Prof. Gadre
Lec No. 9
21/1/10

LECTURE 9

ITERATING THE FILTER BANK FOR $\phi(\cdot)$, $\psi(\cdot)$

$h[n]$: lowpass filter impulse response

$$\phi(t) = \sum_{n=-\infty}^{+\infty} h[n]\,\phi(2t-n)$$

$g[n]$: highpass filter impulse response

$$\psi(t) = \sum_{n=-\infty}^{+\infty} g[n]\,\phi(2t-n)$$

note: ϕ not ψ.

For the Haar MRA,
we had noted:

$$h[n] = \underset{\substack{\uparrow \\ 0}}{1} \qquad 1$$

$$g[n] = \underset{\substack{\uparrow \\ 0}}{1} \quad -1$$

for Haar MRA

Differential equations

example

$$y(t) = a_1 \frac{dx(t)}{dt} + a_2 x(t)$$

Difference equations

$$y[n] = \frac{1}{2}\left\{ x[n] + x[n-1] \right\}$$

example

Dilation equation

$$\phi(t) = \phi(2t) + \phi(2t-1)$$

example

new class of equations!

$$\phi(t) = \sum_{n=-\infty}^{+\infty} h[n]\,\phi(2t-n)$$

Take Fourier Transform

$$\Phi(t) \xrightarrow{\text{Fourier transform}} \hat{\Phi}(\Omega)$$

analog angular frequency variable

$$\hat{\Phi}(\Omega) = \int_{-\infty}^{+\infty} \phi(t)\, e^{-j\Omega t}\, dt$$

$$\int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} h[n]\,\phi(2t-n)\,e^{-j\Omega t}\,dt$$

$$= \sum_{n=-\infty}^{+\infty} h[n] \left\{ \int_{-\infty}^{+\infty} \phi(2t-n)\,e^{-j\Omega t}\,dt \right.$$

$$\int_{-\infty}^{+\infty} \phi(2t - n) \cdot e^{-j\Omega t} \, dt$$

Put $2t - n = \lambda$

$$t = (\lambda + n)/2$$

$$dt = \frac{1}{2} d\lambda$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \phi(\lambda)\cdot e^{-j\Omega\left(\frac{\lambda+n}{2}\right)}\,d\lambda$$

$$= \frac{1}{2}\, e^{-j\frac{\Omega n}{2}} \underbrace{\int_{-\infty}^{+\infty} \phi(\lambda)\cdot e^{-j\frac{\Omega}{2}\lambda}\,d\lambda}_{\hat{\phi}\left(\frac{\Omega}{2}\right)}$$

$$\hat{\Phi}(\Omega)$$

$$= \sum_{n=-\infty}^{+\infty} \underline{h[n]}\,\frac{1}{2}\,\underline{e}^{-j\frac{\Omega}{2}n}\,\hat{\Phi}\left(\frac{\Omega}{2}\right)$$

$$\sum_{n=-\infty}^{+\infty} h[n]\,e^{-j\Omega n} = \text{DTFT of } h[\cdot] \text{ at } \Omega$$

$h[n] \xrightarrow[\text{Discrete Time Fourier Transform (DTFT)}]{} H(\Omega)$

analog variable

Frequency domain dilation equation:

$$\hat{\Phi}(\Omega) = \frac{1}{2} H\left(\frac{\Omega}{2}\right) \hat{\Phi}\left(\frac{\Omega}{2}\right)$$

Replacing $\Omega \leftarrow \dfrac{\Omega}{2}$

$$\hat{\Phi}\left(\frac{\Omega}{2}\right) = \frac{1}{2} H\left(\frac{\Omega}{4}\right) \hat{\Phi}\left(\frac{\Omega}{4}\right)$$

$$\hat{\Phi}(\Omega) = \left(\text{Continue towards } N \to \infty\right)$$

$$\left\{ \prod_{m=1}^{N} \frac{1}{2} \cdot H\left(\frac{\Omega}{2^m}\right) \right\} \cdot \hat{\Phi}\left(\frac{\Omega}{2^N}\right)$$

