Prof Gadre Date

LECTURE 13

CONJUGATE QUADRATURE FILTERS: DAUBECHIES' FAMILY OF MRA

$$H_1(z) = z^{-D} H_0(-z^{-1})$$

$$G_0(z) = H_1(-z)$$

$$G_1(z) = -H_0(-z)$$

$$z^{-D} H_0(-z^{-1})$$

$$z = e^{j\omega}$$

$$e^{-j\omega D} H_0(-e^{-j\omega})$$

$$\left| e^{-j\omega D} \cdot H_0(e^{-j\omega}) \right|$$

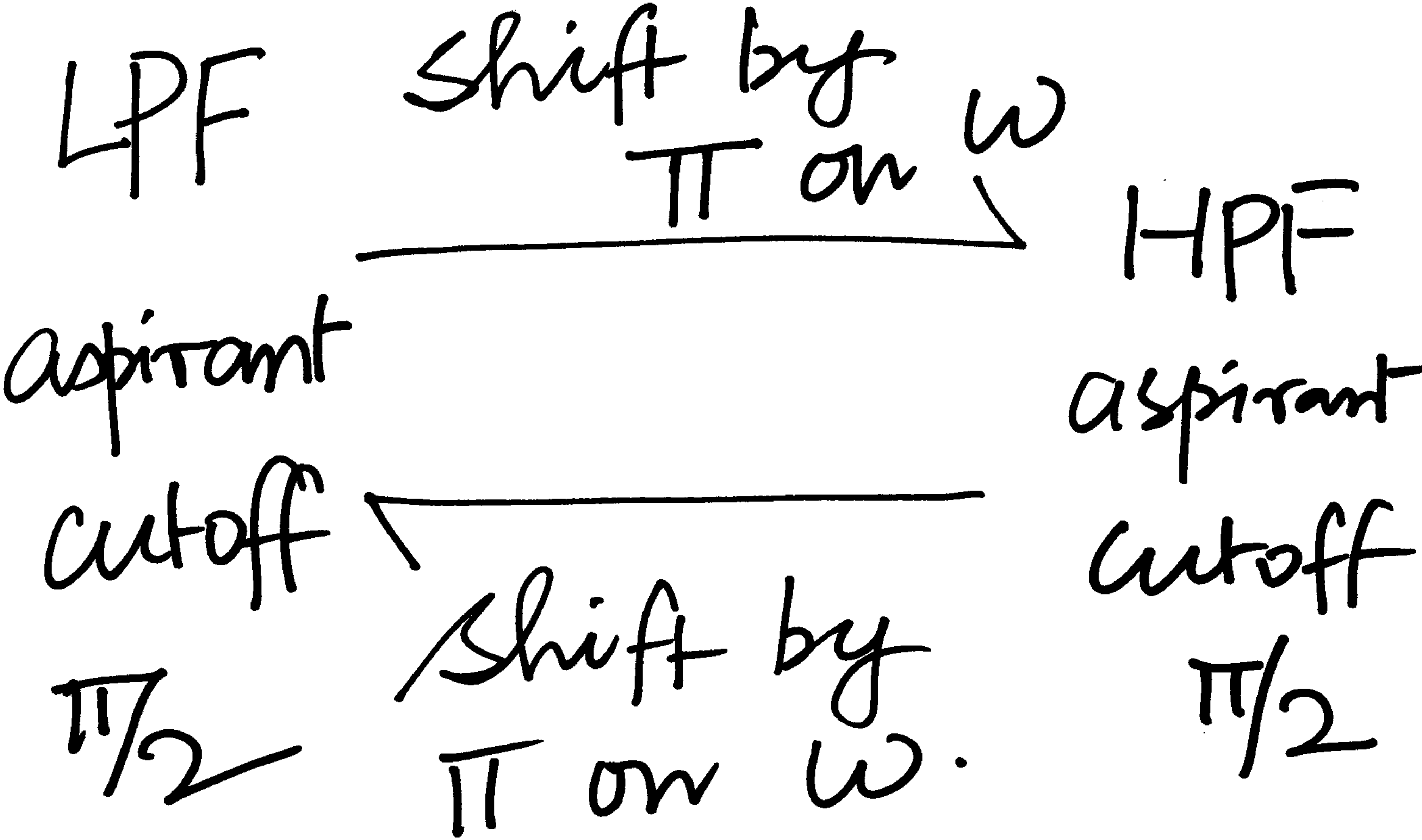
$$= \left| H_0(e^{-j\omega}) \right|$$

$H_0(Z)$: filter with a real impulse response (real coefficients)

