Prof: Gadre-lec
Date: 28/1/10

LECTURE 12

PERFECT RECONSTRUCTION CONJUGATE QUADRATURE

Two channel filter bank

Input

$$x[n] \xrightarrow{\mathcal{Z}} X(Z)$$

Z transform

Output

$$y(n) \xrightarrow{\mathcal{Z}} Y(z)$$

(We suppress Regions of Convergence)

Analysis side

$H_0(Z)$: lowpass

$H_1(Z)$: highpass

Synthesis side

$G_0(z)$: lowpass

$G_1(z)$: highpass

$$Y(z) = T_0(z)\,X(z) + T_1(z)\,\underbrace{X(-z)}_{\text{alias term}}$$

$\zeta(z)$: alias "system function"

$\tau_1(z) = 0$

ALIAS CANCELLATION

System becomes linear shift - invariant

$$\tau_1(z) = \frac{1}{2}\left\{ G_0(z) H_0(-z) + G_1(z) H_1(-z) \right\}$$

$$T_1(z) = 0$$

$$\Rightarrow \frac{G_0(z)}{G_1(z)} = \frac{-H_1(-z)}{H_0(-z)}$$

Simple case: equate ...

numerators:

$$G_0(z) = \pm H_1(-z)$$

$$G_1(z) = \mp H_0(-z)$$

If $\tau_1(z) = 0$ then

$$Y(z) = \tau_0(z) X(z)$$

$\tau_0(z)$ is the <u>system function</u>.

In a perfect reconstruction system we allow

$$T_0(z) = c_0 z^{-D}$$

c_0 = constant.

Ideally we would have:

