/2}\phi(2^{m}t-n)$$

$$\left\{ 2^{-m/2} \phi(2^{m} t - n) \right\}_{n \in \mathbb{Z}}$$

is an <u>orthonormal</u>
basis for V_m

Consider $x(\cdot) \in \mathcal{L}_2(\mathbb{R})$

$$\left\langle x(\cdot), 2^{-m/2} \phi(2^{m} \cdot - n) \right\rangle$$

$$x(t) \longrightarrow \hat{x}(\Omega)$$

$$\hat{x}(\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

FOURIER TRANSFORM

"Component" of $x(t)$ "along" $e^{j\Omega t}$

Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{x}(\Omega) e^{j\Omega t} \, d\Omega$$

"Reconstruction from components"

Parseval's Theorem

$$\langle x(t), y(t) \rangle = \frac{1}{2\pi} \langle \hat{x}(\Omega), \hat{y}(\Omega) \rangle$$

$$\langle x(\cdot), \phi(2^m \cdot - n) \rangle$$

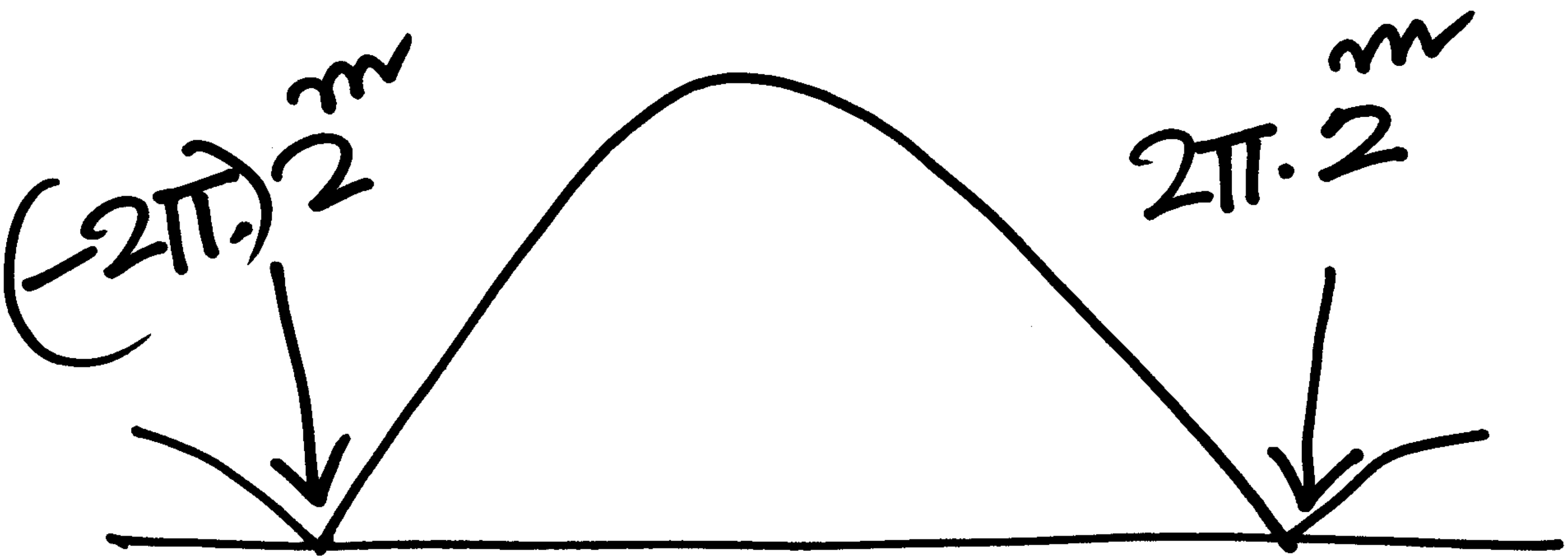
orthonormalizing factor $2^{m/2}$

$$= \frac{1}{2\pi} \left\langle \hat{x}(\Omega), \widehat{2^{-m/2} \phi(2^m t - n)} \right\rangle$$