$$H_0(e^{-j\omega}) = \overline{H_0(e^{j\omega})}$$

$$H_0\left(-e^{-j\omega}\right)$$

$$= H_0\left(e^{-j(\omega \pm \pi)}\right)$$



LPF

aspirant

cutoff

$\pi/2$

Shift by

π on ω

HPF

aspirant

cutoff

$\pi/2$

Shift by

π on ω.

We have shown:

$$H_1(z) = z^{-D} H_0(-z^{-1})$$

is indeed a HPF "aspirant" cutoff $\frac{\pi}{2}$

Perfect reconstruction:

$$\begin{aligned} &G_0(z)H_0(z) \\ &\quad + G_1(z)H_1(z) \\ &\qquad = C_0 z^{-D} \end{aligned}$$

$$H_1(-z)H_0(z) - H_0(-z)H_1(z) = c_0 z^{-D}$$

$$(-1)^{D}\,\underline{\underline{z}}^{-D}\cdot H_0(z^{-1})\,H_0(z)$$

$$-\;H_0(-z)\,\underline{\underline{z}}^{-D}\cdot H_0(-z^{-1})$$

$$=\;c_0\,\underline{\underline{z}}^{-D}\cdot$$

What we need for perfect reconstruction:

$$(-1)^D H_0(z) H_0(z^{-1}) - H_0(-z) H_0(-z^{-1}) = c_0$$

$$H_0(z) = 1 + z^{-1}$$

(Haar) essentially

$$H_0(z^{-1}) = 1 \overset{-}{\cancel{+}} z$$

$$z^{-D} H_0(z^{-1}) : z^{-1}(1 - z)$$

With D odd we essentially have:

$$H_0(z)H_0(\bar{z}^{-1}) + H_0(-z)H_0(-\bar{z}^{-1}) = \text{Constant}$$

With $z = e^{j\omega}$:

$$H_0(e^{j\omega})H_0(e^{-j\omega}) + H_0(-e^{j\omega})H_0(-e^{-j\omega}) = \text{const}$$

For real impulse response $H_0(z)$:

$$H_0(e^{j\omega})\overline{H_0(e^{j\omega})} + H_0(e^{j(\omega\pm\pi)})\overline{H_0(e^{j(\omega\pm\pi)})} = \text{const}$$

$$\underbrace{|H_0(e^{j\omega})|^2 + |H_0(e^{j(\omega\pm\pi)})|^2}_{\text{POWER COMPLEMENT}} = \text{const}$$

Analysis filters are power complementary

So too the synthesis filters

(Exercise: Prove!)

For perfect reconstruction

$$K_0(z) = H_0(z)H_0(z^{-1})$$

$$K_0(z) + K_0(-z) = \text{constant}$$

We are going to choose even length

$H_0(Z)$: $\underset{\underset{0}{\uparrow}}{h_0}\ h_1 \cdots h_D$

$H_0(\bar{z}')$:

$$h_D \cdots h_1 \underset{\overset{\uparrow}{0}}{h_0}$$

$H_0(z)H_0(z^{-1})$ corresponds to their convolution

$$\left(\underset{\underset{0}{\uparrow}}{h_0} \cdots h_D\right) * \left(h_D \cdots \underset{\underset{0}{\uparrow}}{h_0}\right)$$

$h[k]$: $\underset{\overset{\uparrow}{0}}{h_0} \cdots\cdots h_D$

sequence ~~$g[n]$~~: $h_D \cdots\cdots \underset{\overset{\uparrow}{0}}{h_0}$

$g[k]$:

$g[n-k]$: $\underset{\overset{\uparrow}{n}}{h_0}\ h_1 \cdots \underset{\overset{\uparrow}{(n+D)}}{h_D}$

$$\begin{tikzcd} h_0 & \cdots & h_D \\ 0 \arrow[u] & & \end{tikzcd}$$

$$\begin{tikzcd} h_0 & \cdots & h_D \\ n \arrow[u] & \text{(moving)} & \end{tikzcd}$$

$$x_0(z) = H_0(z) H_0(z^{-1})$$

Corresponds to a sequence, whose m^{th} sample is:

