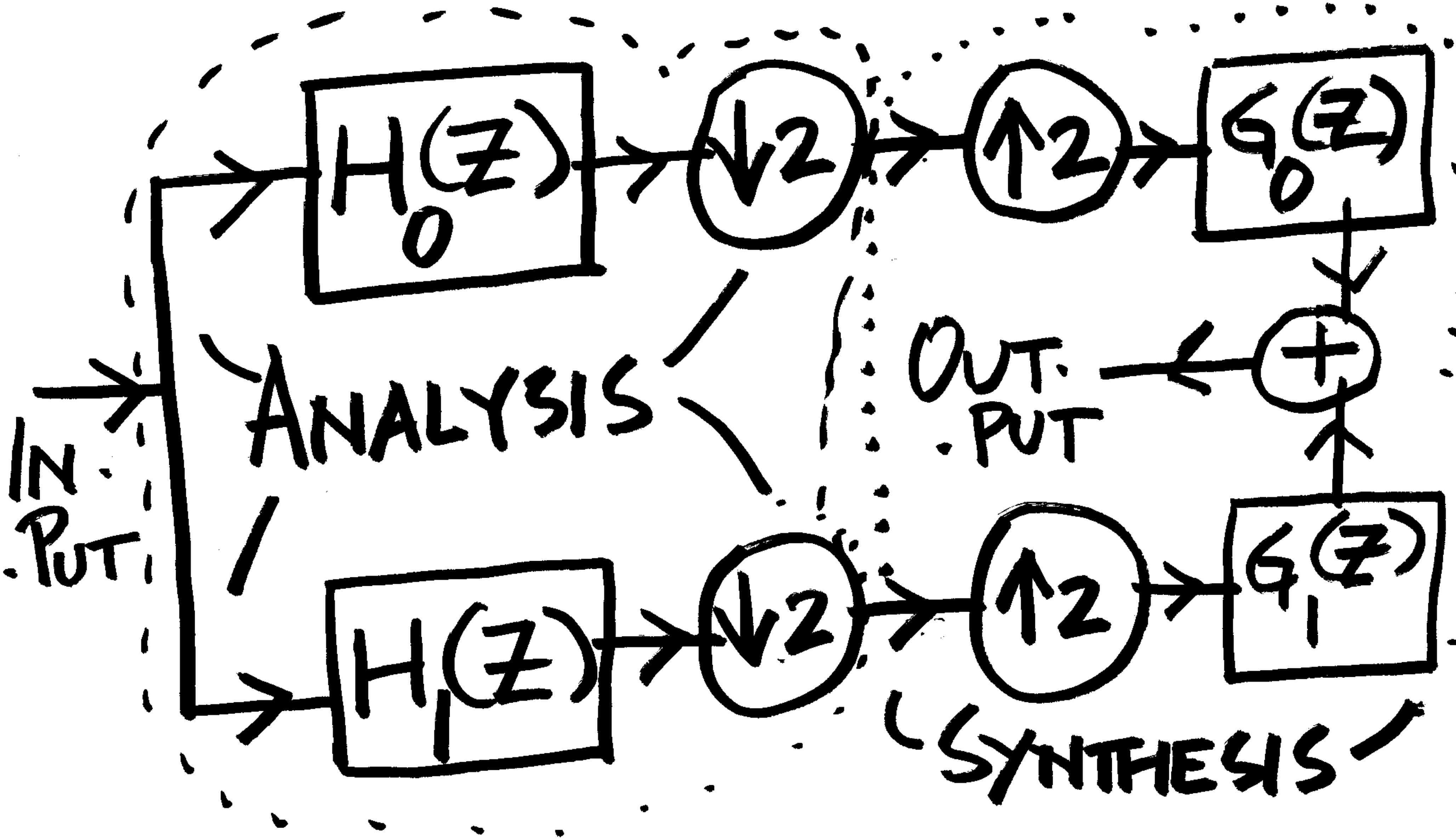
LECTURE 10

Z-DOMAIN ANALYSIS OF MULTIRATE FILTER BANK



IN PUT
$H_0(Z)$
$H_1(Z)$
$\downarrow 2$
$\downarrow 2$
ANALYSIS
$\uparrow 2$
$\uparrow 2$
$G_0(Z)$
$G_1(Z)$
+
OUT. PUT
SYNTHESIS