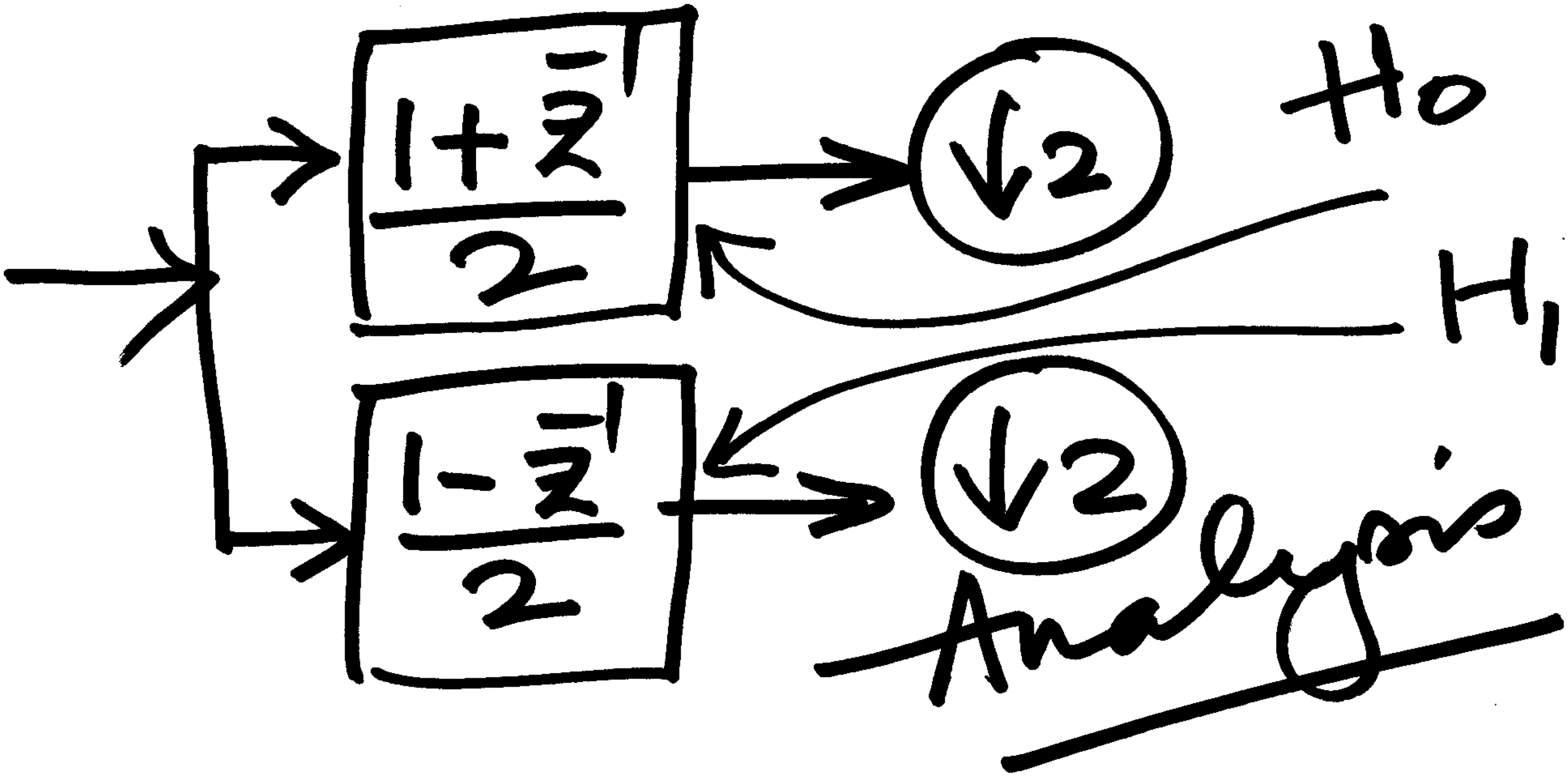
$$\tau_0(z) = 1 \quad \forall z$$

$\mathbb{Z}^D$ (D positive)

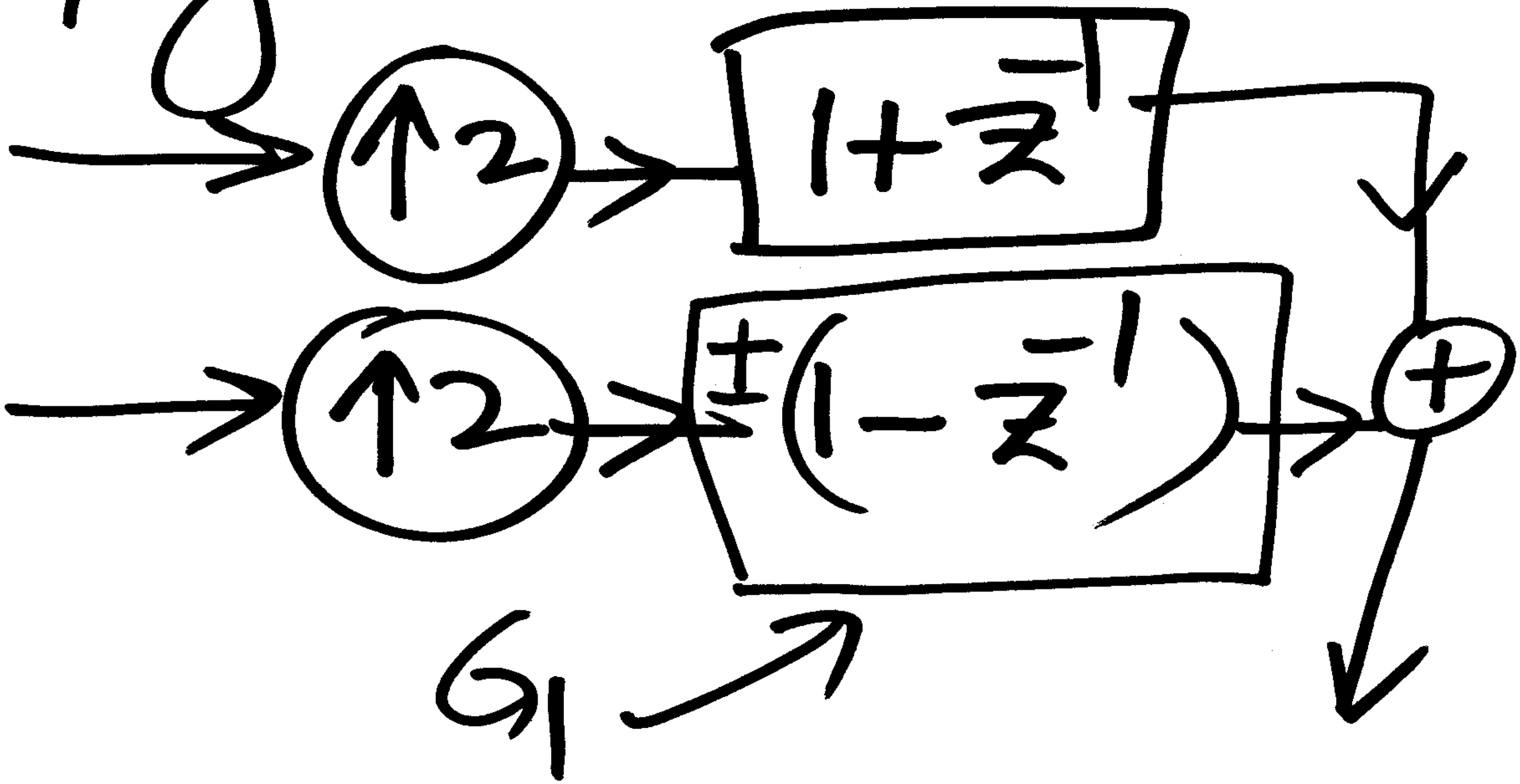
is "allowed"