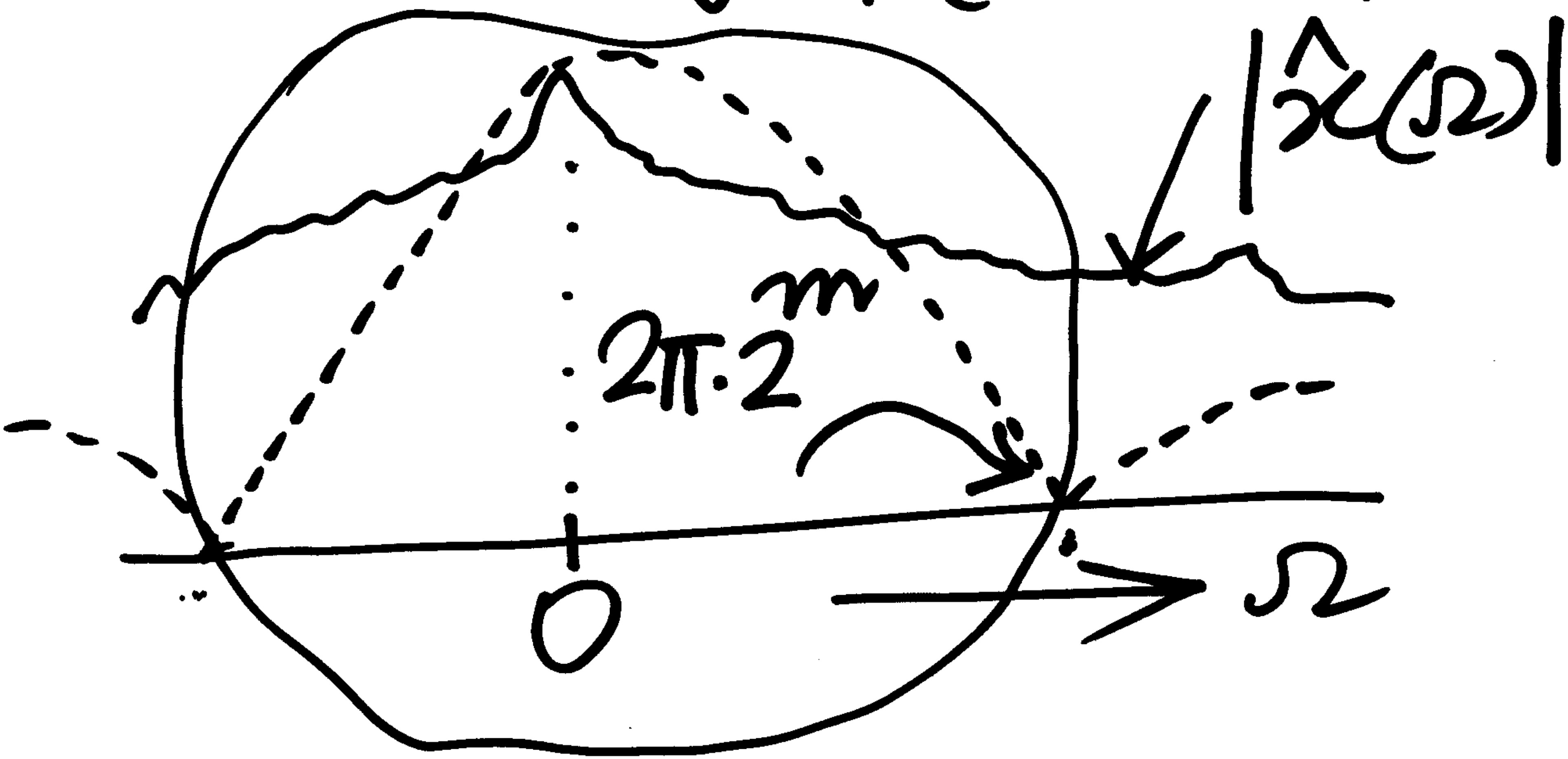
We have just seen

$$\Big|\,\widehat{2^{-\frac{m}{2}}\,\phi(2^{m}t-n)} \Big\rangle$$

It looks like:



$$\left\langle \hat{x}(\cdot),\ 2^{m/2}\phi(2^{m}t-n) \right\rangle$$



Fourier Transform of $\psi(t)$:

$$\psi(t) = \phi(2t) - \phi(2t-1)$$

$$\hat{\psi}(\Omega) = \frac{1}{2}\hat{\phi}\left(\frac{\Omega}{2}\right) - e^{-j\Omega/2}\cdot\frac{1}{2}\cdot\hat{\phi}\left(\frac{\Omega}{2}\right.$$

$$\phi(2t) \longrightarrow \frac{1}{2}\hat{\phi}\left(\frac{\Omega}{2}\right)$$

$$\phi(t-n) \longrightarrow e^{-j\Omega n}\hat{\phi}(\Omega)$$

$$\phi(2t-n)$$

$$\longrightarrow \frac{1}{2}\left\{ e^{-j\Omega n}\,\hat{\phi}(\Omega) \right\}_{\Omega \leftarrow \frac{\Omega}{2}}$$

$$\phi(2t-n) \longrightarrow \frac{1}{2} \cdot e^{-j\frac{\Omega}{2}n} \cdot \hat{\phi}\left(\frac{\Omega}{2}\right)$$