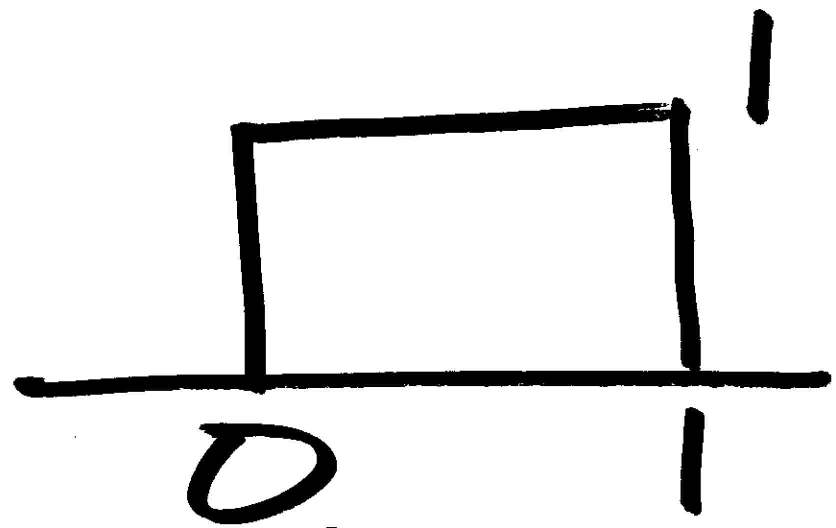
For finite Ω, we have

$$\lim_{N \to \infty} \frac{\Omega}{2^N} = 0$$

$$\hat{\Phi}(\Omega) = \left\{ \prod_{m=1}^{\infty} \frac{1}{2} \cdot H\left(\frac{\Omega}{2^m}\right) \right\} \hat{\Phi}(0)$$

for finite Ω

Haar $\phi(t)$



$$\hat{\phi}(\Omega) = \int_0^1 1 e^{-j\Omega t} dt$$

$$= \left. \frac{e^{-j\Omega t}}{-j\Omega} \right|_0^1$$

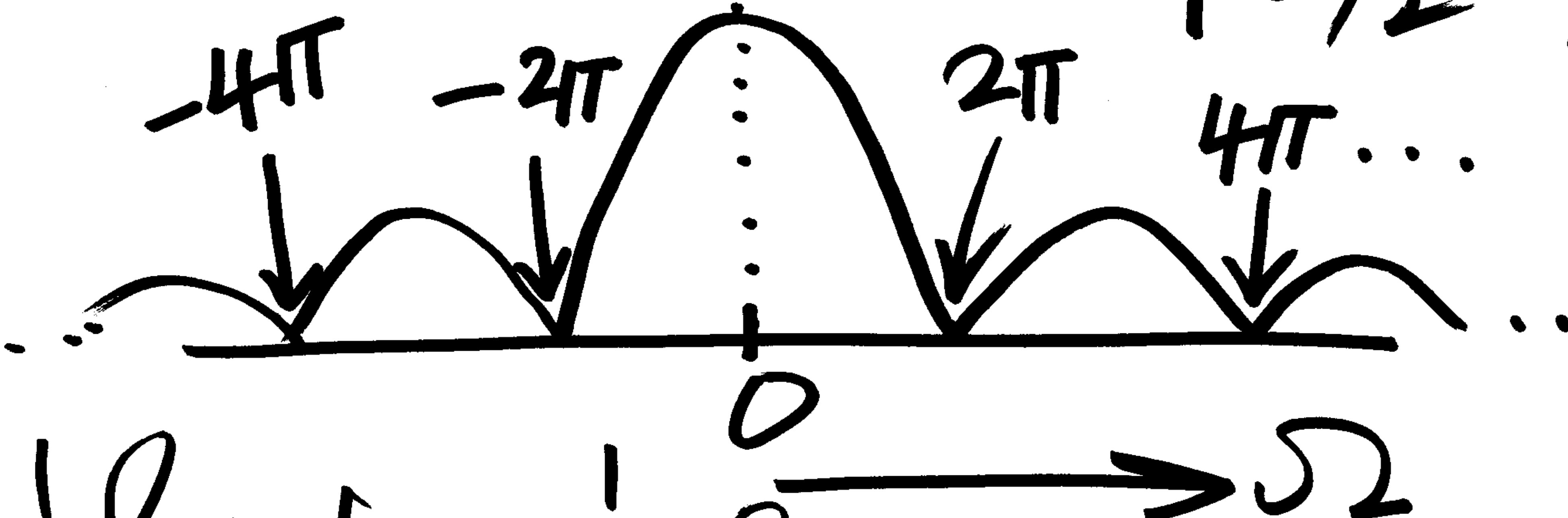
$$= \frac{1}{j\Omega}\left(1 - e^{-j\Omega}\right)$$