"NEW" OR "UNUSUAL" BLOCKS:

$\downarrow 2$ downsampler

$$\cdots\ -3\ -2\ -1\ 0\ 1\ 2\ 3\ 4\ 5\ \cdots$$

$$\cdots\ -1\ 0\ 1\ 2\ \cdots$$

Downsampler:
first "kills" some
samples:

-4	-3	-2	-1	0	1	2	3	4	5	
(-4)	0	(-2)	0	(0)	0	(2)	0	(4)	0	...

then 'compresses' the resultant:

$$\cdots\ 0\ (-2)\ 0\ (0)\ 0\ (2)\ 0\ (4)\ 0\ \cdots$$

$$\cdots\ -1\ \ 0\ \ 1\ \ 2\ \cdots$$

"Upsampler": "expands"

$\uparrow 2$: $\ldots -3\ -2\ -1\ 0\ 1\ 2\ 3\ 4\ 5 \ldots$

$0\ (-4)\ 0\ (-2)\ 0\ (0)\ 0\ (2)\ 0\ (4)\ 0$

The upsampler outdoes
the last 'compression'
step of a downsampler
Hence "*expander*".

Downsampling by M:

Example: $M = 5$.

$\downarrow M$:

$$-5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12$$

$$\cdots \; -5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 10 \; .$$

'Compression' step:

$$\cdots\ (-5)\,0 \,\vdots\, 0\,(0)\,0 \ldots 0\ (5)\,0 \ldots 0\,(10)$$

$$\cdots\ -1 \quad 0 \quad 1 \quad 2 \quad \cdots$$

Zeroes 'thrown away'!

Upsampling by 'M' :

$\uparrow M$:

Example $M = 7$

$$\cdots \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \cdots$$

$$(-14) \cdots (-7)\, 0 \cdots 0\, (0)\, \underbrace{0 \cdots 0}_{6 \text{ zeros}}\, (7)\, 0 \cdots 0\, (14)$$

Z- domain effect of upsampling:

$\uparrow M$: $\underset{\text{Input}}{x[n]} \rightarrow \uparrow M \rightarrow x_U[n]$

Z-transform of $x_v(n)$:

$$\sum_{n=-\infty}^{+\infty} x_v[n] z^{-n}$$

$$= \sum_{k=-\infty}^{+\infty} x[k] z^{-Mk}$$

(occurs at Mk)

$X_U(Z) = Z$ transform of $x_U[n]$

$$= X(Z^M).$$

$$X(z) = X_v(z^{1/M})$$

$\uparrow M$ is <u>invertible</u>.

clearly evident

(↓2) : First "killing" step:

can be viewed as modulation or "multiply by killing sequence"

'WINDOW'

Window periodic sequence: for $\downarrow 2$

$P_W[n] =$

$\ldots 1\ 0\ 1\ 0\ \underset{\substack{\uparrow \\ 0}}{1}\ 0\ 1\ 0\ \ldots$

periodically continued $\longleftarrow$

periodically continued $\longrightarrow$

$\downarrow M$, example of $M=4$:

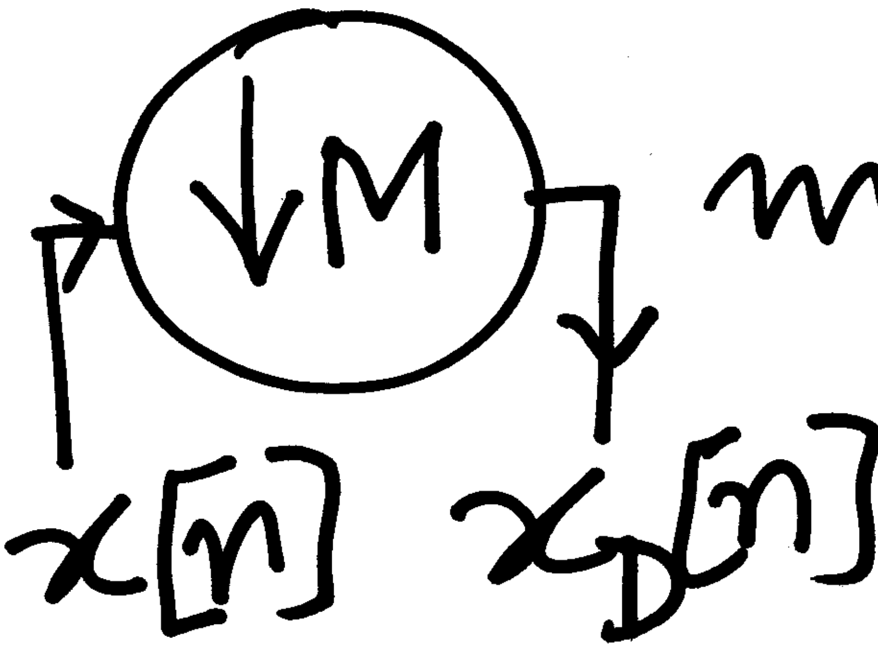
$p_W[n] =$

$\ldots 1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,0\,1 \ldots$

0

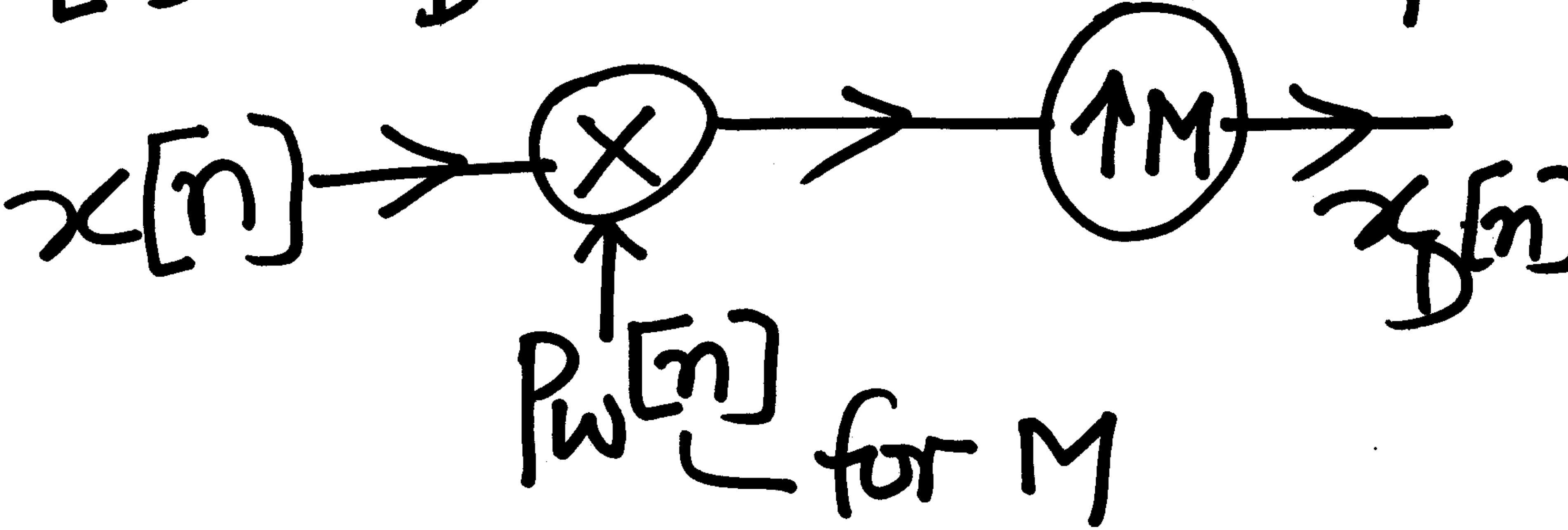
