EXERCISES

A good starting point is to prove all the results not proved in detail during the lectures. Then one can progress to various exercises provided in the reference textbooks.

MODULE 1:

1. Lecture 5: The Dirac equation gives $\rho = \psi^\dagger\psi$ and $\vec{j} = \psi^\dagger\vec{\alpha}\psi$. Show that in the non-relativistic limit, these reduce to the corresponding expressions following from the Schrödinger equation.
2. Lecture 5: Prove the Gordon decomposition identity for arbitrary spinors ψ_1 and ψ_2 satisfying the Dirac equation, $c\overline{\psi}_2\gamma^\mu\psi_1 = \frac{1}{2m}[\overline{\psi}_2 p^\mu\psi_1 - (p^\mu\overline{\psi}_2)\psi_1] - \frac{i}{2m}p_\nu(\overline{\psi}_2\sigma^{\mu\nu}\psi_1)$, where $p^\mu \equiv i\hbar\partial^\mu$ and $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu,\gamma^\nu]$.
   Show that when $\psi_1 = \psi_2$, the two combinations on the r.h.s. (kinetic and spin) satisfy the current conservation condition separately.
3. Lecture 5: Show that $\overline{\psi}\psi = \pm 1$ for free particle solutions of the Dirac equation.
4. Lecture 6: For a relativistic electron, find the energy levels in a constant magnetic field $\vec{B} = B\hat{z}$.
5. Lecture 7: Show that $[\vec{\alpha}\cdot\vec{p}, \beta(\vec{\Sigma}\cdot\vec{L} + \hbar)] = 0$.
6. Lecture 8: The Lenz vector, $\vec{M} = \frac{1}{2m}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - \frac{Ze^2}{r}\vec{r}$, is conserved for a particle moving in a Coulomb potential in classical mechanics (it defines the major axis of the orbit). It is not conserved for a Dirac particle in a Coulomb potential, but show that $\vec{M}\cdot\vec{\Sigma}$ is conserved, and connects states with the same j but opposite parities (e.g. $2s_{1/2}$ and $2p_{1/2}$).
7. Lecture 8: Construct the relativistic spinor eigenstates for the Coulomb problem in the Dirac basis:
   (a) First combine the spherical harmonics Y_{lm} with spin states $s_z = \pm\frac{1}{2}$ to obtain two-component eigenstates $\mathcal{Y}_l^{jm}$ of the total angular momentum $j = l \oplus \frac{1}{2}$.
   (b) Show that $\vec{\sigma}\cdot\hat{r}\mathcal{Y}_{j\mp1/2}^{jm} = -\mathcal{Y}_{j\pm1/2}^{jm}$.
   (c) Then combine $\mathcal{Y}_l^{jm}$ states with appropriate coefficients to obtain four-component eigenstates ψ for given j, m, k.
8. Lecture 10: Show that $i\vec{\Sigma}\cdot\vec{\nabla}\times\vec{E} + 2\vec{\Sigma}\cdot\vec{E}\times\vec{p}$ is Hermitian.
9. Lecture 10: The Dirac equation for a fermion with anomalous magnetic moment has an extra interaction, $\frac{\kappa e\hbar}{4mc}\sigma_{\mu\nu}F^{\mu\nu}$. Consider its non-relativistic expansion, and find the effect of passing a polarised neutron beam through a static non-uniform electric field.
10. Lecture 10: For reflection of a Dirac fermion from a barrier, calculate the reflection and transmission coefficients for the current $j_z = c\psi^\dagger\alpha_z\psi$, i.e. evaluate $(j_z)_r/(j_z)_i$ and $(j_z)_t/(j_z)_i$.
11. Lecture 10: Consider a Dirac fermion in a one dimensional potential well, $V(x) = eA_0(x) = -V_0\theta(a - |x|)$. For a Dirac bound state with spin down:
    (a) Find the wavefunction $\psi(x)$ for $|x| > a$ and $|x| < a$ when $V_0 < mc^2$.
    (b) Give all the boundary conditions to be satisfied by $\psi(x)$ at $x = \pm a$.
    (c) For given fixed a, find the minimum value of V_0 that just binds the n^{th} bound state (i.e. $E_n = mc^2$). What is the wavefunction for such a state?