$$= \frac{1}{j\Omega} \cdot e^{-j\frac{\Omega}{2}} \cdot \left(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}\right)$$

$$= \frac{1}{j\Omega} \cdot e^{-j\frac{\Omega}{2}} \cdot 2j\sin\frac{\Omega}{2}$$

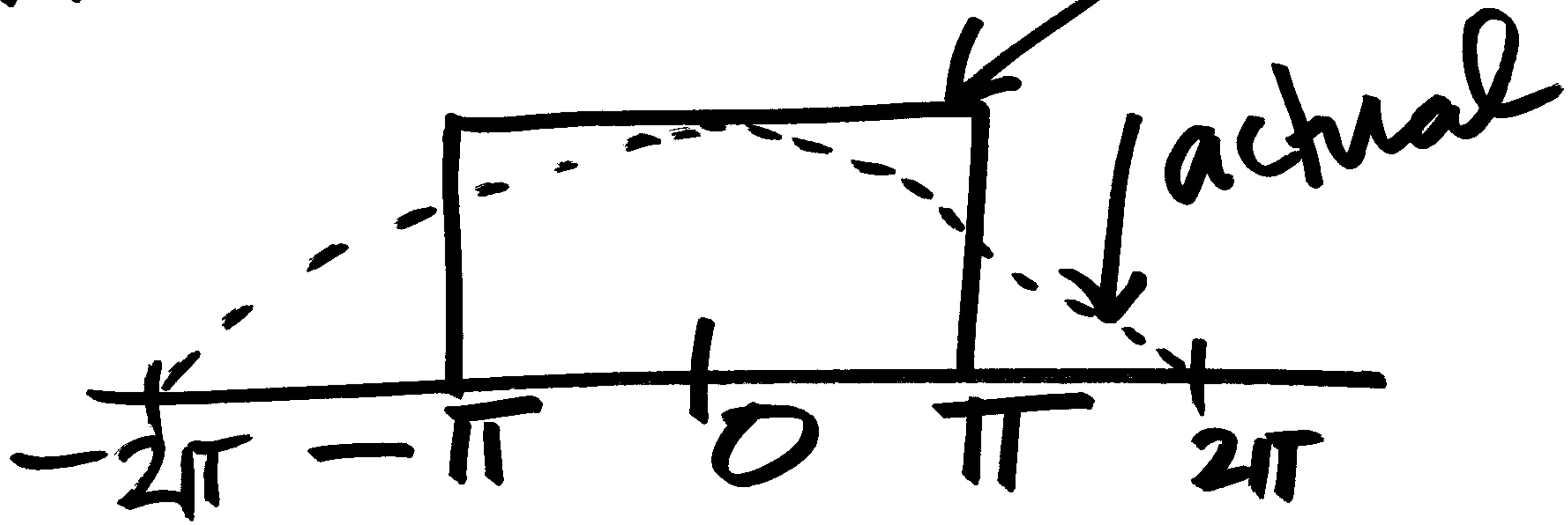
$$= e^{-j\frac{\Omega}{2}} \cdot \left(\frac{\sin \Omega/2}{\Omega/2} \right)$$

$$\left|\hat{\phi}(\Omega)\right| = \left|\frac{\sin \Omega/2}{\Omega/2}\right|$$



'lowpass' function

$\hat{\Phi}(\Omega)$ is moving towards the ideal



We need to focus on

$$\prod_{m=1}^{\infty} \frac{1}{2} H\left(\frac{\Omega}{2^m}\right)$$

$\hat{\phi}(0)$ is just a constant

The sequence $h[n]$ can be thought of, as a train of impulses at integer locations!