Dot product of impulse response corresponding to $H_0(\cdot)$, and the same shifted by m samples (forward)

$\langle a[\cdot], b[\cdot] \rangle$ = dot product of a, b

m^{th} sample of $K_0(\cdot)$

$= \langle h_0[\cdot], h_0[\cdot \pm m] \rangle$

$h_0 \quad h_1 \quad h_2 \quad h_3 \cdots$

$\qquad\qquad\qquad\quad h_0 \quad h_1 \cdots$

$h_0 h_2 + h_1 h_3$

$$h_0 \; h_1 \; h_2 \; h_3$$

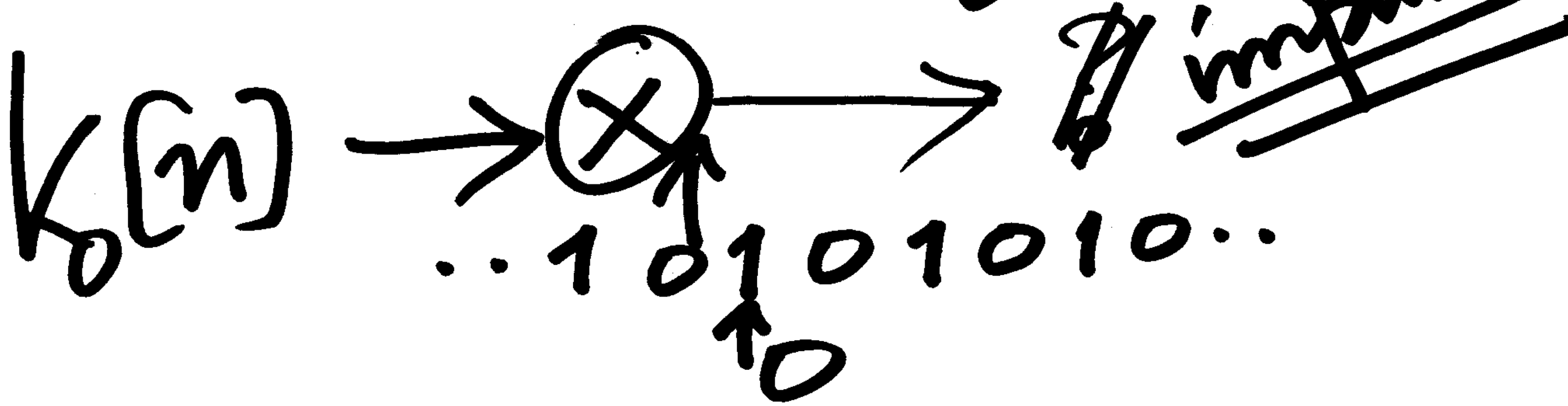
$$\because \; h_2 \, h_3$$

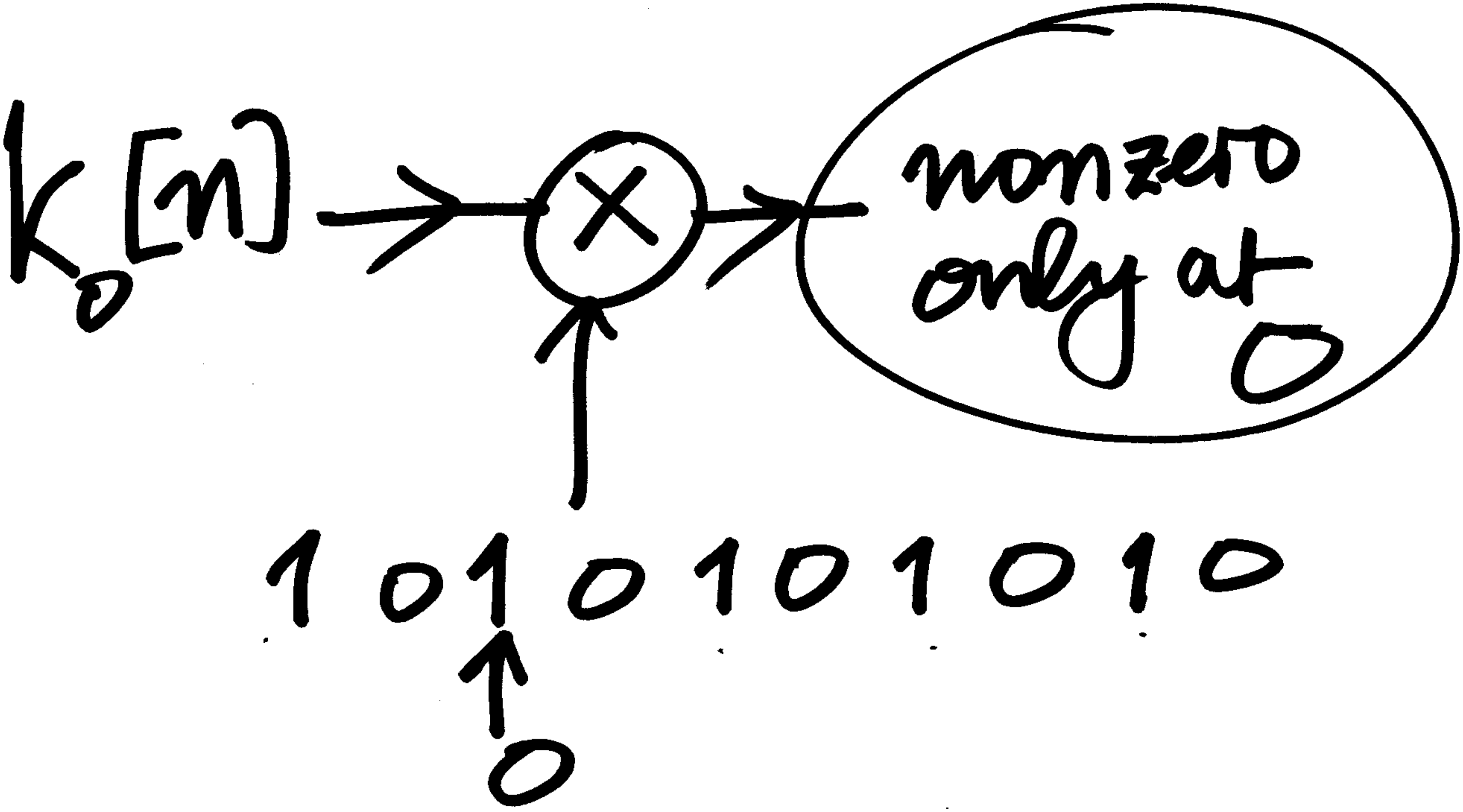
$$h_0 h_2 + h_1 h_3$$

$$K_0(z) + K_0(-z) = \mathrm{Const}$$

$$\frac{1}{2}\left\{K_0(z) + K_0(-z)\right\} = \mathrm{Const}$$

Let $K_0(Z)$ correspond to the sequence $k_0[n]$




$x_0[n]$
×
nonzero only at 0
1 0 1 0 1 0 1 0 1 0
0

The surprise is at
even locations

$$m = 2\ell$$

$\ell \in \mathbb{Z}$!

$m \neq 0$

Daubechies' filter with length 4:

$$\underset{\overset{\uparrow}{0}}{h_0} \quad h_1 \quad h_2 \quad h_3$$

Haar: One $(1-z^{-1})$
in HPF

$\Rightarrow$ This length 4
filter: Two $(1-z^{-1})$
in HPF

Corresponding lowpass filter:
has factor $(1+z^{-1})^2$

$$H_0(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3}$$

3 zeros to choose

2 already chosen

One free : B_0

$$H_0(Z) = \left(1 + Z^{-1}\right)^2 \left(1 + B_0 Z^{-1}\right)$$

$$H_0(z) =$$

$$(1+2z^{-1}+z^{-2})(1+B_0 z^{-1})$$

$$\begin{array}{cccc}
1 & +\ 2z^{-1} & +\ z^{-2} & \\
 & B_0 z^{-1} & +\ 2B_0 z^{-2} & +\ B_0 z^{-3} \\
\hline
1 & 2+B_0 & 1+2B_0 & B_0 \\
\hline
\end{array}$$