periodically continued

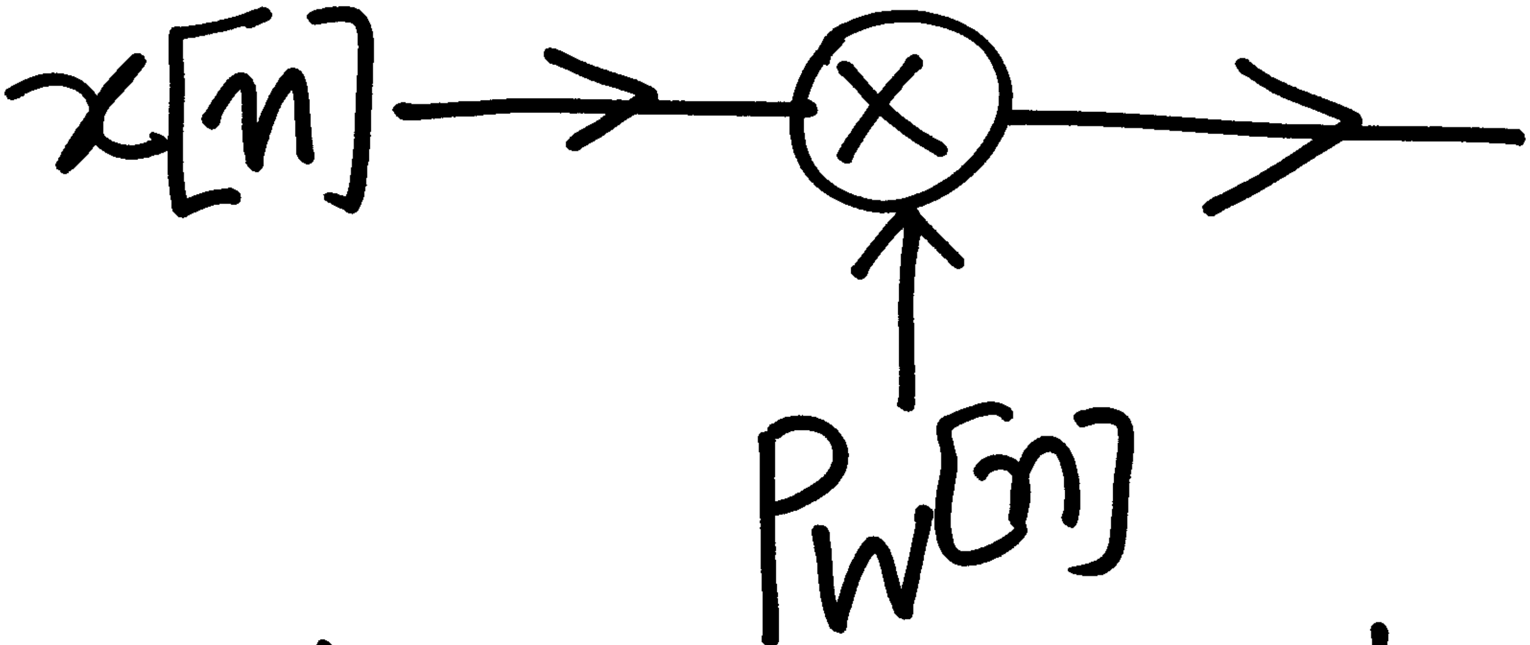
The 'Killing' process is essentially multiplication by $P_W[n]$ for appropriate M.