because of <u>causality</u>

Haar Filter Bank



Synthesis:



$$T_1(z) = \frac{1}{2}\left\{ G_0(z)H_0(-z) + G_1(z)H_1(-z) \right\}$$

RHS =

$$\frac{1}{2}\left\{(1+z^{-1})\left(\frac{1-z^{-1}}{2}\right) + (1-z^{-1})\left(\frac{1+z^{-1}}{2}\right)\right\}$$

We want $= 0$

For alias cancellation it is very clear that $G_1(z) = -(1 - z^{-1})$

"Frozen" Haar:

$$H_0(z) = \frac{1}{2}(1+z^{-1})$$

$$H_1(z) = \frac{1}{2}(1-z^{-1})$$

$$G_0(z) = (1 + z^{-1})$$

$$G_1(z) = -(1 - z^{-1})$$

Verify $T_0(z)$:

$$\tau_0(z) = \frac{1}{2}\left\{ G_0(z)H_0(z) + G_1(z)H_1(z) \right\}$$

$$= \frac{1}{2}\left\{ \frac{(1+z^{-1})^2}{2} - \frac{(1-z^{-1})^2}{2} \right\}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \left\{ \left(1+z^{-1}\right)^2 - \left(1-z^{-1}\right)^2 \right\}$$

$$= \frac{1}{4} \cdot \left\{ \left(1+z^{-1}+1-z^{-1}\right) \left(2z^{-1}\right) \right\} = 1$$

$$G_0(z) = z^{-1}$$

Only a delay of one sample

$$G_0(z) = \pm H_1(-z)$$

Haar case:

$$H_1(z) = \frac{1}{2}\left(1 - z^{-1}\right)$$

$$H_1(-z) = \frac{1}{2}\left(1 + z^{-1}\right).$$

More generally for alias cancellation

$$G_0(z) = \pm R(z) H_1(-z)$$

$$G_1(z) = \mp R(z) H_0(-z).$$

In particular for the Haar case we have chosen

$$R(z) = 2$$

(constant)