12. Lecture 13: Using the chiral symmetry (i.e. γ_5 symmetry) of the Dirac operator $(i\not{D} - m)$, show that if λ is an eigenvalue of $i\not{D}$, so is $-\lambda$.
13. Lecture 13: Argue that any 4×4 complex matrix can be expressed as a linear combination (with complex coefficients) of the 16 γ-matrices $\{S, V, T, A, P\}$.
14. Lecture 13: Compare the projection operators for helicity and chirality. Show that they are the same for massless Dirac particles, but are different for massive Dirac particles.
15. Lecture 14: Fermi statistics is defined in terms of the creation and annihilation operators, $a^\dagger$ and a respectively, with the rules: $a|0\rangle = 0$, $a^\dagger|0\rangle = |1\rangle$, $a^2 = 0$, $(a^\dagger)^2 = 0$, $aa^\dagger + a^\dagger a = 1$. With mixing of particle and antiparticle modes, it is possible to have an operator $c = c^\dagger = a + a^\dagger$ for Majorana fermions. Show that $c|0\rangle = |1\rangle$, $c|1\rangle = |0\rangle$, $c^2 = 1$. Find the analogue of the number operator $a^\dagger a$ for Majorana fermions?
16. Lecture 15: Construct a Lorentz invariant, Hermitian and probability conserving interaction term for a Dirac fermion with anomalous electric dipole moment. What happens to the parity and time reversal transformations in the presence of this term?
17. Lecture 15: Construct operators for parity, charge conjugation and time reversal symmetries in (a) Weyl and (b) Majorana representations. Which of these discrete symmetries are satisfied by (a) Weyl and (b) Majorana fermions, and which ones are not? What is the combined PCT operation in the two cases?
18. Lecture 17: Low energy electrons in graphene have pseudo-spin (related to orbital angular momentum) as well as the usual spin. Consider the situation when a magnetic field B is applied perpendicular to the graphene sheet. Derive the Landau level spectrum, i.e. $E_n = \sqrt{2\hbar|e|Bv_F^2(n + \frac{1}{2} \pm \frac{1}{2})}$, and the pattern of Zeeman splitting of the energy levels.

MODULE 2:

1. Lecture 19: Unitary representations of continuous groups are conveniently parametrised in terms of their Hermitian generators as $U(\alpha) = \exp(i\sum_a \alpha_a T_a)$. Then the group composition rule demands that the product $U(\beta)U(\alpha)U(-\beta)U(-\alpha)$ must be of the form $U(\gamma)$. Expand around identity and deduce that the group algebra must close, i.e. $[T_a, T_b] = if_{abc}T_c$ with real structure constants f_ijk.
2. Lecture 19: Show that the Lorentz group generators, $J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$, satisfy the algebra, $[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})$.
3. Lecture 22: Derive the commutation rules for the Pauli-Lubanski vector, $[W^\mu, W^\nu] = i\epsilon^{\mu\nu\lambda\sigma}W_\lambda P_\sigma$.
4. Lecture 23: For the Wigner rotation, $W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p)$, show that $W(R, p) = R$ when Λ is an ordinary rotation R.
5. Lecture 23: For a massless particle, light-front coordinates are convenient, i.e. $x^\pm = x^0 \pm x^3$. Show that $W^- = 0$ in this case, while W^1, W^2, W^+ satisfy the algebra of the two dimensional Euclidean group $E(2)$.
6. Lecture 23: Obtain the algebra of the Galilean group from the non-relativistic limit of the Poincaré group generators $\vec{J}, \vec{K}, \vec{P}, H$. Find the generators, and their allowed values of quantum numbers, that label the single particle states in this case.