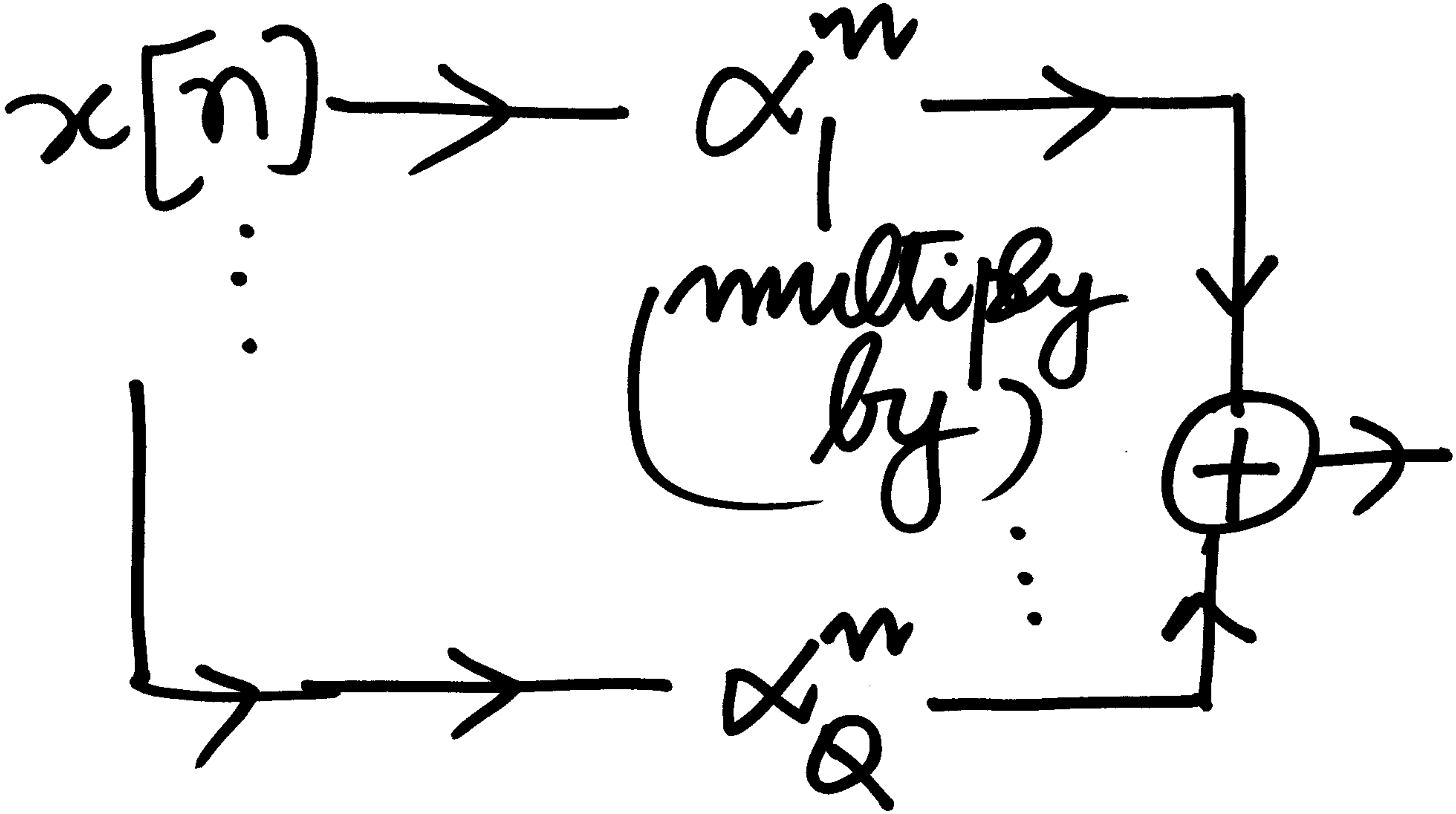
means:

'inverse upsample'





needs to be replaced by:



x[n]
α₁ⁿ
(multiply by)
α_Qⁿ
+

In other words we need to express

$$p_w[n] = \sum_{k=1}^{Q} c_k \alpha_k^n$$

Time-limited sequence:

$$b_0 \cdots b_{N-1}$$
$$\uparrow$$
$$0$$

Its Discrete Fourier Transform (DFT)

$$B[k] = \sum_{n=0}^{N-1} b[n] e^{-j\frac{2\pi}{N}nk}$$

$k = 0, \ldots, N-1$

We can reconstruct $b[n]$:

$$b[n] = \frac{1}{N} \cdot \sum_{k=0}^{N-1} B[k] e^{j\frac{2\pi}{N}kn}$$

$$n = 0, \dots, N-1$$

Inverse DFT (IDFT)

The expression

$$\sum_{k=0}^{N-1} B[k]\, e^{j\frac{2\pi}{N}kn} = \tilde{b}[n]$$

generates a periodic sequence

$$\tilde{b}[n+lN] = \sum_{k=0}^{N-1} B[k]\, e^{j\frac{2\pi}{N}k(n+lN)}$$

$$= \sum_{k=0}^{N-1} B[k]\, e^{j\frac{2\pi}{N}kn} \cdot \underbrace{\cancel{e^{j2\pi l}}}_{(1)}$$

$$\tilde{b}[n+lN] = \tilde{b}[n] \quad \forall n$$

Therefore periodic with period N.