$$G_1(z) = -2H_0(-z)$$

Indeed

$$-2H_0(-z)$$

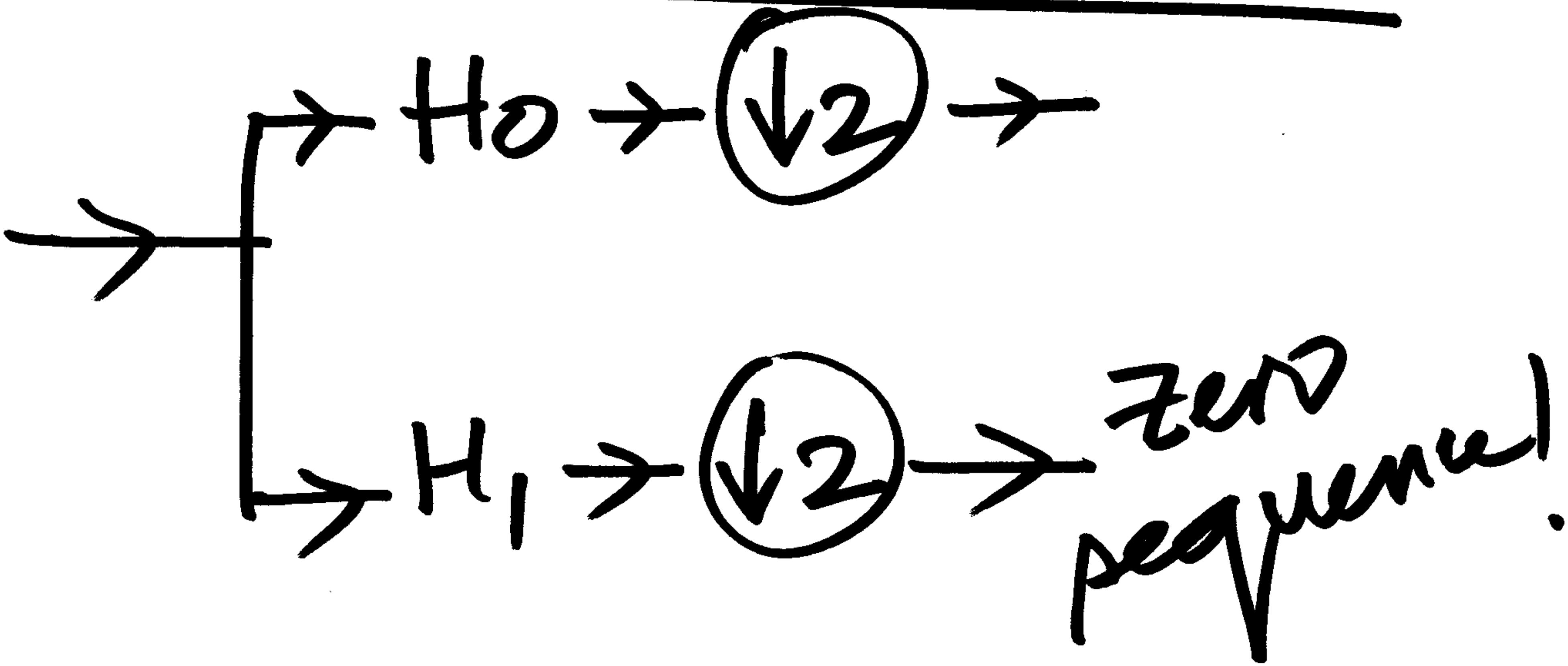
$$= (-2)\frac{1}{2}.(1 - z^{-1})$$

Correct

What does the Haar MRA do to constant sequences?

Consider $x[n] = c_1 \ \forall n$

Haar Filter Bank



$$H_1(z) = \frac{1}{2}(1 - z^{-1})$$

$$\frac{x[n] - x[n-1]}{2}$$

$$= 0 \quad \forall n$$

Consider a cascade of $(1 - z^{-1})$ terms

$$\to \boxed{1 - z^{-1}} \to \boxed{1 - z^{-1}} \to \cdots$$

Every instance of $(1 - z^{-1})$ in the cascade reduces a polynomial to one lower degree.