7. Lecture 25: Show that the spinors of the free fermion basis obey:
(a) $\omega^{r\dagger}(\epsilon_r\vec{p})\ \omega^{r'}(\epsilon_{r'}\vec{p}) = \frac{E}{mc^2}\delta_{rr'}$
(b) $\sum_{r=1}^{4} \omega^r_\alpha(\epsilon_r\vec{p})\ \omega^{r\dagger}_\beta(\epsilon_r\vec{p}) = \frac{E}{mc^2}\delta_{\alpha\beta}$

MODULE 3:

1. Lecture 27: Show that the Feynman propagator for a Dirac fermion, $S_F(p) = \frac{\not{p}+m}{p^2-m^2+i\epsilon}$, $\epsilon \to 0_+$, propagates the positive energy component forward in time and the negative energy component backward in time.
2. Lecture 27: Suppose the vacuum is replaced by a Fermi gas with Fermi momentum k_F. How does that modify the Feynman propagator? What is the change in the low density limit?
3. Lecture 29: Prove the following identities:
(a) $\gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}$
(b) $\gamma_\mu \not{a}\not{b}\not{c}\not{d}\gamma^\mu = 2(\not{d}\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}\not{d})$
4. Lecture 31: Consider the local gauge transformation, $A_\mu(x) \to A_\mu(x) - \partial_\mu\Lambda(x)$, $\psi(x) \to U(x)\ \psi(x)$, $U(x) = e^{ie\Lambda(x)/\hbar c}$. Show that the gauge connection transforms as $\exp[-\frac{ie}{\hbar c}\int_{x_1}^{x_2} A_\mu(x)\ dx^\mu] \to U(x_2)\ \exp[-\frac{ie}{\hbar c}\int_{x_1}^{x_2} A_\mu(x)\ dx^\mu]\ U^{-1}(x_1)$. Deduce that $\exp[-\frac{ie}{\hbar c}\oint A_\mu(x)\ dx^\mu]$ and $\overline{\psi}(x_2)\ \exp[-\frac{ie}{\hbar c}\int_{x_1}^{x_2} A_\mu(x)\ dx^\mu]\ \psi(x_1)$ are gauge invariant quantities.
5. Lecture 32: The polarisation vector for photons ϵ^μ has four components. For free photons, argue that complete gauge fixing to radiation gauge keeps only two transverse components as physical degrees of freedom.
6. Lecture 32: Construct the polarisation vector for a virtual photon describing a static electric field.
7. Lecture 38: Expressions for unpolarised Compton scattering and crossing symmetry can be used to obtain results for $e^+e^- \to \gamma\gamma$. The $1s$ state of positronium (bound state of e^+ and e^-) decays by annihilation into photons. There are two states, depending on the spin pairing, corresponding to $S = 0$ and $S = 1$. Using discrete symmetries, deduce which of the two states can decay into two photons (the other has to decay into three photons), and obtain its lifetime.
8. Lecture 39: Calculate the leading order (tree level) unpolarised cross-section for Bhabha scattering, $e^+e^- \to e^+e^-$.
9. Lecture 44: Find the leading order amplitude for scattering of photons (i.e. $\gamma\gamma \to \gamma\gamma$) using the Feynman rules. Evaluate the Dirac trace using current conservation. Show that the result is finite, without evaluating the momentum integral. For photons of energies much smaller than the electron mass, express the scattering amplitude in terms of powers of the electromagnetic field strength $F_{\mu\nu}$.
10. Lecture 44: Consider a charged scalar field $\Phi = \rho e^{i\phi}$, which under the electromagnetic gauge symmetry transforms as $\Phi(x) \to e^{iq\Lambda(x)/\hbar c}\Phi(x)$ ($q = 2e$ in the case of Cooper pairs). Superconductivity arises when this field condenses, i.e. $\rho(x) \approx$ constant. Show that in the condensed phase, the gauge invariant term $|D_\mu\Phi|^2$ becomes proportional to $(qA_\mu + \partial_\mu\phi)^2$. Find the resultant value of the photon mass.