For $M = 2$

we need the periodic sequence:

$\ldots 1\ 0\ 1\ 0\ 1\ 0\ \ldots$

one period

DFT of 1 0

$B[k]$

$$= 1 \cdot e^{-j\frac{2\pi}{2} \cdot 0 \cdot 0} + 0 = 1.$$

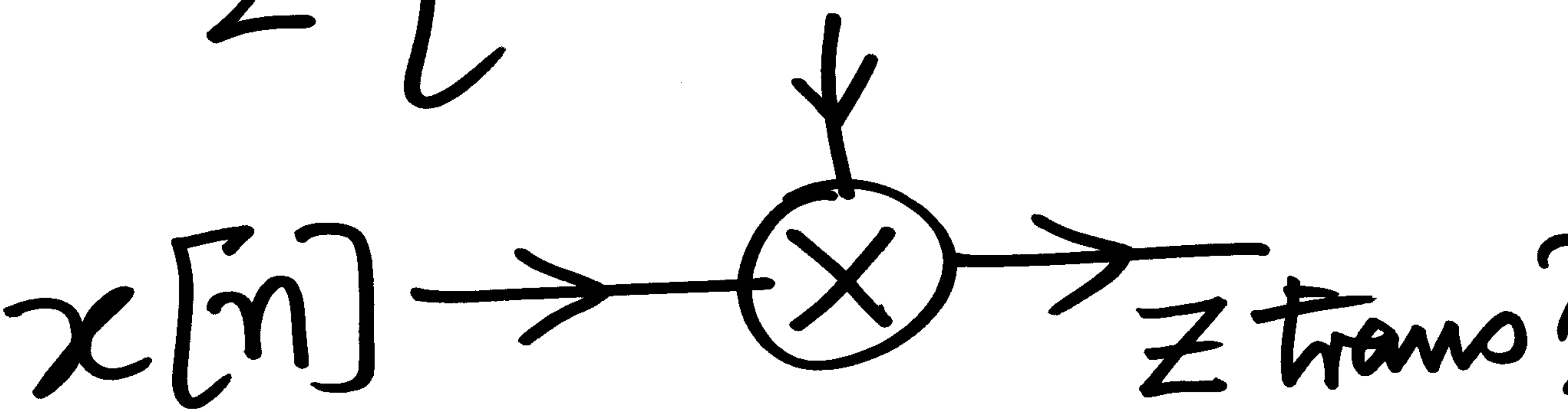
$k = 0, 1.$

$p_w[n]$ for $M = 2$

$$= \frac{1}{2} \sum_{k=0}^{1} B[k] e^{j\frac{2\pi}{2}kn}$$

$$= \frac{1}{2} \sum_{k=0}^{1} e^{j\pi kn}$$

$$= \frac{1}{2}\left\{ 1^n + (-1)^n \right\}.$$



$$\sum_{n=-\infty}^{+\infty} x[n] \cdot \frac{1}{2}\left\{1^{n} + (-1)^{n}\right\} \cdot z^{-n}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{+\infty} x[n]\, z^{-n} + \frac{1}{2}\cdot\sum_{n=-\infty}^{+\infty} x[n](-z$$

$$= \frac{1}{2}\underbrace{\left\{ X(z) + X(-z) \right\}}_{\downarrow}$$

goes thru an "inverse" $\uparrow 2 \rightarrow$

$$x[n], \; X(z) \longrightarrow \boxed{\downarrow 2} \longrightarrow x_D[n]$$

$$X_D(z) = \frac{1}{2}\left\{ X\left(z^{1/2}\right) + X\left(-z^{1/2}\right)\right.$$