$H(\Omega)$ is the Fourier transform of this continuous time function

$$h(t) \xrightarrow[\text{Fourier transform}]{} H(\Omega)$$

$$H(\Omega) = \int_{-\infty}^{+\infty} h(t)\, e^{-j\Omega t}\, dt$$

Consider: $\alpha H(\alpha\Omega)$,

$\alpha > 0$:

$$= \alpha \int_{-\infty}^{+\infty} h(t) e^{-j\alpha\Omega t} \, dt$$

Put $\alpha t = \lambda$

$$= \alpha \int_{-\infty}^{+\infty} h\left(\frac{\lambda}{\alpha}\right) e^{-j\Omega\lambda} \frac{d\lambda}{\alpha}$$

$$= \int_{-\infty}^{+\infty} h\left(\frac{\lambda}{\alpha}\right) \cdot e^{-j\Omega\lambda} \, d\lambda.$$

$$h(t) \xrightarrow{\text{Fourier transform}} H(\Omega)$$

$$h\left(\frac{t}{\alpha}\right) \longrightarrow \alpha H(\alpha\Omega)$$

$\alpha > 0$.

Train of impulses
Corresponding to
the impulse response
$h[n] : h(t)$

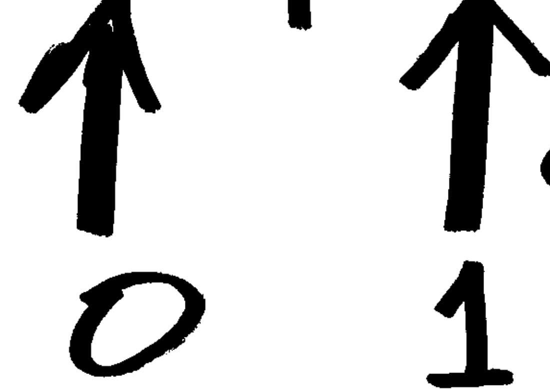
Its continuous time Fourier transform

$= H(\Omega)$

$\frac{1}{2}H\left(\frac{\Omega}{2}\right)$ is then the continuous time Fourier transform of $h(2t)$

For the Haar case

for example:

$h(t) =$ 

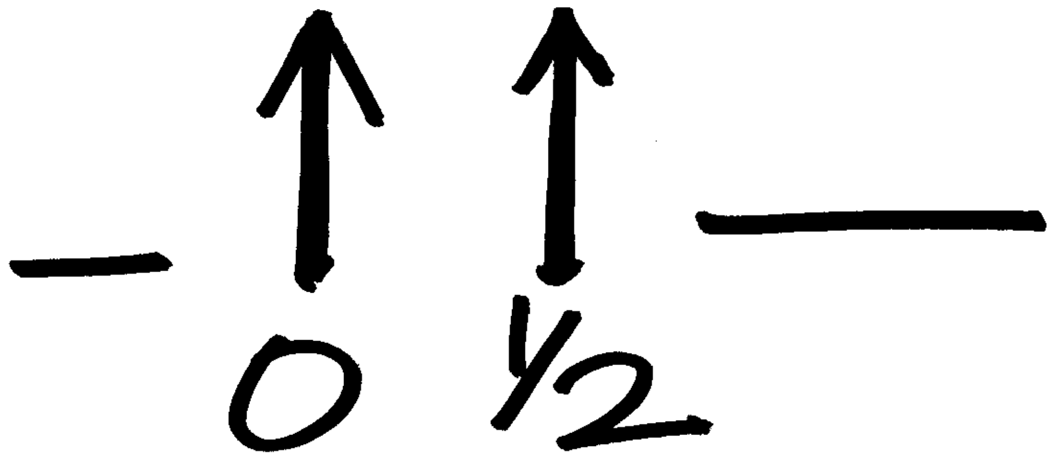
← impulse

$$\frac{1}{2} H\left(\frac{\Omega}{2}\right) \cdot \frac{1}{2} H\left(\frac{\Omega}{4}\right)$$