$$a_0 n^M + a_1 n^{(M-1)} + \cdots + a_M$$

$=$ polynomial input $x[n]$

$$a_0 n^M + a_1 n^{M-1} + \cdots + a_M$$

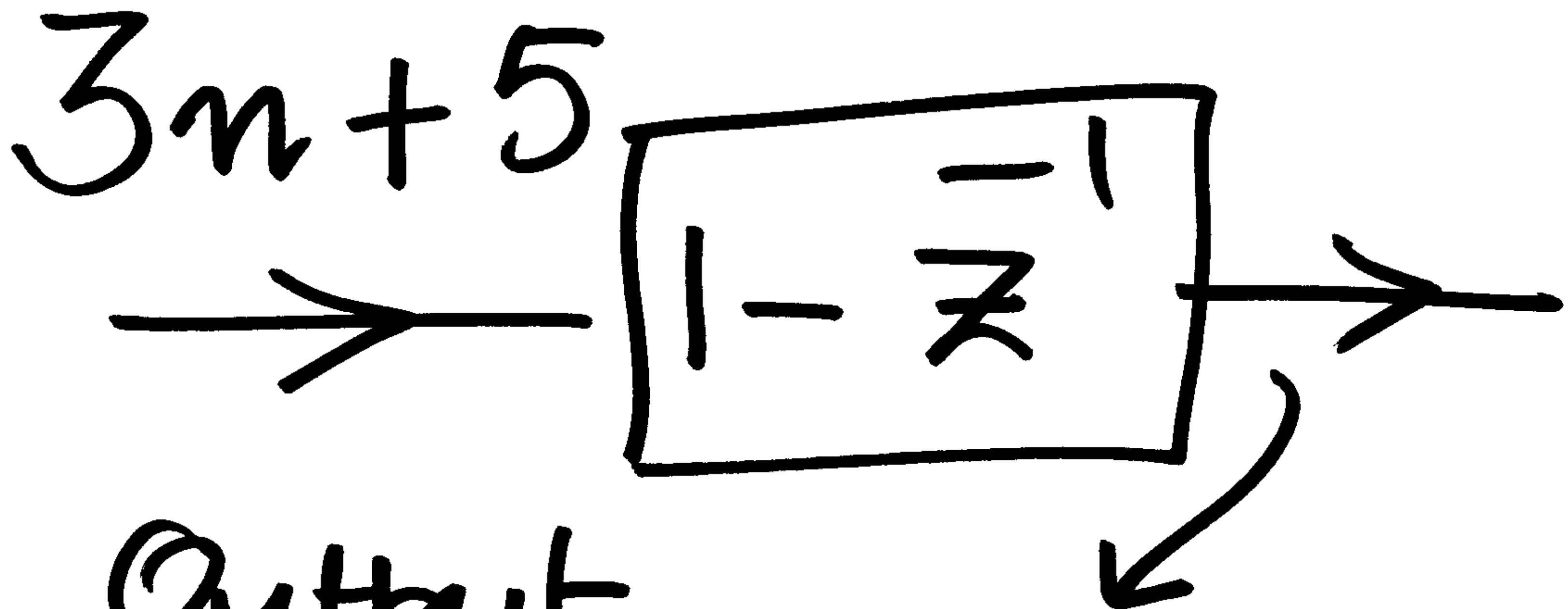
$$- \left\{ a_0 (n-1)^M + a_1 (n-1)^{M-1} \right.$$

$$\left. \cdots + a_M \right\}.$$

Expanding,

Coeff of x^M :

$$a_0 - a_0 = \underline{0} .$$



Output

$$= 3n+5-\left\{3(n-1)+5\right\}$$

$3n+5$

$-3n+3-5$

$= \underline{3} \quad \forall n$

$$\left(1 - z^{-1}\right)\Big|_{z=e^{j\omega}}$$

MUST BE HIGHPASS

$$= 1 - e^{-j\omega}$$

At $\omega = 0$ $\quad \underline{= 0}$

We now intend to build the <u>Daubechies'</u> MRAS

(INGRID DAUBECHIES).

As we go to
increasing "seniority"
in this family,
more and more
$(1-\bar{z}^{1})$

CONJUGATE QUADRATURE

Alias cancellation:

$$G_0(z) = \pm H_1(-z)$$
$$G_1(z) = \mp H_0(-z)$$

Let us choose (inspired by Haar):

$$G_1(z) = -H_0(-z)$$

$$G_0(z) = +H_1(-z)$$

$\tau_1(z) = 0$ by construction

$$\tau_0(z) =$$

$$\frac{1}{2}\left\{ H_1(-z)H_0(z) + (-H_0(-z))(H_1(z)) \right\}$$

If we consider
Haar,
relation bet H_0
and H_1 !

In the Haar case

$$H_1(-z) = \frac{1+z^{-1}}{2} = H_0(z).$$

We shall in general note that $H_0(z)$ should be related to $H_1(-z)$.

Choose $H_1(z)$ to be slightly modified from $H_0(-z)$.

Let us choose:

$$H_1(z) = z^{-D} H_0(-z^{-1})$$

"allowance of delay"

Haar:

$$z^{-1} H_0(-z^{-1})$$

$$H_0(-z^{-1})(z^{-1}) = \left(\frac{1-z}{2}\right) z^{-1}$$

$$= \left(\frac{z^{-1}-1}{2}\right).$$

Consider

$$H_1(z) = z^{-D} H_0(-z^{-1})$$

$$T_0(z) = \frac{1}{2}\left\{ H_0(z)(-1)^{-D} H_0(z^{-1}) z^{-D} - H_0(-z)\, z^{-D} H_0(-z^{-1}) \right\}$$