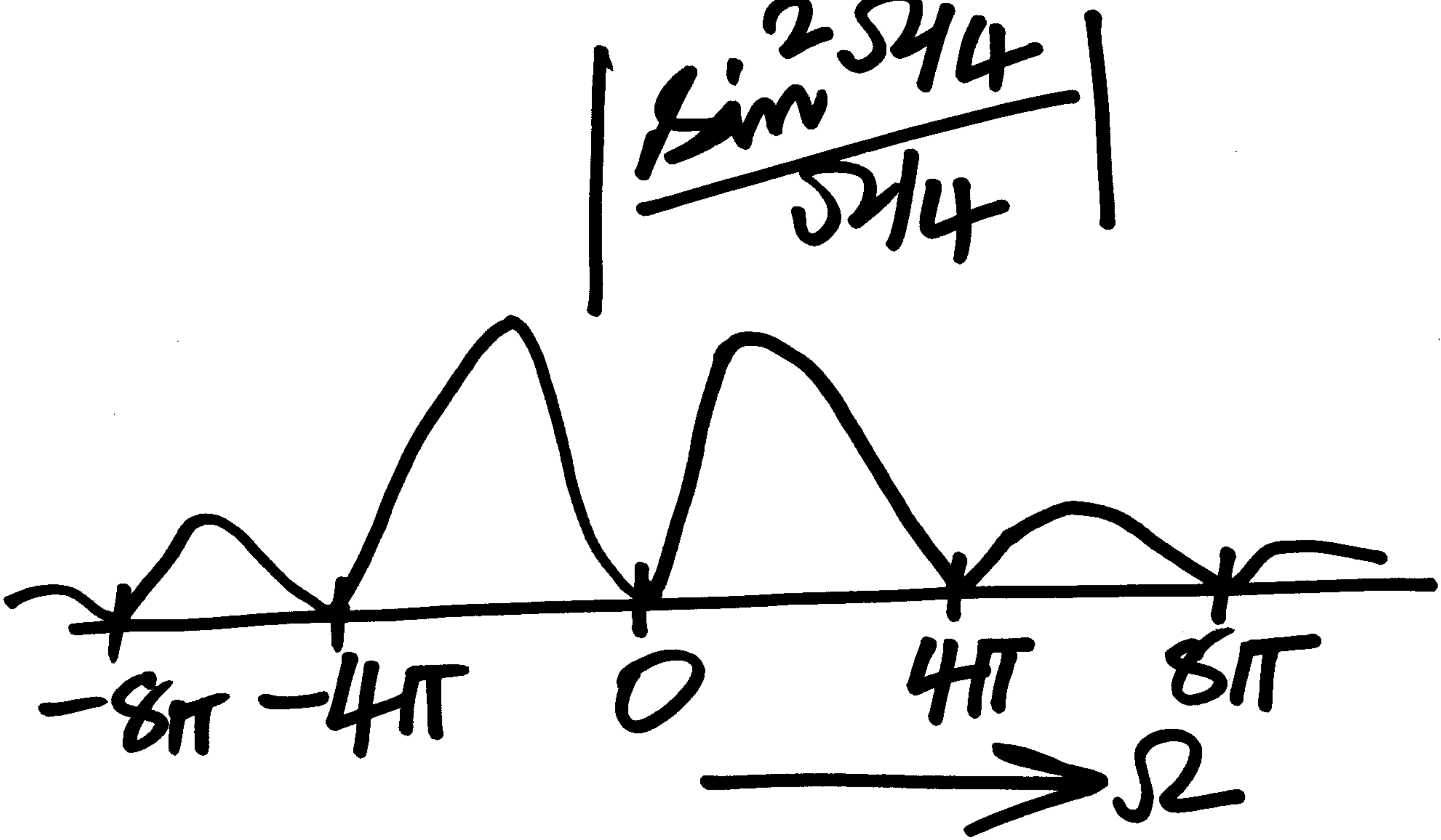
$$\psi(t) = \phi(2t) - \phi(2t-1)$$

$$\Big\downarrow \text{Fourier Trans}$$

$$\frac{1}{2}\hat{\phi}\left(\frac{\Omega}{2}\right) - \frac{1}{2} \cdot e^{-j\frac{\Omega}{2}} \cdot \hat{\phi}\left(\frac{\Omega}{2}\right)$$

$$= \frac{1}{2}\left(1 - e^{-j\frac{\Omega}{2}}\right) \cdot \hat{\phi}\left(\frac{\Omega}{2}\right)$$

$$= \left\{\frac{1}{2} \cdot e^{-j\frac{\Omega}{4}} \cdot 2j\sin\frac{\Omega}{4}\right\} \frac{1}{2} e^{-j\frac{\Omega}{4}} \cdots$$



$\left| \frac{\sin^2 \Omega/4}{\Omega/4} \right|$
-8π
-4π
0
4π
8π
Ω

$$\langle x(t), \psi(t-n) \rangle$$

magnitude sense:

First multiply

$$\hat{x}(\Omega), \quad \widehat{\psi(t-n)}$$

W_0 (instead of W_0', go to W_1):

$\psi(2t - n)$

$$\left|\widehat{\psi(2t-n)}\right|$$