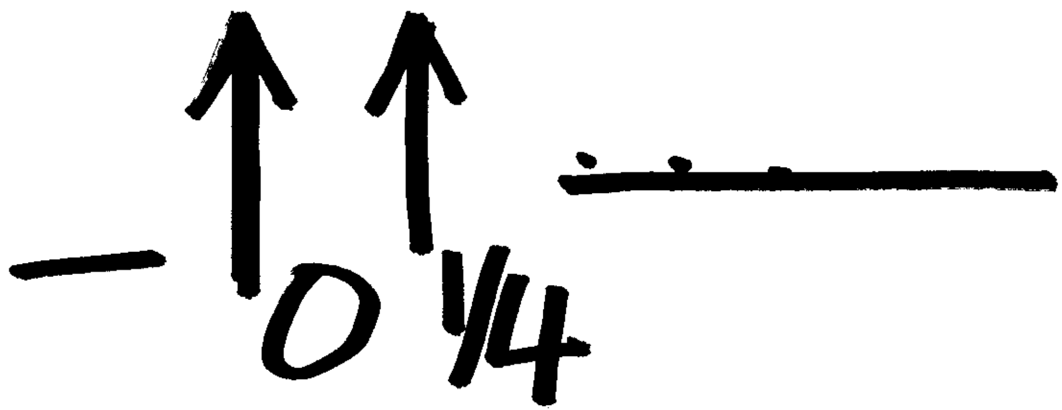
$$\equiv h(2t) * h(4t)$$

Continuous time convolution

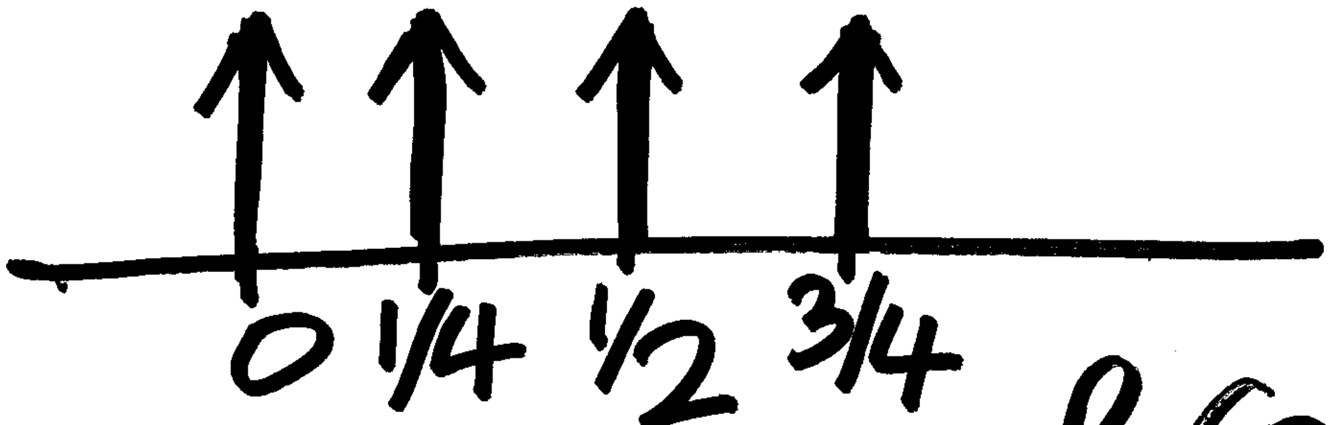
$h(2t)$:



$h(4t)$:

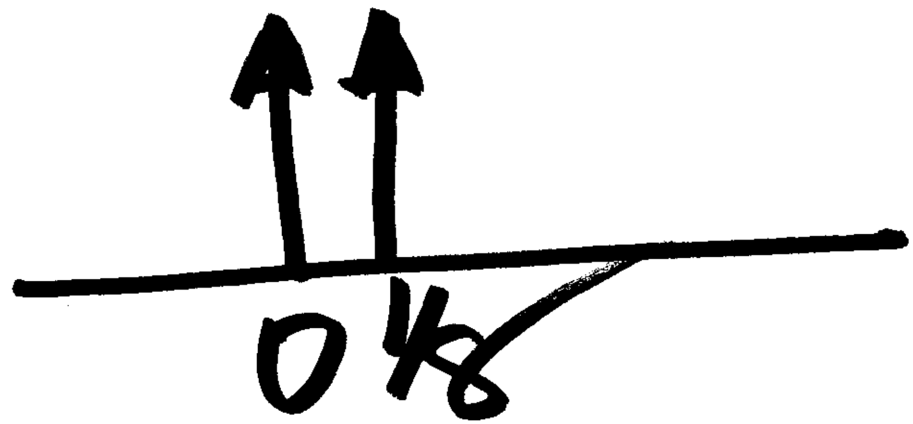


$h(2t) * h(4t)$



now $* \; h(8t)$

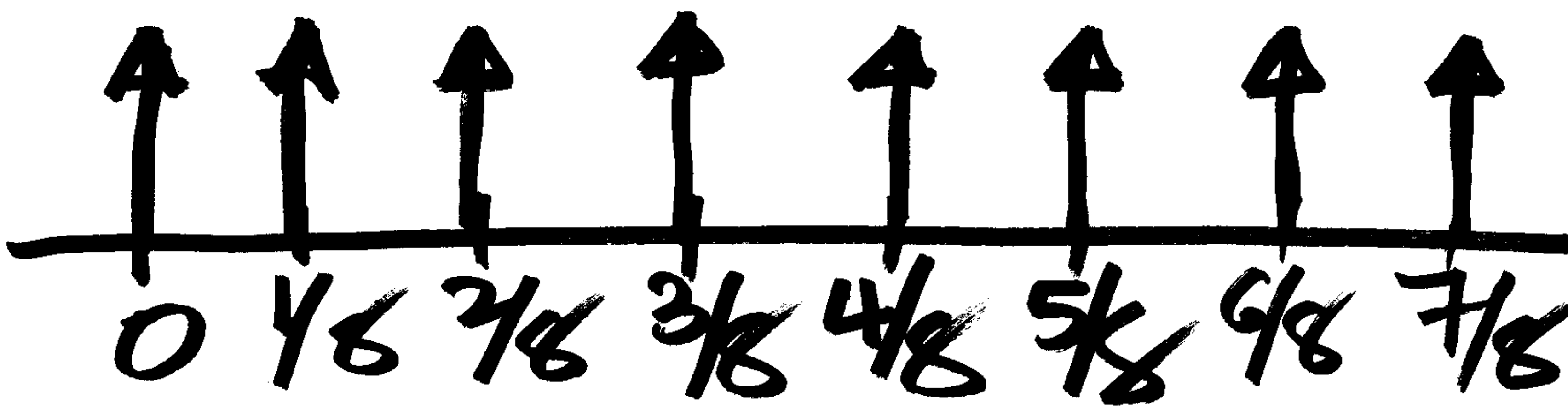
$h(8t)$



Convolving

$$\left(h(2t) * h(4t)\right) * h(8t)$$

$$h(2t) * h(4t) * h(8t)$$



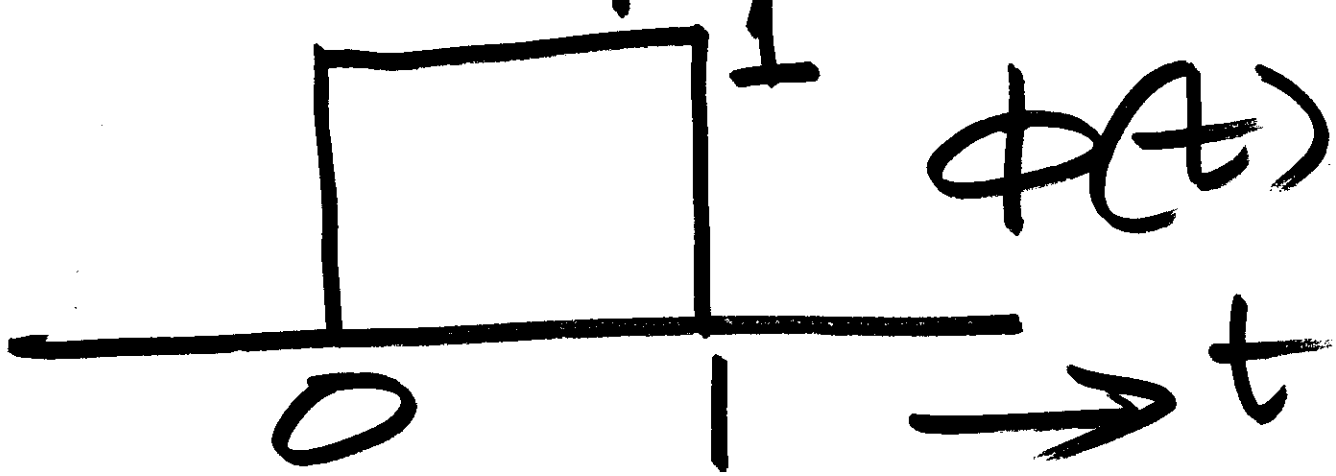
$h(2t) * h(4t) \cdots$

$* \, h(2^N t)$

$\equiv 2^N$ impulses located at $k/2^N$; $k = 0 : 2^N - 1$

The last of these impulses goes closer and closer to 1.

We are, obviously moving towards the continuous function

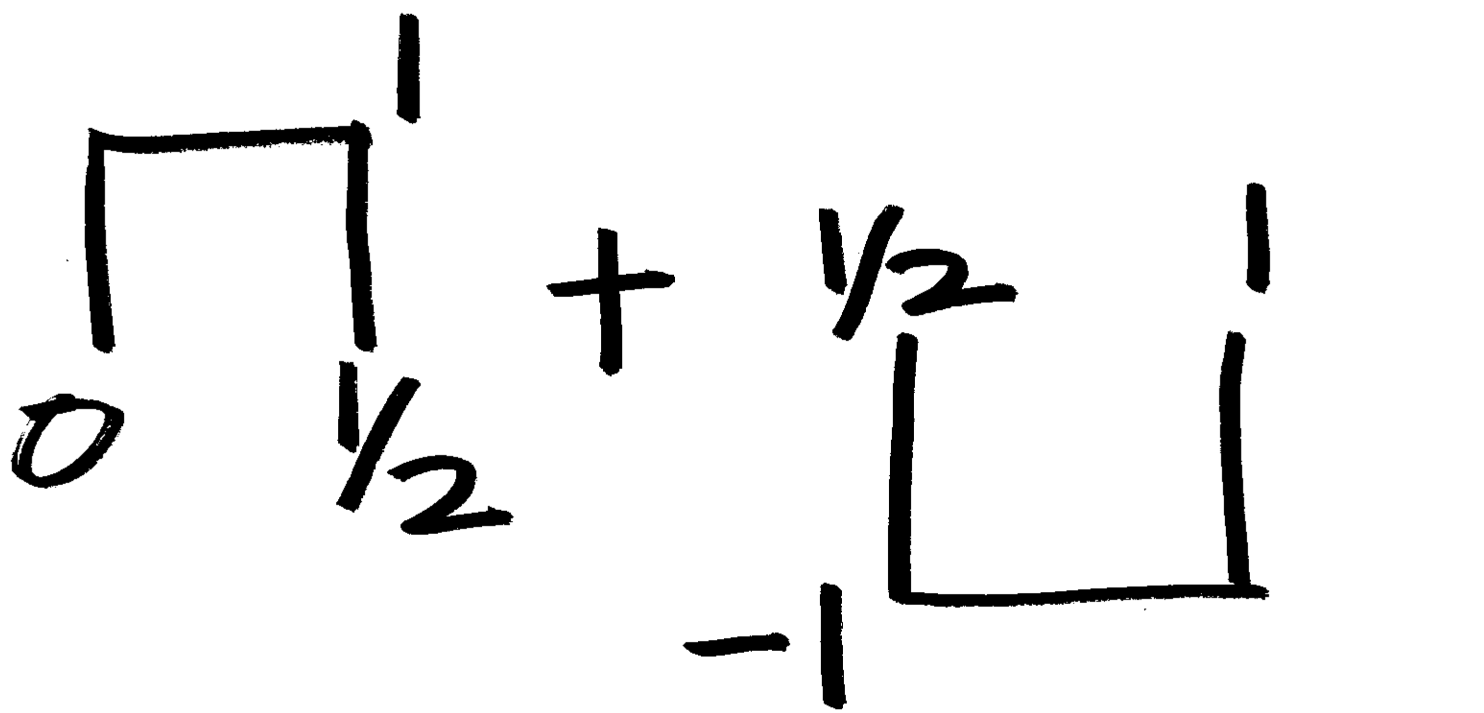


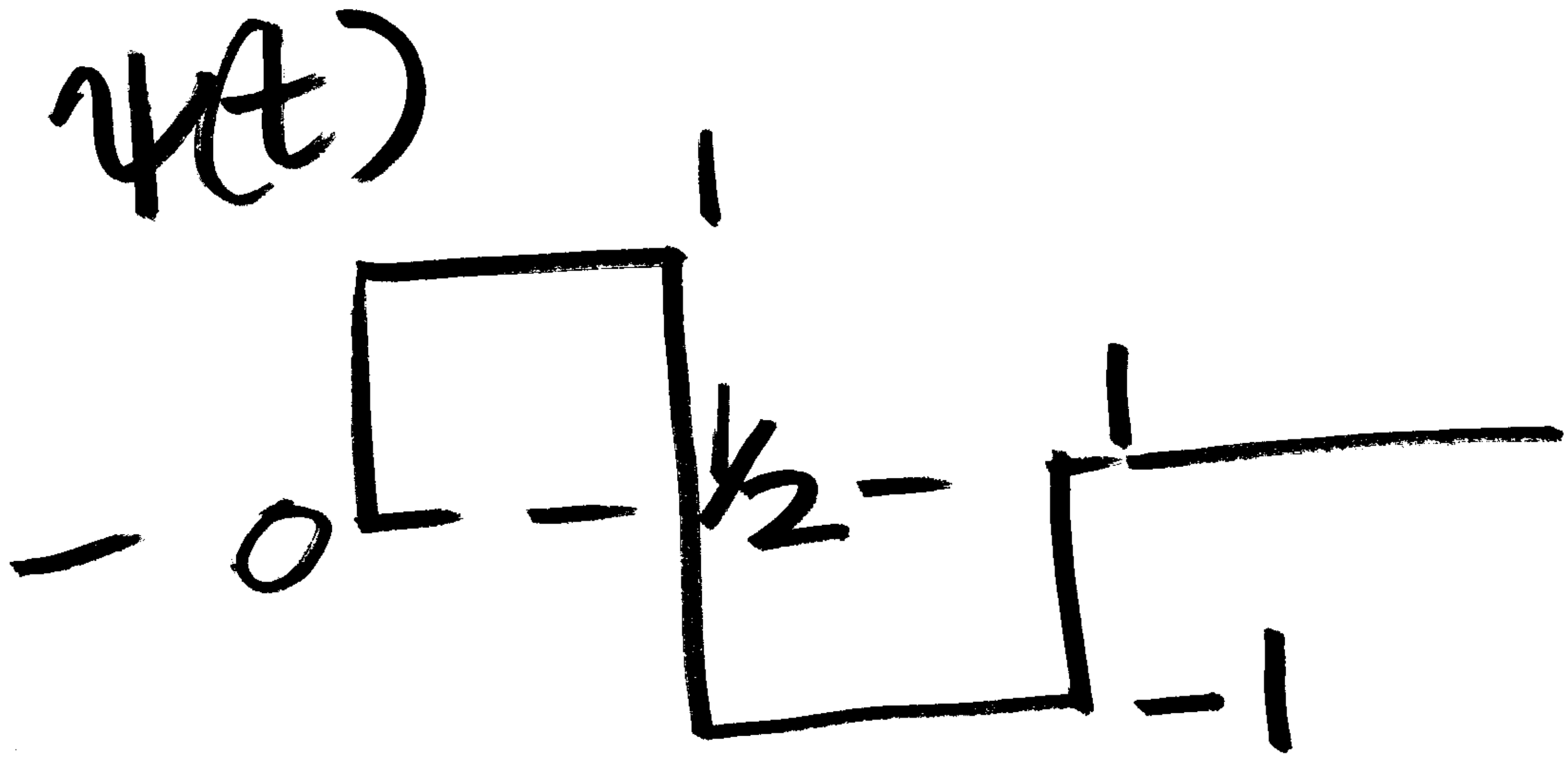
We have $\phi(\cdot)$

$$\psi(t) = \sum_{n=-\infty}^{+\infty} g[n]\,\phi(2t-n)$$

Haar case $g[n] = \underset{\substack{\uparrow \\ 0}}{1} \quad -1$

$$\phi(2t) \dot{-} \phi(2t-1)$$




ψ(t)
1
0
1/2
1
-1